Locally conformal cosymplectic manifolds and time-dependent Hamiltonian systems*

Domingo Chinea, Manuel de León, Juan C. Marrero

Abstract. We show that locally conformal cosymplectic manifolds may be seen as generalized phase spaces of time-dependent Hamiltonian systems. Thus we extend the results of I. Vaisman for the time-dependent case.

Keywords: cosymplectic manifold, locally conformal cosymplectic manifold, Hamiltonian systems

Classification: 53C15, 58F05, 70H05

1. Introduction.

There is a well known connection between symplectic geometry and mathematical physics, particularly mechanics. A Hamiltonian dynamical system is given by a symplectic manifold (the phase space) and a function on it (the Hamiltonian function). In [7], I. Vaisman shows that locally conformal symplectic manifolds may be seen as generalized phase spaces of Hamiltonian dynamical systems, since the form of the Hamilton equations is preserved by homothetic transformations.

The purpose of this paper is to extend the results of Vaisman to the case of time-dependent Hamiltonian systems. In this case, the phase space is a cosymplectic manifold (the odd dimensional analogue of a symplectic manifold) and the Hamiltonian function a function on it. Then we prove that locally conformal cosymplectic manifolds may be seen as generalized phase spaces of time-dependent Hamiltonian systems.

2. Cosymplectic manifolds and time-dependent Hamiltonian systems.

An almost cosymplectic manifold is a triple (M, Ω, η), where M is a $(2n+1)$-dimensional manifold and Ω and η are a 2-form and a 1-form respectively on M such that $\eta \wedge \Omega^n \neq 0$. If, in addition, Ω and η are closed, i.e., $d\eta = 0, d\Omega = 0$, then M is said to be a cosymplectic manifold ([3], [6]).

Let $C^\infty(M)$ be the ring of differentiable functions on M, and $\Xi(M), \Lambda^1(M)$ the $C^\infty(M)$-modules of differentiable vector fields and 1-forms on M, respectively. If M is an almost cosymplectic manifold, then there exists an isomorphism of $C^\infty(M)$-modules

$$A : \Xi(M) \longrightarrow \Lambda^1(M) \tag{1}$$

* Partially supported by DGICYT–Spain, Proyecto PB88–0012 and Consejería de Educación del Gobierno de Canarias–Spain

defined by

$$A(X) = i_X\Omega + \eta(X)\eta$$

(see [6]). The Reeb vector field ξ is given by $\xi = A^{-1}(\eta)$. Thus ξ is characterized by the identities $i_\xi\Omega = 0$, $\eta(\xi) = 1$. Now, let $f : M \to R$ be a differentiable function defined on M. Then there exists a unique vector field X_f on M such that $A(X_f) = df - \xi(f)\eta + \eta$, i.e., X_f is the vector field characterized by the identities

$$i_{X_f}\Omega = df - \xi(f)\eta, \quad \eta(X_f) = 1.$$

If (M, Ω, η) is cosymplectic, we call X_f the **Hamiltonian vector field** with the energy function f. In fact, this construction generalizes the corresponding one for Hamiltonian vector fields on a symplectic manifold ([1], [4], [5]).

Next, we show that the phase space for a time-dependent Hamiltonian system is a cosymplectic manifold.

Let (S, ω) be a $2n$-dimensional symplectic manifold. Consider the product manifold $R \times S$ and denote by $\pi : R \times S \to S$ the canonical projection defined by $\pi(t, x) = x$, $t \in R$, $x \in S$. We set $\tilde{\omega} = \pi^*\omega$, $\eta = dt$. Then $(R \times S, \tilde{\omega}, \eta)$ is a cosymplectic manifold. If $H : R \times S \to R$ is a function, let X_H be the Hamiltonian vector field for H, i.e., X_H is the unique vector field on $R \times S$ characterized by the identities

$$i_{X_H}\tilde{\omega} = dH - \frac{\partial H}{\partial t}\, dt, \quad dt(X_H) = 1,$$

since the Reeb vector field is $\partial/\partial t$.

Consider the canonical coordinates (q^i, p_i) on S and (t, q^i, p_i) the induced coordinates on $R \times S$, where t is the canonical global coordinate on R. Since $\tilde{\omega} = dq^i \wedge dp_i$, we deduce that

$$X_H = \frac{\partial}{\partial t} + \frac{\partial H}{\partial p_i}\,\frac{\partial}{\partial q^i} - \frac{\partial H}{\partial q^i}\,\frac{\partial}{\partial p_i}\,.$$

Hence the integral curves $\sigma(t) = (t, q^i(t), p_i(t))$ of X_H satisfy the Hamilton equations:

$$\frac{dq^i}{dt} = \frac{\partial H}{\partial p_i}, \quad \frac{dp_i}{dt} = -\frac{\partial H}{\partial q^i}\,.$$

We remark that the *Hamilton equations may be also obtained, if one considers a function* $H : M \to R$ *on an arbitrary cosymplectic manifold* (M, Ω, η). In fact, there exists a local coordinate system (t, q^i, p_i) on M such that

$$\Omega = dq^i \wedge dp_i, \quad \eta = dt.$$

So, the cosymplectic manifolds provide a good geometric framework for time-dependent Hamiltonian systems.

3. Locally conformal cosymplectic manifolds and time-dependent Hamiltonian systems.

An almost cosymplectic manifold (M, Ω, η) is said to be **locally conformal cosymplectic** (l.c.c.), if there exist an open covering $\{U_\alpha\}_{\alpha \in A}$, and a system of functions $\sigma_\alpha : U_\alpha \to R$ such that

$$d(e^{2\sigma_\alpha}\Omega) = 0, \quad d(e^{\sigma_\alpha}\eta) = 0. \tag{2}$$

The local 1-forms $d\sigma_\alpha$ glue up to a closed 1-form θ satisfying

$$d(\Omega) = -2\Omega \wedge \theta, \quad d(\eta) = \eta \wedge \theta. \tag{3}$$

Conversely, if there exists a 1-form satisfying (3), we obtain a family $\{U_\alpha, \sigma_\alpha\}$ satisfying (2) (see [2]). We call θ the **Lee form**. Clearly, if (M, Ω, η) is cosymplectic, then it is l.c.c. and its Lee form vanishes.

Let (M, Ω, η) be a l.c.c. manifold with Lee form θ. Then we may define a vector field B on M by

$$A(B) = \theta - \theta(\xi)\eta,$$

where $A : \Xi(M) \to \Lambda^1(M)$ is the isomorphism given by (1) and ξ is the Reeb vector field of M. We call B the **canonical vector field**.

Next, let $f : M \to R$ be a function on M. Then there exists a unique vector field X_f on M defined by

$$X_f = A^{-1}(df - \xi(f)\eta + \eta) + fB,$$

i.e., X_f is characterized by the identities

$$i_{X_f}\Omega = df - \xi(f)\eta + f(\theta - \theta(\xi)\eta), \quad \eta(X_f) = 1.$$

We call X_f **Hamiltonian vector field** with energy function f. This definition generalizes the corresponding one for cosymplectic manifolds (see Section 2).

Now, consider a time-dependent dynamical system with n degrees of freedom. Its phase space is a $(2n+1)$-dimensional cosymplectic manifold M. We know that the dynamics consists of the orbits of a vector field X_H (the Hamiltonian vector field for the energy H) on M (see Section 2). In fact, we have an open covering of coordinate neighborhoods $\{U_\alpha\}_{\alpha \in A}$ with local coordinates $(t_\alpha, q^i_\alpha, p^\alpha_i)$, $1 \leq i \leq n$. The Hamiltonian H and the cosymplectic structure (Ω, η) restrict to each U_α given by a local Hamiltonian $H_\alpha = H_\alpha(t_\alpha, q^i_\alpha, p^\alpha_i)$ and a cosymplectic structure $\Omega_\alpha = dq^i_\alpha \wedge dp^\alpha_i$, $\eta_\alpha = dt_\alpha$, respectively. Then X_H restricted to U_α is precisely X_{H_α}. Thus, the orbits are given by the Hamilton equations

$$\frac{dq^i_\alpha}{dt_\alpha} = \frac{\partial H_\alpha}{\partial p^\alpha_i}, \quad \frac{dp^\alpha_i}{dt_\alpha} = -\frac{\partial H_\alpha}{\partial q^i_\alpha}\,. \tag{4}$$

(We notice that $t_\alpha = t$ is a global coordinate.)

Now, let M be an arbitrary $2n+1$-dimensional manifold endowed with an open covering of coordinate neighborhoods $\{U_\alpha\}_{\alpha \in A}$ with local coordinates $(t_\alpha, q^i_\alpha, p^\alpha_i)$, $1 \leq i \leq n$ and consider the coordinate transformations

$$(5) \qquad t_\beta = t_\beta(t_\alpha, q^i_\alpha, p^\alpha_i), \quad q^i_\beta = q^i_\beta(t_\alpha, q^i_\alpha, p^\alpha_i), \quad p^\beta_i = p^\beta_i(t_\alpha, q^i_\alpha, p^\alpha_i).$$

Since the dynamic information is given by a global vector field (the Hamiltonian vector field) we are only interested in the case when (5) preserves the form of the Hamilton equations (4). Clearly, if (5) implies

$$\Omega_\beta = dq^i_\beta \wedge dp^\beta_i = dq^i_\alpha \wedge dp^\alpha_i = \Omega_\alpha, \ \eta_\beta = dt_\beta = dt_\alpha = \eta_\alpha, \ H_\beta = H_\alpha \,,$$

where $H_\alpha : U_\alpha \to R$, $\alpha \in A$, then (5) preserves (4). But this also happens, if (5) implies

$$(6) \qquad \Omega_\beta = \lambda^2_{\beta\alpha}\Omega_\alpha, \quad \eta_\beta = \lambda_{\beta\alpha}\eta_\alpha, \quad H_\beta = \lambda_{\beta\alpha}H_\alpha \,,$$

where $\lambda_{\beta\alpha} = \mathit{const.} \neq 0$. In fact, from (6) we obtain

$$X_{H_\alpha} = \lambda_{\beta\alpha} H_{H_\beta} \,,$$

and hence the integral curves of X_{H_α} and X_{H_β} are the same. Further (6) implies the cocycle condition

$$(7) \qquad \lambda_{\beta\alpha}\lambda_{\alpha\gamma} = \lambda_{\beta\gamma} \,.$$

We know that (7) implies the existence of the local functions $\sigma_\alpha : U_\alpha \to R$ satisfying

$$\lambda_{\beta\alpha} = \frac{e^{\sigma_\beta}}{e^{\sigma_\alpha}} \,.$$

Thus

$$\Omega = e^{-2\sigma_\alpha}\Omega_\alpha, \quad \eta = e^{-\sigma_\alpha}\eta_\alpha \,,$$

are globally defined on M. Hence (M, Ω, η) is a locally conformal cosymplectic manifold. Moreover, we may define a global function $H : M \to R$ by $H = e^{-\sigma_\alpha}H_\alpha$, and the corresponding Hamiltonian vector field X_H is given by

$$X_H = e^{\sigma_\alpha} X_{H_\alpha} \,.$$

Therefore, **the locally conformal cosymplectic manifolds may be considered as natural phase spaces of time-dependent Hamiltonian dynamical systems**.

References

[1] Abraham R., Marsden J., *Foundations of Mechanics*, 2^{nd} ed., Benjamin, New York, 1978.
[2] Chinea D., Marrero J.C., *Locally conformal cosymplectic manifolds*, preprint.
[3] Libermann P., *Sur les structures presque complexes et autres structures infinitésimales régulières*, Bull. Soc. Math. France **83** (1955), 195–224.
[4] Libermann P., Marle Ch., *Symplectic Geometry and Analytical Mechanics*, Reidel Publ., Dordrecht, 1987.
[5] de León M., Rodrigues P., *Methods of Differential Geometry in Analytical Mechanics*, North-Holland Mathematical Studies No. 158, Amsterdam, 1989.
[6] Takizawa S., *On contact structures of real and complex manifolds*, Tôhoku Math. J. **15** (1963), 227–252.
[7] Vaisman I, *Locally conformal symplectic manifolds*, Internat. J. Math. & Math. Sci. **8, 3** (1985), 521–536.

Departamento de Matemática Fundamental, Universidad de La Laguna, La Laguna, Tenerife, Canary Islands, Spain

Unidad de Matemáticas, Consejo Superior de Investigaciones Científicas, Serrano, 123, 28006 Madrid, Spain

Departamento de Matemática Fundamental, Universidad de La Laguna, La Laguna, Tenerife, Canary Islands, Spain

(*Received* November 22, 1990)