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Sets invariant under projections onto one dimensional subspaces

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Abstract. The Hahn–Banach theorem implies that if m is a one dimensional subspace of a t.v.s. E, and B is a circled convex body in E, there is a continuous linear projection P onto m with $P(B) \subseteq B$. We determine the sets B which have the property of being invariant under projections onto lines through 0 subject to a weak boundedness type requirement.

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Definition. Let $B \subseteq \mathbb{R}^n$. We say B is invariant under projections onto lines to mean for all lines m through 0 there is a linear projection P from $\mathbb{R}^n$ onto m with $P(B) \subseteq B$.

Notation. We will first let $B \subseteq \mathbb{R}^2$. We talk about the projection onto m along x, for $x \neq 0$, to mean the linear projection onto m with $N(P) \ni x$. For $\theta \in \mathbb{R}$, let $x(\theta) = (\cos\theta, \sin\theta) \in \mathbb{R}^2$, and let $\alpha(\theta) = \{\gamma \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right]$: the projection P onto $\mathbb{R}x(\theta)$ along $x(\gamma)$ satisfies $P(B) \subseteq B\}$. We let $S(\theta) = \{t > 0 : tx(\theta) \in B\}$.

Lemma 1. *Let B be a closed nonempty subset of $\mathbb{R}^2$ which is invariant under projections onto lines. For some θ, suppose there is a sequence $\varphi_n \to \theta$ and $\lambda_n \in \alpha(\varphi_n)$ and $\mu \in \alpha(\theta)$ such that $\lambda_n \neq \mu$ and $\liminf \sin^2(\lambda_n - \theta) > 0$ (i.e. λ_n stays away from θ (mod π)). Then $S(\theta)$ is $(0,\infty)$ or $(0,M]$ or $[M,\infty)$ for some $M > 0$.*

Proof: Suppose $0 < a < b < \infty$ with a, b in $S(\theta)$ but $(a,b) \cap S(\theta) = \emptyset$. Let P be the projection onto $\mathbb{R}x(\theta)$ along $x(\mu)$, and let P_n be the projection onto $\mathbb{R}x(\varphi_n)$ along $x(\lambda_n)$. Then $P^{-1}((a,b)x(\theta)) \cap B$ is empty and so, if $C_n = P_n^{-1}(P^{-1}(a,b)x(\theta) \cap \mathbb{R}x(\varphi_n)) \cap (0,\infty)x(\theta)$, then $C_n \cap B = \emptyset$. Because $\lambda_n \neq \mu, C_n \neq (a,b)x(\theta)$, and because λ_n stays away from θ (mod π), $C_n \to (a,b)x(\theta)$ as $n \to \theta$. Thus, since C_n is a multiple of $(a,b)n(\theta)$, C_n contains either $ax(\theta)$, or $bx(\theta)$, a contradiction.

Thus $S(\theta)$ is an interval. Suppose $S(\theta) = [a,b]$ with $0 < a < b < \infty$. Then $P_n([a,b]x(\theta)) \subseteq B$ and if $V_n = P(P_n([a,b]x(\theta))$, then $V_n \subseteq B$. However, $V_n \neq [a,b]x(\theta)$ since $\lambda_n \neq \mu$ and $V_n \to [a,b]x(\theta)$ as $n \to \infty$ since λ_n stays away from θ (mod π). Thus V_n being a multiple of $[a,b]x(\theta)$, contains points of $(0,\infty)(\theta)$ not in $[a,b]x(\theta)$, a contradiction.

Hence $S(\theta) = (0,M], [M,\infty)$ or $(0,\infty)$. □

Definition. We call an angle $\theta \in \mathbb{R}$ surrounded, if there are $\theta_n \to \theta, \theta_{2n} < \theta, \theta_{2n+1} > \theta$, and $\gamma \neq \theta$ so that $\gamma \in \alpha(\theta_n)$ for all n.

Lemma 2. *Let B be a closed nonempty subset of $\mathbb{R}^2$ which is invariant under projections onto lines. For all $\theta \in \mathbb{R}$, one of the following holds.*

(a) $\lim_{\varphi \to \theta^+} \sin(\alpha(\varphi) - \theta) = 0$,
(b) $\lim_{\varphi \to \theta^-} \sin(\alpha(\varphi) - \theta) = 0$,
(c) $S(\theta) = (0, M], [M, \infty)$ *or* $(0, \infty)$ *for some* $M > 0$,
(d) θ *is surrounded.*

Proof: If (a) and (b) do not hold, there is $\theta_n \to \theta, \theta_{2n} < \theta, \theta_{2n+1} > \theta$ with $\liminf \sin^2(\lambda_n - \theta) > 0$ for some $\lambda_n \in \alpha(\theta_n)$. Unless there is $\gamma \in \alpha(\theta)$ such that $\lambda_n = \gamma$ for all large n, in which case θ is surrounded, Lemma 1 shows that (c) holds. □

Lemma 3. *Let B be a nonempty closed subset of $\mathbb{R}^2$ which is invariant under projections onto lines. The set of θ such that* (a) *or* (b) *of Lemma* 1.2 *hold, is nowhere dense in $\mathbb{R}^2$.*

Proof: If there were a sequence θ_n of angles of type (a) so that $\{\theta_n : n \in \mathbb{N}\}$ was dense in an open interval I, then for each j, $\sin^2(\alpha(\varphi) - \varphi) < j^{-1}$, if $\varphi \in (\theta_n, \theta_n + \varepsilon_{jn})$, where $\varepsilon_{jn} > 0$. Thus in a dense G_δ set in I, we have $\sin^2(\alpha(\theta) - \theta) = 0$, which is impossible. For (b), take $(\theta_n - \varepsilon_{jn}, \theta_n)$. □

Lemma 4. *Let B be a nonempty closed subset of $\mathbb{R}^2$ invariant under projections onto lines. Suppose I is a nonempty open interval of angles and every $\theta \in I$ is surrounded. Then either*

(a) *some* $S(\theta) = (0, M], [M, \infty)$, *on* $(0, \infty)$, *or else*
(b) *there is* γ *so that* $\alpha(\theta) = \{\gamma\}$ *for all* $\theta \in I$.

Proof: Assume (a) false, so that by Lemma 1, if $\varphi \in I, \varphi_{2n+1} \downarrow \varphi, \varphi_{2n} \uparrow \varphi$, with $\gamma_\varphi \in \alpha(\varphi_n)$ for all n, with $\gamma_\varphi \neq \varphi$, then $\alpha(\varphi) = \{\gamma_\varphi\}$.

Let $\gamma_0 \in I, \alpha(\varphi_0) = \{\gamma\}$. Without loss of generality let $\gamma_0 > \varphi_0 > \gamma_0 - \pi$. For $\xi \in (\varphi_0, \gamma_0) \cap I$, let $\theta = \sup\{\lambda < \xi : \alpha(\lambda) \ni \gamma_0\}$. Either (a) $\theta = \xi$, or (b) $\theta < \xi$ and $\alpha(\theta) \ni \gamma_0$, or (c) $\theta < \xi, \gamma_0 \notin \alpha(\theta)$, but $\theta_n \uparrow \theta$ with $\gamma_0 \in \alpha(\theta_n)$. If (b) holds, then $\gamma_0 = \gamma_\theta$, contradiction θ being a sup. If (c) holds, by Lemma 1, $\theta = \gamma_0$ contradicting $\xi < \gamma_0$. Hence (a) holds and $\alpha(\xi) = \{\gamma_0\}$, since $\gamma_0 > \xi$. Similarly for $\xi \in I, \xi \in (\gamma_0 - \pi, \gamma_0)$, we have $\alpha(\xi) = \{\gamma_0\}$. Now I does not include γ_0 (modulo π) since, if it did, there would be $\theta_n \uparrow \gamma_0$ (or $\theta_n \downarrow \gamma_0 - \pi$) with $\gamma_{\gamma_0} \in \alpha(\theta_n), \gamma_{\gamma_0} \neq \gamma_0$, contradicting $\alpha(\theta_n) = \gamma_0$, since $\theta_n \in I$. □

Lemma 5. *Let B be a nonempty closed subset of $\mathbb{R}^2$ which is invariant under projections onto lines. If there is an open interval of angles which are surrounded, then B is a union of parallel lines, or B is contained in a line through* 0, *or there is θ, and $M, N > 0$, such that $(0, M] \subseteq S(\theta) \subseteq (0, W]$, or $[M, \infty) \subseteq S(\theta) \subseteq [N, \infty)$.*

Proof: Let I be an open interval of angles with $\alpha(x) = \{\gamma\}$ for each $x \in I$. Assume B is not a union of parallel lines or a subset of $\mathbb{R}x(\gamma)$. We can find an angle $\theta \neq \gamma$

(mod π) with $\alpha(\theta) \ni \psi, \psi \neq \gamma$. Let P be the projection onto $\mathbb{R}x(\theta)$ along $x(\psi)$, and P be the projection onto $\mathbb{R}x(\theta)$ along $x(\gamma)$. Then $PP_\theta(x(\theta)) = w_\theta x(\theta)$ for some w_θ. The set $\{w_\theta : \theta \in I\}$ is an open interval $(w_0, w_1), w_0 < w_1$, so that if $w \in (w_0, w_1)$, then $wS(\theta) \subseteq S(\theta)$.

Suppose $(w_0, w_1) \cap (1, \infty) \neq \emptyset$. Then there are w_2 and w_3 in $(w_0, w_1), 1 < w_2 < w_3$, and $n \in \mathbb{N}$ with $w_2^{n+1} = w_3^n$. Then $[w_2^n, w_2^{n+1}] = [w_2^n, w_3^n]$ so for each $x \in [w_2^n, w_2^{n+1}]$, we have $xS(\theta) \subseteq S(\theta)$. Since $w_2 S(\theta) \subseteq S(\theta)$, we have $x \in [w_2^{n+1}, w_2^{n+2}]$ implying $xS(\theta) \subseteq S(\theta)$, and so on, giving $xS(\theta) \subseteq S(\theta)$ for all $x \geq w_2^n$. Note $S(\theta) \neq \emptyset$, so taking $y \in S(\theta), [w_2^n y, \infty) \subseteq S(\theta)$.

If $(w_0, w_1) \cap (-\infty, -1) \neq \emptyset$, then $(w_0^2, w_1^2) \cap (1, \infty) \neq \emptyset$ and we apply the argument above with w_0^2 and w_1^2 instead of w_0 and w_1.

If $(w_0, w_1) \cap (-1, 1) \neq \emptyset$, then a similar argument gives $(0, w^n y] \subseteq S(\theta)$ for $y \in S(\theta)$. Now the complement $S(\theta)'$ is nonempty and invariant under $\{w^{-1},; w \in (w_0, w_1)\}$. Hence when $S(\theta) \supseteq (0, M], S(\theta)' \supseteq [N, \infty)$ for some $N \in \mathbb{R}$, and when $S(\theta) \supseteq [N, \infty), S(\theta)' \supseteq (0, M]$. □

Lemma 6. *Let B be a nonempty closed subset of $\mathbb{R}^2$ which is invariant under projections onto lines. Let B contain $(0, \varepsilon)x$ for some $x \neq 0, \varepsilon > 0$. Then B is one of* (a), (b) *or* (c) *of Theorem* 1.

PROOF: We may suppose none of these hold. Hence there is a projection P onto $\mathbb{R}y$ for some $\mathbb{R}y \neq \mathbb{R}x$, not along x, giving $\varepsilon_y > 0$ with $(0, \varepsilon_y]y \subseteq B$, replacing y by $-y$ if required.

Let $K = \{y : [0,1]y \subseteq B\}$. Suppose $y, z \in K$, linearly independent. Let P be a projection on $\mathbb{R}(y+z), P(B) \subset B$. $P^{-1}((y+z)/2)$ intersects $(0,1]y$ or $(0,1]z$, giving $(y+z)/2 \in K$. If $y, z \in K$ and are linearly dependent, then $(y+z)/2 \in K$, so K is a closed convex set invariant under projections onto lines.

Suppose there is $w \neq 0, \lambda_n \downarrow 0, \lambda_n w \notin K$. Then let us project onto $\mathbb{R}w$ along s. We find $K \subseteq (-\infty, 0]w + \mathbb{R}s$. But since for all y, y or $-y$ is in the cone generated by K, we have $(-\infty, 0) + \mathbb{R}s$ contained in this cone. It follows that $(-\varepsilon, \varepsilon)s \subseteq K$ for some $\varepsilon > 0$, and all projections onto $\mathbb{R}t \neq \mathbb{R}s$ are along s, a contradiction. Hence, for all $w \neq 0$, there exists $\varepsilon > 0, [0, \varepsilon]w \subseteq K$, and $0 \in \operatorname{int} K$.

Now K contains no lines since B doesn't. Hence $K \cap -K$ is a bounded convex neighborhood of 0, with boundary D say. Now $D \cap \partial K \neq \emptyset$, ∂K is connected, and $D \cap \partial K$ is closed in ∂K, so to show $D = \partial K$, we want $D \cap \partial K$ open in ∂K. If we parametrize D and ∂K by polar coordinates, giving radius r as a function of angle θ, they are absolutely continuous, and a.e. (θ) we have the derivative of r with respect to θ unique and equal for both curves since for all θ there exist supporting lines to K and $K \cap -K$ which are parallel.

We claim $K = B$. Since K is a convex bounded neighborhood of 0, $\alpha(\theta)$ is nondecreasing, apart from a jump from $\frac{\pi}{2}$ to $\frac{-\pi}{2}$, and has period π. We may take θ so that $\alpha(\theta)$ is not constant on a neighborhood of θ. And if $\varphi_n \to \theta, \lambda_n \in \alpha(\varphi_n), \lambda_n \neq \mu \in \alpha(\theta)$, then λ_n stays away from θ (mod π) since $\operatorname{int}(K) \neq \emptyset$. By Lemma 1, $S(\theta)$ is an interval $(\theta, \varepsilon]$. Here, the line $\mathbb{R}x(\theta)$ intersects ∂K at a point not in the relative interior of a line segment of ∂K, we have $\alpha(\theta) = \gamma$ for

$\theta_1 \le \theta \le \theta_2$ with $S(\theta_1) = (0, \varepsilon_1]$ and $S(\theta_2) = (0, \varepsilon_2]$. Hence $S(\theta)$ is an interval for each $\theta \in [\theta_1, \theta_2]$. $\square$

Lemma 7. *Let B be a nonempty closed subset of $\mathbb{R}^2$ which is invariant under projections onto lines. Suppose there is $w_0 \in \mathbb{R}^2 \setminus \{0\}$ and $\lambda_n \to \infty$ such that either for all n, $\lambda_n^{-1} w_0 \in B$, or for all n, $\lambda_n w_0 \notin B$. Then B is either:*

(a) *contained in a line $\mathbb{R}x$,*
(b) *a union of parallel lines, or*
(c) *for every nonzero w in $\mathbb{R}^2$, there is $\lambda_n \to \infty$ with either $\lambda_n^{-1} w \in B$ for all n, or $\lambda_n w \notin B$ for all n.*

Proof: Assume neither (a) nor (b) hold.

(i) Suppose $\lambda_n w_0 \notin B, \lambda_n \to \infty$. We claim this holds for all $w \neq 0$. Suppose not. Let $S = \{v \neq 0 : \text{there exists } M > 0, [M, \infty)v \subseteq B\}$, so $S \neq \emptyset$, and let $z_0 \in S$. Take P a projection into $\mathbb{R}w_0$ along s, $P(B) \subseteq B$. Then $S \subseteq \mathbb{R}s + (-\infty, 0]w_0$. Since (a) and (b) do not hold, there is $y \notin \mathbb{R}z_0, y \in S$. Hence for all $v \neq 0$, v or $-v$ is in S, and so the open half plane $\mathbb{R}s + (-\infty, 0)w_0 \subseteq S$. It follows that s and $-s$ are in S. Hence the projection onto $\mathbb{R}x \neq \mathbb{R}s$ is along s, giving (b).
(ii) Suppose $\lambda_n^{-1} w_0 \in B$ for all n. Let $S = \{s \in \mathbb{R}^2 \setminus \{0\}$: there exists $\varepsilon_n \downarrow 0$, $\varepsilon_n s \in B\}$. Suppose, to derive a contradiction, there is v_0 with $(0, \varepsilon)v_0 \notin B$, for some $\varepsilon > 0$, we argue as in (i) to find $S = \mathbb{R}t + (-\infty, 0]v_0$, if we project onto $\mathbb{R}v_0$ along t, giving (b). $\square$

Theorem 8. *Let B be a closed nonempty subset of $\mathbb{R}^2$ and suppose there is $w \in \mathbb{R}^2, w \neq 0$, and $\lambda_n \to \infty$, such that $\lambda_n^{-1} w \in B$ or $\lambda_n w \notin B$.*

For every one dimensional subspace m, there exists a linear projection $P : \mathbb{R}^2 \to m$ with $P(B) \subseteq B$ iff B is one of:

(a) *a subset, containing* 0*, of a line through* 0*,*
(b) *a union of parallel lines, containing* 0*,*
(c) *a bounded convex symmetric neighborhood of* 0*.*

Proof: This follows from Lemmas 1 to 7. $\square$

Proposition 9. *Let B be a nonempty closed subset of $\mathbb{R}^n$, such that for all w in an $n-1$ dimensional subspace W, there is a sequence (w_k) in $(0, \infty)w \cap B$ tending to* 0*, or a sequence (w_k) in $(0, \infty)w \cap B'$, $\|w_k\| \to \infty$.*

B is invariant under projections onto lines iff B is one of:

(a) *$S + E, E$ a subspace, $0 \in S \subseteq \ell, \ell$ a 1 dimensional subspace, $\ell \cap E = \{0\}, S$ not convex and symmetric,*
(b) *$B + E, B$ the unit ball in a subspace M, given by a norm, and E a subspace with $M \cap E = \{0\}$.*

Proof: $\Longleftarrow$ Straightforward.

$\Longrightarrow$ Suppose (b) does not hold. We claim there is $e_1 \neq 0$ with $B \cap \mathbb{R}e_1$ not convex or not symmetric about 0.

If B is not symmetric, this is immediate. Suppose B is not convex, so there are a, b in B with $(a+b)/2 \notin B$. We may assume $\{a, b\}$ linearly independent. There is $w \neq 0$ in $(\mathbb{R}a + \mathbb{R}b) \cap W$. Hence $B \cap (\mathbb{R}a + \mathbb{R}b)$ is a union of parallel lines on a subset of a line, and is not convex, giving e_1.

Let F be the linear span of B, of dimension m. Suppose $b \in B \setminus \mathbb{R}e_1$. Then $B \cap (\mathbb{R}e_1 + \mathbb{R}b)$ is a union of parallel lines, $S + \mathbb{R}e_2$ say, since $B \cap \mathbb{R}e_1$ is not symmetric or not convex. If $m > 2$, take $b \notin \mathbb{R}e_1 + \mathbb{R}e_2, b \in B$, giving $e_3 \notin \mathbb{R}e_1 + \mathbb{R}e_2$ with $S + \mathbb{R}e_3 \subseteq B$.

Continuing, we have a basis $(e_1, \dots e_m)$ of F with $S + \mathbb{R}e_1 \subseteq B$ for $i \geq 2$. We see that $S + E \subseteq B$, where $E = \mathbb{R}e_2 + \cdots + \mathbb{R}e_m$. But if $P(B) \subset B$ and P projects F on $\mathbb{R}e_1$, then $P(E) = \{0\}$, so $B \subseteq S + E$, giving $B = S + E$. $\square$

Example 10. We give the simplest example of a closed nonempty subset B of $\mathbb{R}^n$ which is invariant under projections onto lines, but which has, for all $x \neq 0, (0, \varepsilon)x \subseteq B'$ for some $\varepsilon > 0$ and $[M, \infty)x \subseteq B$ for some M.

$$B = \bigcap_{i=1}^{n} \{x : x_i \in (-\infty, -1] \cup \{0\} \cup [1, \infty)\}.$$

Problem 11. How can one describe all such sets as the above (by other than their defining property of being invariant under projections onto lines)?

Theorem 12. *Let B be a nonempty closed subset of a real locally convex topological vector space E, whose closed subspaces are barrelled. Suppose for all w in a hyperplane W, there is a sequence $\lambda_k \to \infty$ with $\lambda_k w \notin B$ or $\lambda_k^{-1} w \in B$.*

For all one dimensional subspaces m, there exists a continuous linear projection $P : E \to m$ such that $P(B) \subseteq B$ is one of:

(a) *a closed convex circled subset whose linear hull is closed,*
(b) *$S + F$, where $0 \in S, S$ a closed subset of a one dimensional subspace ℓ, S not both convex and symmetric, F a closed linear subspace not containing ℓ.*

Proof: $\Longrightarrow$ Suppose for all finite dimensional subspaces X of E, $B \cap X$ is a closed convex circled set. Then B is a closed convex circled set. Let G denote its linear hull. If G is not closed, we can take a one dimensional subspace $m \subseteq \overline{G}$ with $m \cap G = \{0\}$. Let P be a projection on m with $P(B) \subseteq B$. Since $P(B) \subseteq m \cap B = \{0\}, P = 0$ on G by linearity and on $\overline{G}$ by continuity, contradicting P being the identity on m. Hence G is closed.

Otherwise, by Theorem 1.9, there is a finite dimensional subspace X with $B \cap X = S + F_X$, where S is a subset of a 1 dimensional subspace ℓ, not both convex and symmetric, and F_X is a linear subspace, $S \subsetneq F_X$. For Y a finite dimensional subspace, $Y \supseteq X$, we have $B \cap Y = S + F_Y, F_Y$ a linear subspace, $S \subsetneq F_Y$. Let $F = \mathrm{cl}(V)\{F_Y : Y \geq X\}$. Now claim $B = S + F$ and $\ell \subsetneq F$. Projecting onto ℓ with $P, P(B) \subseteq B$, we have $F_Y \subseteq N(P)$ for all Y, and $N(P)$ is closed, giving $F \subseteq N(P)$ and $\ell \subsetneq F$. If $b \in B$, take Y a finite dimensional subspace containing b and X, so $b \in S + F_Y \subseteq S + F$. Since for all Y, $S + F_Y \subseteq B$ and B is closed, $S + F \subseteq B$, proving the claim.

$\Longleftarrow$ Let H be the linear hull of B. Note H is closed. Suppose $m \subsetneq H, m = \mathbb{R}x_m$ say. Take a nonempty convex open neighborhood A of x_m not intersecting H. By Mazur's theorem, a geometrical version of Hahn–Banach, ([1, II, Theorem 3.1]), there is a closed hyperplane in E containing M and not intersecting A. This gives a continuous linear $f : E \to \mathbb{R}$ with $f(H) = 0, f(x_m) = 1$, and put $Py = f(y)x_m$.

Suppose $m \subseteq B, m = \mathbb{R}x_m$ say, take a continuous linear $f : E \to \mathbb{R}$ with $f(x_m) = 1$ and put $Py = f(y)x_m$. Now suppose $m \subseteq H, m \subsetneq B$. In case (a), since H is barrelled, B is a neighborhood of 0 in H, being a barrel in it. We let $m = \mathbb{R}x_m$ where x_m is in the boundary of B in H. By the First Separation Theorem ([1, II, Theorem 9.1, Corollary]), there is a closed real hyperplane in H supporting B at x_m, giving $f : H \to \mathbb{R}$ linear, continuous, with $f(x_m) = 1$. Extending f to E [1, II, Theorem 4.2]) gives $Py = f(y)x_m$ as required.

In case (b), take a closed hyperplane in H containing F, but not x_m, by Mazur's theorem as above, i.e. a continuous linear $f : H \to \mathbb{R}$ with $f(x_m) = 1$. Extending f to E gives $Py = f(y)x_m$ as required. $\square$

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