Classification of tensor products of symmetric graphs

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Abstract. In the category of symmetric graphs there are exactly five closed tensor products. If we omit the requirement of units, we obtain twelve more.

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The aim of this article is an inventarization of (closed) tensor products in the category *Gph* of symmetric graphs. More precisely, we consider regular closed tensor products, i.e. those in which the vertex set is the Cartesian product of the vertex sets of the factors. It is known (see [8]) that there are exactly four such that the corresponding exponentiation $H(B,C)$ is carried by the set of all graph homomorphisms; on the other hand, it is also well-known that *Gph* is a Cartesian closed category with exponentiation carried otherwise. Here we will show that there exactly five regular tensor products in *Gph*. If one does not require a unit, one has seventeen.

Constructions will be quite explicit and one only needs a basic preliminary knowledge of category theory, as presented e.g. in the introductory chapters of [4].

This paper is related to [2]. Unlike that, we do not (and cannot) consider product variants with a non-trivial role of non-edges, and, of course, we put much stronger constraints on the products.

1. (Closed) tensor products.

1.1. A tensor structure in a category K is constituted by functors $\otimes : K \times K \to K$, $H : K^{op} \times K \to K$ and an object E of K satisfying

$$(1)\qquad (A \otimes B) \otimes C \cong A \otimes (B \otimes C),$$

$$(2)\qquad A \otimes B \cong B \otimes A,$$

$$(3)\qquad E \otimes A \cong A, \quad \text{and}$$

$$(4)\qquad K(A \otimes B, C) \cong K(A, H(B,C)).$$

1.2. **Remarks:** 1. In a general category, one has to require certain coherence conditions concerning the indicated natural equivalences (cf. [1], [3], [5]). In the category we are dealing with (and in similar ones) they are, however, satisfied automatically (see [6]).

2. Sometimes one considers non-commutative tensor products. Then one has to consider a couple of distinct hom-products, one adjoint to $-\otimes B$ and the other adjoint to $A\otimes -$.

Associativity is also very important. In fact, it is equivalent to the possibility to replace (4) by

$$H(A\otimes B, C)\cong H(A, H(B,C)).$$

Thus, one may argue that only under (1) and (2) one can justly think of H as an exponentiation.

3. The existence of the unit E seems to be slightly less essential. Therefore, we will also discuss the case with (3) dropped.

4. Of course, $\otimes$ determines H and vice versa.

1.3. As all $A\otimes -$ and $-\otimes B$ are left adjoints, they preserve colimits. In the next section, we will show how the objects of *Gph* can be obtained in a canonical way as colimits of diagrams featuring two objects only. Thus, a tensor product is determined by values in these objects and mappings between them. This observation will be used in Section 3 and further.

1.4. We will be concerned with the category *Gph* of symmetric graphs (with possible, but not mandatory loops) and edge preserving mappings. Denote by

$$U : Gph \to \mathrm{Set}$$

the functor reducing a graph to its vertex set. We will be concerned with *regular* tensor products, namely those $\otimes$, for which

$$U(A\otimes B)\cong U(A)\times U(B).$$

2. The comma-construction.

2.1. In this section, we will recall in some detail the well-known fact that there is a very small dense subcategory of *Gph*.

Let us introduce the following notation: P is the graph $(\{0\},\emptyset)$ with one vertex and no edge, whereas I is the graph $(\{0,1\},\{01\})$, i.e. two vertices joined by an edge.

We have the homomorphisms

$$\iota_i : P\to I, \iota_i(0)=i; \quad \varepsilon : I\to I,\ \varepsilon(i)=1-i$$

which, together with the identities of P, I constitute a full subcategory $\mathcal{A}$ of *Gph*.

For a graph $A\in Gph$, define the category

$$\mathcal{C}_A$$

the objects of which are morphisms $\alpha : X \to A$ $(X = P, I)$ and the morphisms $\alpha \to \beta$ of which are the commuting triangles

$$\begin{tikzcd} X \arrow[rr, "\alpha"] \arrow[d, "\gamma"'] & & A \\ Y \arrow[urr, "\beta"'] & & \end{tikzcd} \quad ;$$

this will be indicated as (α, γ, β).
(This is the comma-category determined by $\mathcal{A}$ and A in *Gph*—cf. [4].)

2.2. Define diagrams

$$D_A : \mathcal{C}_A \to Gph$$

by putting $D_A(X \overset{\alpha}{\to} A) = X,\ D_A(\alpha, \gamma, \beta) = \gamma$. Moreover, put

$$\lambda_\alpha = \alpha : D_A(\alpha) \to A.$$

It is well-known (and easy to check) that $(\lambda_\alpha)_\alpha$ is a colimit of D_A.

3. The matrix of a tensor product.

3.1. In the sequel, we will use the following easy (and well-known) observation:
Let $F : \mathcal{C}_1 \times \mathcal{C}_2 \to \mathcal{C}$ be a diagram. Consider functors $F_x : \mathcal{C}_2 \to \mathcal{C}$ defined by $F_x(y) = F(x, y), F_x(\varphi) = F(1_x, \varphi)$ and form colimits

$$(\kappa_y^x : F_x(y) \to G(x))_y \,.$$

For $\psi : x \to x'$ in $\mathcal{C}_1$, we have the natural transformations $\psi' : F_x \to F_{x'}$ defined by $\psi'_y = F(\psi, 1_y)$; now the system of equations

$$G(\psi) \circ \kappa_y^x = \kappa_y^{x'} \circ \psi'_y \quad (y \in \operatorname{obj} \mathcal{C}_2)$$

defines a unique $G(\psi) : G(x) \to G(x')$. Obviously, a functor $G : \mathcal{C}_2 \to \mathcal{C}$ is thus obtained. Let $(\mu_y : G(y) \to c)_y$ be a colimit of G. Then

$$(\mu_y \circ \kappa_y^x : F(x, y) \to c)_{x,y}$$

is a colimit of F.

3.2. Suppose we have a tensor product $\otimes$ on *Gph*. Since (recall 1.3) $- \otimes A$ and $A \otimes -$ preserve colimits, it follows by Section 2 that $\otimes$ is determined by the values in $\mathcal{A} \times \mathcal{A}$. Thus, consider a functor

$$M : \mathcal{A} \times \mathcal{A} \to Gph$$

such that $UM(X,Y) = U(X) \times U(Y)$ and $M(X,Y) \cong M(Y,X)$. We will show that we can extend M to a functor $\otimes$ by colimits (that is, we will see that, when using 2.2 and taking the natural extension, we will indeed have $M = \otimes \mid \mathcal{A} \times \mathcal{A}$). This functor will satisfy (2) of 1.1. In the next section we will show that it has an adjoint exponentiation. Thus, to inventorize our tensor products it will suffice to check the functors for associativity and unit, which will be done in Section 5.

3.3. For $A, B \in \operatorname{obj} Gph$ consider the functor

$$F^{AB} = M(D_A(-), D_B(-)) : \mathcal{C}_A \times \mathcal{C}_B \to Gph,$$

and define $A \otimes B$ as the top of a colimit

$$\kappa^{AB}_{\alpha\beta} : F^{AB}(\alpha, \beta) \to A \otimes B.$$

Now let us consider $\varphi : A \to A', \psi : B \to B'$. Obviously

$$\tau_{\alpha\beta} = \kappa^{A'B'}_{\varphi\alpha,\psi\beta} : F^{AB}(\alpha, \beta) \to A' \otimes B'$$

constitutes an upper bound of F^{AB} and hence there is a uniquely defined

$$\varphi \otimes \psi : A \otimes B \to A' \otimes B'$$

satisfying

$$(\varphi \otimes \psi) \circ \kappa^{AB}_{\alpha\beta} = \kappa^{A'B'}_{\varphi\alpha,\psi\beta}\,.$$

It is easy to check that thus a functor

$$\otimes : Gph \times Gph \to Gph$$

is defined.

From $M(X,Y) \cong M(Y,X)$ it is straightforward to infer that also $A \otimes B \cong B \otimes A$.

3.4. Let K be a small category with a top, that is, with an element k_o such that for each $k \in \operatorname{obj} K$ there is exactly one morphism $\gamma_k : k \to k_o$. Then obviously $(\gamma_k : D(k) \to D(k_o))_k$ is a colimit.

Such is the case with $\mathcal{C}_A \times \mathcal{C}_B$ for A, B in $\mathcal{A}$, where obviously $(1_A, 1_B)$ is the top. Consequently, we see that we can put $A \otimes B = M(A, B)$. Also, we see that inside $\mathcal{A} \times \mathcal{A}$ we then also have $M(\varphi, \psi) = \varphi \otimes \psi$.

3.5. It is perhaps not immediately obvious that $A \otimes -$ and $- \otimes A$ preserve colimits. From 3.1 (and 3.4), however, we can at least infer that these functors preserve the colimits of the diagrams $D_B : \mathcal{C}_B \to Gph$.

3.6. **The concrete form of $A \otimes B$**: It is a matter of immediate checking that the colimit $A \otimes B$ can be obtained as follows: Take the Cartesian product of the set of vertices $U(A) \otimes U(B)$. For any edge a_0a_1 in A and b_0b_1 in B fit into $\{a_0a_1\} \otimes \{b_0b_1\}$ edges a_ib_j for all edges ij in $M(I, I)$ (note that this includes also loops in A resp. B). For isolated points (without loops) fill in similarly the edges (resp. loops) according to those in $M(P, P), M(P, I)$ and (I, P).

One also easily checks that in this choice of the colimit $\varphi \otimes \psi$ is carried by the Cartesian product of the maps of vertices.

In particular, note that

3.6.1. $(\iota_i \otimes 1_B)(P \otimes B)$ cover the vertices of $I \otimes B$.

4. The adjunction.

4.1. If an H satisfying (4) in 1.1 exists, the homomorphisms $P \to H(B,C)$ are in a natural one-to-one correspondence with $P \otimes B \to C$ and the homomorphisms $I \to H(B,C)$ corresponds to $I \otimes B \to C$. Thus, up to natural equivalence, there is only one candidate for exponentiation:

We put

$$H(B,C) = (Gph(P \otimes B, C), R)$$

with $\varphi_0 R \varphi_1$ iff there is a $\psi : I \otimes B \to C$ satisfying

$$\varphi_i = \psi \circ (\iota_i \otimes 1_B), i = 0, 1.$$

Furthermore, for $\beta : B' \to B, \gamma : C \to C'$ define

$$H(\beta,\gamma)(\varphi) = \gamma \circ \varphi \circ (1_P \otimes \beta).$$

It is easy to check that $H(\beta,\gamma)$is indeed a homomorphism $H(B,C) \to H(B',C')$.

4.2. Theorem. *Let $\otimes$ be defined as in* 3.3 *and let H be defined as in* 4.1. *Then there is a natural equivalence*

$$Gph(A \otimes B, C) \cong Gph(A, H(B,C)).$$

Proof: Recall 3.5 and 2.2. For $f : A \otimes B \to C$ define $\bar{f} : A \to H(B,C)$ by putting $\bar{f}(x) = (P \otimes B \xrightarrow{\xi_x \otimes 1_B} A \otimes B \xrightarrow{f} C)$ where $\chi_x : P \to A$ sends 0 to x. Let xy be an edge in A.

Consider $\alpha_{xy} : I \to A$ sending 0 to x and 1 to y. As $\alpha_{xy} \circ \iota_0 = \xi_x$ and $\alpha_{xy} \circ \iota_1 = \xi_y$, $\bar{f}(x)\bar{f}(y)$ is an edge by virtue of the morphism $f \circ (a_{xy} \otimes 1_B)$.

On the other hand, let $g : A \to H(B,C)$ be a homomorphism. Define $\tau_\alpha : D_A(\alpha) \otimes B \to C$ for $\alpha \in \mathrm{obj}\, \mathcal{C}_\mathcal{A}$ as follows: If $\alpha = \xi_x : P \to A$, put $\tau_\alpha = g(x)$. If $\alpha = \alpha_{xy} : I \to A$, there is $\psi : I \otimes B = D_A(\alpha) \otimes B \to C$ such that $\psi \circ (\iota_0 \otimes 1_B) = g(x)$ and $\psi \circ (\iota_1 \otimes 1_B) = g(y)$; by 3.6.1 it is uniquely determined and we put $\tau_\alpha = \psi$. Obviously, $(\tau_\alpha)_\alpha$ is an upper bound of $D_A(-) \otimes B$ and we have a uniquely determined $\tilde{g} : A \otimes B \to C$ such that $\tilde{g} \circ (\lambda_\alpha \otimes 1_B) = \tau_\alpha$ (λ from 2.2). We immediately see that by this definition

$$g^{\sim -}(x) = g(x).$$

On the other hand, consider $f : A \otimes B \to C$ and $g = \bar{f}$. We have $f \circ (\xi_x \otimes 1_B) = \bar{f}(x) = \tau_{\xi_x}$ by the definition above and also $f \circ (\alpha_{xy} \otimes 1_B) = \tau_{\alpha_{xy}}$ by 3.6.1. Thus, also

$$f^{-\sim} = f$$

and we see that $f \mapsto \bar{f}$ and $g \mapsto \tilde{g}$ are mutually inverse correspondences.

The naturality is straightforward. $\square$

5. Associativity.

5.1. Let us first make an inventory of the candidates for the functor M we have so far. We must introduce some notation:

D is the discrete graph with vertices $0, 1$; if A is a graph then $\overline{A}$ is obtained from A by adding loops to all vertices; the following are special graphs with vertices $\{0, 1\} \times \{0, 1\}$:

Δ $\qquad$ U $\qquad$ V $\qquad$ W

The disjoint sum of two graphs A, B will be denoted by $A + B$, in case of several graphs A_i we write $\sum A_i$.

Later on, we will decompose a graph A into the sum $A^0 + A^1$, where A^0 consists of all isolated vertices (a vertex with loop is not considered isolated even if it is not connected with any other).

5.2. For the homomorphisms of $\mathcal{A}$ we have $M(\varphi, \psi) = \varphi \otimes \psi$. Since these are carried by the Cartesian product of the mappings and since these have to be homomorphisms, the value $M(P, P)$ limits the $M(P, I)$ and $M(I, P)$ and these in turn limit the $M(I, I)$ in the obvious way. Further,taking into account the commutativity and the morphism $\varepsilon : I \to I$, we obtain the following possibilities:

($M1$) $P \otimes P = P$, $P \otimes I = D$,
$I \otimes I$ any of $\Delta, U, V, W, \overline{\Delta}, \overline{U}, \overline{V}, \overline{W}$;

($M2$) $P \otimes P = P$, $P \otimes I = I$,
$I \otimes I$ any of $U, W, \overline{U}, \overline{W}$;

($M3$) $P \otimes P = P$, $P \otimes I = \overline{I}$,
$I \otimes I$ either $\overline{U}$ or $\overline{W}$;

($M4$) $P \otimes P = \overline{P}$, $P \otimes I = \overline{D}$,
$I \otimes I$ any of $\overline{\Delta}, \overline{U}, \overline{V}, \overline{W}$;
out of these, the one with $\overline{\Delta}$ will differ from all the others and we will refer to it as ($M4.1$);

($M5$) $P \otimes P = \overline{P}$, $P \otimes I = \overline{I}$,
$I \otimes I$ either $\overline{U}$ or $\overline{W}$.

5.3. To check the associativity, realize first that sum is a colimit construction and hence preserved by tensor products. Thus,

$$A \otimes (B \otimes C) \cong \sum_{i,j,k=0,1} A^i \otimes (B^j \otimes C^k) \quad \text{and}$$

$$(A \otimes B) \otimes C \cong \sum_{i,j,k=0,1} (A^i \otimes B^j) \otimes C^k \, .$$

As $I \otimes I$ is a part of a well-known associative product in all cases (cf. [2]) we always have $A^1 \otimes (B^1 \otimes C^1) \cong (A^1 \otimes B^1) \otimes C^1$. Thus, we only have to consider the summands with at least one discrete factor. We will refer to them as special summands.

$(M1)$: All cases yield associative products since all special summands are discrete.
$(M2)$: All cases yield well-known associative products (see [8], cf. also [7]).
$(M3)$: This is a less trivial case, but we still easy see the associativity when realizing that $P \otimes A = \overline{A}$ whenever A has no discrete points, and that $\overline{A} \otimes B = A \otimes B = \overline{A \otimes B}$, if A, B have no discrete points.
$(M4)$: Case $(M4.1)$ is associative: all this product does it putting loops into the Cartesian product of vertices.

But none of the other three is. Indeed, we have

while

$(M5)$: Here, again, both cases yield associative products. Indeed, all what happens is adding all possible loops into well-known associative cases.

5.4. Not all of the products have a unit, though. Let us realize that, because of the assumption of $U(A \otimes B) = U(A) \times U(B)$, the only two candidates for a unit are P and $\overline{P}$. This excludes $(M3), (M4)$ and $(M5)$: indeed, P is obviously no unit and neither is $\overline{P}$ as $\overline{P} \otimes I$ has to have loops.

In all cases of $(M2)$ we have a unit, namely P. Finally, in $(M1)$, we have a unit, namely $\overline{P}$, only in the case $I \otimes I = V$. In case Δ, resp. $\overline{\Delta}$, we have $\overline{P} \otimes I = D$, resp. $\overline{D}$, and in all the remaining ones $\overline{P} \otimes I = \overline{I}$.

5.5. **Summary:** We have thus proved that there are five tensor products in *Gph*. If we do not require units, we obtain seventeen of them.

5.6. **Remarks:** 1. All what was said in Section 1–4 can be easily modified to fit the category of oriented graphs (and, of course, other locally presentable categories as well). Checking the associativity is not quite so easy, though. It is known that there are 52 tensor products of oriented graphs such that $H(B, C)$ is carried by the homomorphism set ([7]); those 52 had to be sorted from among the 256 candidates by a rather tedious procedure. It should be advisable to leave the sorting of the remainder to computers.

2. One sees that in all cases one can obtain loops in $H(B, C)$ even if B, C have none. One may well ask whether there are closed tensor structures in the popular category of symmetric graphs without loops. There are none, and this may indicate that this category is not quite so nice after all.

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