A two weight weak inequality for potential type operators

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Abstract. We give conditions on pairs of weights which are necessary and sufficient for the operator $T(f) = K * f$ to be a weak type mapping of one weighted Lorentz space in another one. The kernel K is an anisotropic radial decreasing function.

Keywords: integral operator, anisotropic potential, weighted Lorentz space

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1. Introduction.

In [5], [2], a complete description is given for such pairs of weights (w,v) that the anisotropic potential is a bounded mapping of a weighted Lebesgue space L^p_v into a weak space L^q_w, $1 < p < q < \infty$. These results were extended in [4], [6], [3] to the case of weighted Lorentz spaces. At the same time, a sufficient condition was established in [6], for a two weight weak type inequality for integral operators with arbitrary positive kernels. This condition appears also necessary in some cases which are important for applications (see [3]).

In [7], under additional assumptions on the positive kernel, a condition for pairs of weights (w, v) was proved which is necessary and sufficient for the corresponding integral operator to be a bounded mapping of L^p_v in the weak L^q_w, where $1 < p \leq q < \infty$.

The aim of the present paper is to generalize the latter result to the case of weighted Lorentz spaces and of kernels which are anisotropic radial decreasing functions from the class A_1. In the last section, we extend the results for more general kernels and compare the condition for (w, v) with the other one obtained by Gabidzashvili and Kokilashvili [2], [6].

Fix $\alpha = (\alpha_1, \dots, \alpha_n)$ such that $\alpha_i > 0$ for $i = 1, \dots, n$ and $\sum_i \alpha_i = n$. For $x \in \mathbb{R}^n$, we put

$$|x|_\alpha = \max_i |x_i|^{\alpha_i} .$$

This is a quasi-norm which satisfies the inequalities

$$a^{-1}|x|_\alpha - |y|_\alpha \leq |x + y|_\alpha \leq a(|x|_\alpha + |y|_\alpha), \tag{1.1}$$

where $a = 2^{\alpha_0 - 1}, \alpha_0 = \max \alpha_i$.

We shall assume that K is an α-anisotropic radial decreasing (a.r.d) function, i.e. $K(x) = k(|x|_\alpha), x \in \mathbb{R}^n$, where k is a positive non-increasing function on $[0, \infty)$.

Throughout the paper, the symbol Q denotes the anisotropic ball

$$Q = Q(x,r) = \{y \in \mathbb{R}^n : |y-x|_\alpha \le r\},\ x \in \mathbb{R}^n,\ r > 0.$$

The Lebesgue measure of a measurable set E in $\mathbb{R}^n$ will be denoted by $|E|$. Note that $|Q(x,r)| = (2r)^n$.

We shall say that a function K belongs to the (anisotropic) Muckenhoupt class A_1, if there exists a constant $c > 0$ such that the inequality

$$\frac{1}{|Q|}\int_Q K(y)\,dy \le cK(x) \tag{1.2}$$

holds for every $Q = Q(z,r)$ and for a.e. $x \in Q$.

Let $1 \le s \le p < \infty$ and let μ be a Borel measure on $\mathbb{R}^n$. The Lorentz space $L^{ps}(\mu)$ is the set of all measurable functions f with the finite norm

$$\|f\|_{L^{ps}(\mu)} = \big(s\int_0^\infty \mu(\{x : |f(x)| > \lambda\})^{s/p}\lambda^{s-1}\,d\lambda\big)^{1/s}.$$

Note that $L^{pp}(\mu)$ is the usual Lebesgue space $L^p(\mu)$.

For a measurable function f and a Borel measure μ, we define

$$T(f\mu)(x) = \int_{\mathbb{R}^n} K(x-y)f(y)\,d\mu(y), \tag{1.3}$$

particularly

$$T(f)(x) = \int_{\mathbb{R}^n} K(x-y)f(y)\,dy.$$

Theorem 1. *Let $1 < p \le q < \infty$ and $1 \le s \le p$. Let ω, μ be Borel measures on $\mathbb{R}^n$, μ non-trivial, and let ψ be a positive measurable function. Consider the operator T defined by (1.3), where K is a positive α-anisotropic radial decreasing function from the Muckenhoupt class A_1. Then the inequality*

$$\omega(\{x : T(f\psi\mu)(x) > \lambda\})^{1/q} \le \frac{A}{\lambda}\|f\|_{L^{ps}(\mu)} \tag{1.4}$$

holds for every f and $\lambda > 0$ if and only if the inequality

$$\|\chi_Q T(\chi_Q\omega)\psi\|_{L^{p's'}(\mu)} \le B\omega(Q)^{1/q'} < \infty \tag{1.5}$$

holds for every $Q = Q(x,r), x \in \mathbb{R}^n, r > 0$.

Moreover, the ratio A/B of the optimal constants is bounded from below and above by positive numbers which do not depend on f, μ, ω and ψ.

2. Proof of Theorem 1.

In the proof, we follow the ideas of Sawyer [7]. To generalize his result for anisotropic potentials in Lorentz spaces, we make use of the Hölder type inequalities

$$c_0^{-1}\|f\|_{L^{ps}(\mu)} \le \sup\{|\int_{\mathbb{R}^n} fg\,d\mu| : \|g\|_{L^{p's'}(\mu)} \le 1\} \le c_0\|f\|_{L^{ps}(\mu)} \tag{2.1}$$

with $1/p+1/p' = 1/s+1/s' = 1$ (see [1]) and of two assertions concerning coverings. The first assertion is a Whitney type covering lemma.

Lemma 1. *Let Ω be a non-empty open proper subset in $\mathbb{R}^n$. Let $\tau > 1$ and $\eta > a\tau$, where a is the constant from (1.1). Then there exist sets $Q_k = Q(x_k, r_k), k = 1, 2, \ldots,$ and a number $\vartheta = \vartheta(n, a, \tau, \eta)$ such that*

$$\Omega = \bigcup_k Q_k\,, \tag{2.2}$$

$$Q(x_k, \eta r_k) \setminus \Omega \ne \emptyset, \quad k = 1, 2, \ldots, \tag{2.3}$$

and

$$\sum_k \chi_{Q(x_k,\tau r_k)} \le \vartheta\chi_\Omega\,. \tag{2.4}$$

Proof: For $x \in \Omega$, we put $d(x) = \inf_{y\in\partial\Omega} |x - y|_\alpha$ and $r(x) = d(x)/\eta$. Since Ω is a proper subset in $\mathbb{R}^n$, there exists $x_0 \in \mathbb{R}^n \setminus \Omega$. Without loss of generality, we can suppose that x_0 is the origin. We put $d_0 = 0, d_j = [a^2(\eta + \tau)/(\eta - a\tau)]^{j-1}$, $j = 1, 2, \ldots,$ and

$$\Omega_j = \{x \in \Omega : d_{j-1} < |x|_\alpha < d_j\}.$$

Fix $0 < \delta < 1$ and $j = 1, 2, \ldots$. Since $R_{j,1} := \sup_{x\in\Omega_j} r(x) \le d_j/\eta < \infty$, there exists $x_{j,1} \in \Omega_j$ such that $r_{j,1} := r(x_{j,1}) > \delta R_{j,1}$. We proceed by induction. If $x_{j,1}, \ldots, x_{j,m}$ are already chosen and

$$\Omega_j \subset \bigcup_{k=1}^m Q_{j,k}\,, \tag{2.5}$$

where $Q_{j,k} = (x_{j,k}, r_{j,k})$, we stop. If (2.5) does not hold, we put $R_{j,m+1} = \sup\{r(x) : x \in \Omega_j \setminus \bigcup_{k=1}^m Q_{j,k}\}$ and find $x_{j,m+1} \in \Omega_j \setminus \bigcup_{k=1}^m Q_{j,k}$ such that $r_{j,m+1} := r(x_{j,m+1}) > \delta R_{j,m+1}$. If the sequence $\{x_{j,k}\}_k$ obtained in this way is finite, then

$$\Omega_j \subset \bigcup_k Q_{j,k}\,. \tag{2.6}$$

Suppose that $\{x_{j,k}\}_k$ is infinite. Fix $k, m \in \mathbb{N}$ such that $k > m$. Then

$$x_{j,k} \notin Q_{j,m},\ r_{j,m} > \delta R_{j,m} \ge \delta R_{j,k} > \delta r_{j,k}\,,$$

and every $y \in Q(x_{j,m}, \gamma r_{j,m})$ with $\gamma = \frac{\delta}{a(1+\delta)}$ satisfies

$$|y - x_{j,k}|_\alpha \geq a^{-1}|x_{j,k} - x_{j,m}|_\alpha - |y - x_{j,m}|_\alpha > (a^{-1} - \gamma)\delta r_{j,k} = \gamma r_{j,k} \,.$$

Thus $y \notin Q(x_{j,k}, \gamma r_{j,k})$ and so the sets $Q(x_{j,k}, \gamma r_{j,k})$ are pairwise disjoint. On the other hand,

$$|y|_\alpha \leq a(|x_{j,m}|_\alpha + \gamma r_{j,m}) \leq a(1 + \gamma\eta^{-1})d_j \,,$$

i.e.

$$Q(x_{j,m}, r_{j,m}) \subset Q(0, a(1 + \gamma\eta^{-1})d_j).$$

Now, suppose that (2.6) does not hold. Then there exists $x \in \Omega_j \setminus \bigcup_k Q_{j,k}$, and the inequalities $r_{j,m} > \delta R_{j,m} \geq \delta r(x) > 0$ hold for every m. Hence, the bounded set $Q(0, a(1+\gamma\eta^{-1})d_j)$ contains infinite number of pairwise disjoint sets of volume $|Q(x_{j,m}, \gamma r_{j,m})| > (2\gamma\delta r(x))^n$. This is a contradiction, and so (2.6) holds again.

To estimate $\sum_k \chi_{Q(x_{j,k}, \tau r_{j,k})}$, fix $y \in \Omega$ and consider k such that

$$y \in Q(x_{j,k}, \tau r_{j,k}) \setminus Q(x_{j,k}, \gamma r_{j,k}). \tag{2.7}$$

Denote by m the minimal index k for which (2.7) is satisfied. We have

$$\begin{aligned} a(\tau + \eta)r_{j,k} &\geq a(|y - x_{j,k}|_\alpha + d(x_{j,k})) \geq d(y) \geq \\ &\geq a^{-1}|y - x_{j,k}|_\alpha - d(x_{j,k}) \geq (a^{-1}\eta - \tau)r_{j,m} \end{aligned}$$

and

$$|z - y|_\alpha \leq a(|z - x_{j,k}|_\alpha + |y - x_{j,k}|_\alpha) \leq \delta^{-1}a(\tau + \gamma)r_{j,m}, \;\; z \in Q(x_{j,k}, \gamma r_{j,k}).$$

Thus

$$Q(x_{j,k}, \frac{\gamma(\eta - a\tau)}{a^2(\tau + \eta)} r_{j,m}) \subset Q(x_{j,k}, \gamma r_{j,k}) \subset Q(y, \delta^{-1}a(\tau + \gamma)r_{j,m}) \tag{2.8}$$

and so the number of those indices k which satisfy (2.7) is not greater than

$$\left(\frac{a^3(\tau + \eta)(\tau + \gamma)}{\gamma\delta(\eta - a\tau)}\right)^n,$$

since the sets on the left hand side of (2.8) are pairwise disjoint. Choosing δ sufficiently close to 1 and taking into account that at most one of the sets $Q(x_{j,k}, \gamma r_{j,k})$ may contain the point y, we come to the estimate

$$\sum_k \chi_{Q(x_{j,k}, \gamma r_{j,k})} \leq 2 + \left(\frac{a^3(\tau + \eta)(2a\tau + 1)}{\eta - a\tau}\right)^n. \tag{2.9}$$

Let $\{Q_k\}$ be a renumeration of $\{Q_{j,k} : j,k = 1,2,\dots\}$. The inclusion (2.2) follows from (2.6). To prove (2.4), we observe that any $y \in Q(x_{j-1,k}, \tau r_{j-1,k})$ and $z \in Q(x_{j+1,\ell}, \tau r_{j+1,\ell})$, $j \geq 2$, satisfy

$$\begin{aligned} |y|_\alpha \leq & a(|x_{j-1,k}|_\alpha + \tau\eta^{-1} d(x_{j-1,k})) \leq a(1+\tau\eta^{-1}) d_{j-1} = \\ = & (a^{-1} - \tau\eta^{-1}) d_j < a^{-1}|x_{j+1,\ell}|_\alpha - \tau\eta^{-1} d(x_{j+1,\ell}) \leq |z|_\alpha \,, \end{aligned}$$

i.e. $Q(x_{j-1,k}, \tau r_{j-1,k}) \cap Q(x_{j+1,\ell}, \tau r_{j+1,\ell}) = \emptyset$ for every k and ℓ. Using (2.9), we conclude

$$\vartheta \leq 4 + 2\left(\frac{a^3(\tau+\eta)(2a\tau+1)}{\eta - a\tau}\right)^n .$$

The relation (2.3) is obvious. □

Lemma 2 (see [1])**.** *If $1 \leq s \leq p < \infty$ and if $\{E_j\}$ is a sequence of measurable sets in $\mathbb{R}^n$ such that*

$$\sum_j \chi_{E_j} \leq \vartheta,$$

then the inequality

$$\sum_j \|\chi_{E_j} f\|^p_{L^{ps}(\mu)} \leq \vartheta \|f\|^p_{L^{ps}(\mu)}$$

holds for every $f \in L^{ps}(\mu)$.

We are ready to prove Theorem 1.

Necessity. Suppose that (1.4) holds. Let $E \subset \mathbb{R}^n$ and Q be such that $0 < \mu(E \cap Q) \leq \mu(E) < \infty$. Then the properties of K yield

$$T(\chi_{E\cap Q}\psi\mu)(x) \geq \inf_{z,y\in Q} K(z-y) \int_{E\cap Q} \psi \, d\mu := 2\lambda > 0, \;\; x \in Q$$

and so

$$\begin{aligned} \omega(Q) \leq \omega(\{x : T(\chi_{E\cap Q}\psi\mu)(x) > \lambda\}) \leq (A/\lambda)^q \|\chi_{E\cap Q}\|^q_{L^{ps}(\mu)} = \\ = (A/\lambda)^q \mu(E\cap Q)^{q/p} < \infty \,. \end{aligned}$$

Further, using (2.1) and (1.4), we obtain

$$\begin{aligned} \|\chi_Q T(\chi_Q\omega)\psi\|_{L^{p's'}(\mu)} \leq & c_0^{-1} \sup\{|\int_{\mathbb{R}^n} T(\chi_Q\omega)\psi f \, d\mu| : \|f\|_{L^{ps}(\mu)} \leq 1\} \leq \\ \leq & c_0^{-1} \sup\{\int_Q |T(f\psi)| \, d\omega : \|f\|_{L^{ps}(\mu)} \leq 1\} = \\ = & c_0^{-1} \sup\{\int_0^\infty \omega(\{y \in Q : |T(f\psi\mu)(y)| > \lambda\}) \, d\lambda : \\ & : \|f\|_{L^{ps}(\mu)} \leq 1\} \leq \\ \leq & c_0^{-1} \int_0^\infty \min\{\omega(Q), (A/\lambda)^q\} \, d\lambda = c_0^{-1} q' A \omega(Q)^{1/q'} \,. \end{aligned}$$

Hence, the inequality (1.5) holds with $B \leq c_0^{-1} q' A$.

Sufficiency. Assume that (1.5) holds. Let $\lambda > 0$ and f be given. Without loss of generality, we can suppose that $f \geq 0$, $\int |f|^p \, d\mu < \infty$ and that the support of f is compact, say

$$\operatorname{supp} f \subset Q(0, r). \tag{2.10}$$

The function $T(f\psi\mu)$ satisfies the condition (1.2) with the same constant c as for K and so

$$M(T(f\psi\mu)) \leq cT(f\psi\mu), \tag{2.11}$$

where M is the anisotropic maximal operator

$$Mg(x) = \sup_{x \in Q} \frac{1}{|Q|} \int_Q |g(y)| \, dy.$$

Since $M(T(f\psi\mu))$ is lower semicontinuous, the set

$$\Omega_\lambda = \{x : M(T(f\psi\mu))(x) > c\lambda\}$$

is open.

At first, assume the case when $\Omega_\lambda \neq \mathbb{R}^n$. We can use Lemma 1 to write

$$\Omega_\lambda = \bigcup_k Q_k \,,$$

where $Q_k = Q(x_k, r_k)$ and x_k, r_k satisfy (2.3), (2.4) with $\tau = 2a$ and $\eta > 2a^2$. For the sake of brevity, we shall denote $Q(x_k, \tau r_k)$ by Q_k^*. We observe that

$$K(x) \leq ct^n K(y) \quad \text{for every} \quad t \geq 1 \quad \text{and} \quad x, y \quad \text{with} \quad |y|_\alpha \leq t|x|_\alpha \,.$$

Indeed, the condition (1.2) implies that

$$\frac{1}{|Q|} \int_Q K(z) \, dz \leq cK(y') \quad \text{for every } Q \text{ and for a.e. } y' \in Q.$$

Since K is a.r.d., we have

$$\begin{aligned} K(x) \leq & \frac{1}{|Q(0, |x|_\alpha)|} \int_{Q(0,|x|_\alpha)} K(z) \, dz \leq \\ \leq & \frac{t_1^n}{|Q(0, t_1|x|_\alpha)|} \int_{Q(0,t_1|x|_\alpha)} K(z) \, dz \leq ct_1^n K(y') \end{aligned}$$

for every $t_1 > t$ and for a.e. $y' \in Q(0, t_1|x|_\alpha)$, and (2.12) follows.

For $x, z \in Q_k$ and $y \in \mathbb{R}^n \setminus Q_k^*$, the estimate

$$|z - y|_\alpha \leq a(1 + 2a)|x - y|_\alpha$$

holds and using (2.12), we get

$$\begin{aligned} T(\chi_{\mathbb{R}^n \setminus Q_k^*} f\psi\mu)(x) = & \int_{y \notin Q_k^*} K(x-y) f(y)\psi(y)\, d\mu(y) \leq \\ \leq & c_1 \int_{y \notin Q_k^*} \left(\frac{1}{|Q_k|} \int_{Q_k} K(z-y)\, dz \right) f(y)\psi(y)\, d\mu(y) \leq \\ \leq & \frac{c_1}{|Q_k|} \int_{Q_k} T(f\psi\mu)(z)\, dz \leq \\ \leq & c_1 \eta^n \frac{1}{|Q(x_k, \eta r_k)|} \int_{Q(x_k, \eta r_k)} T(f\psi\mu)(z)\, dz, \end{aligned}$$

where $c_1 = ca^n(1+2a)^n$. Since $Q(x_k, \eta r_k) \setminus \Omega_\lambda \neq \emptyset$ by (2.3), the estimate

$$T(\chi_{\mathbb{R}^n \setminus Q_k^*} f\psi\mu)(x) \leq c_1 \eta^n c\lambda$$

follows and we conclude

$$\{x : T(f\psi\mu)(x) > \gamma\lambda\} \cap Q_k \subset \{x : T(\chi_{Q_k^*} f\psi\mu)(x) > \frac{\gamma}{2}\lambda\} \cap Q_k$$

for $\gamma > 2c_1\eta^n c$. Hence, if k is such that $\omega(Q_k^*) > 0$ and

$$\frac{1}{\omega(Q_k^*)} \int_{Q_k^*} T(\chi_{Q_k^*} f\psi\mu)\, d\omega \leq \beta\lambda, \tag{2.13}$$

where $\beta \in (0,1)$ is a fixed number, then

$$\omega(\{x : T(f\psi\mu)(x) > \gamma\lambda\} \cap Q_k) \leq \frac{2}{\gamma\lambda} \int_{Q_k} T(\chi_{Q_k^*} f\psi\mu)\, d\omega \leq \frac{2\beta}{\gamma} \omega(Q_k^*). \tag{2.14}$$

If k is such that $\omega(Q_k^*) > 0$ and (2.13) fails, then using the Hölder inequality (2.11) and the condition (1.5), we obtain

$$\begin{aligned} \lambda^q \omega(Q_k^*) < & \beta^{-q} \omega(Q_k^*)^{1-q} \left(\int_{Q_k^*} T(\chi_{Q_k^*} f\psi\mu)\, d\omega \right)^q = \\ = & \beta^{-q} \omega(Q_k^*)^{1-q} \left(\int_{Q_k^*} T(\chi_{Q_k^*} \omega)\psi f\, d\mu \right)^q \leq \\ \leq & c_0 \beta^{-q} \omega(Q_k^*)^{1-q} \|\chi_{Q_k^*} T(\chi_{Q_k^*}\omega)\psi\|^q_{L^{p's'}(\mu)} \|\chi_{Q_k^*} f\|^q_{L^{ps}(\mu)} \leq \\ \leq & c_0 \beta^{-q} B^q \|\chi_{Q_k^*} f\|^q_{L^{ps}(\mu)}, \end{aligned}$$

i.e.

$$\omega(\{x : T(f\psi\mu)(x) > \gamma\lambda\} \cap Q_k) \leq \omega(Q_k^*) \leq (\beta\lambda)^{-q} B^q \|\chi_{Q_k^*} f\|^q_{L^{ps}(\mu)} . \tag{2.15}$$

Summing the inequalities (2.14) and (2.15) over the appropriate indices k and using the overlapping condition (2.4) together with Lemma 2, we get the estimate

$$\omega\{x : T(f\psi\mu)(x) > \gamma\lambda\} \leq \frac{2}{\gamma}\vartheta\beta\omega(\Omega_\lambda) + (\beta\lambda)^{-q} B^q \vartheta^{q/p} \|f\|^q_{L^{ps}(\mu)}$$

which, according to (2.9), yields

$$\begin{aligned}(\gamma\lambda)^q \omega(\{x : T(f\psi\mu)(x) > \gamma\lambda\}) \leq& \\ \leq 2\vartheta\gamma^{q-1}\beta\omega(\{x : T(f\psi\mu)(x) > \lambda\}) &+ \gamma^q B^q \vartheta^{q/p} \beta^{-q} \|f\|^q_{L^{ps}(\mu)} .\end{aligned} \tag{2.16}$$

In the case when $\Omega_\lambda = \mathbb{R}^n$, we use (2.11), (2.1) and (1.5) and for every Q which contains the support of f, we obtain

$$\begin{aligned}\omega(Q) <& \frac{1}{\lambda}\int_Q T(f\psi\mu)\, d\omega = \frac{1}{\lambda}\int_Q T(\chi_Q\omega)\psi f\, d\mu \leq \\ \leq& \frac{c_0}{\lambda}\|\chi_Q T(\chi_Q\omega)\psi\|_{L^{p's'}(\mu)} \|f\|_{L^{ps}(\mu)} \leq \\ \leq& \frac{c_0}{\lambda} B\omega(Q)^{1/q'} \|f\|_{L^{ps}(\mu)} .\end{aligned}$$

Hence,

$$\omega(Q)^{1/q} \leq \frac{c_0 B}{\lambda}\|f\|_{L^{ps}(\mu)} \tag{2.17}$$

and we can replace Q by $\mathbb{R}^n$ because Q may be arbitrarily large and the right hand side does not depend on Q.

This and (2.16) yield the "good λ inequality"

$$(c\gamma\lambda)^q \omega(\{x : T(f\psi\mu)(x) > c\gamma\lambda\}) \leq c_2\beta\omega(\{x : T(f\psi\mu)(x) > \lambda\}) + c_3^q B^q \|f\|^q_{L^{ps}(\mu)}$$

for every $\lambda > 0$, where $c_2 = 2\vartheta\gamma^{q-1}c^q$ and $c_3 = \max\left\{\gamma\vartheta^{1/p}\beta^{-1}, cc_0\gamma\right\}$. Taking the supremum over $0 < \lambda \leq \frac{t}{c\gamma}$, we obtain

$$\begin{aligned}\sup_{0<\lambda\leq t} \lambda^q \omega(\{x : T(f\psi\mu)(x) > \lambda\}) \leq& \\ \leq c_2\beta \sup_{0<\lambda\leq t} \lambda^q \omega(\{x : T(f\psi\mu)(x) > \lambda\}) &+ c_3^q B^q \|f\|^q_{L^{ps}(\mu)} ,\end{aligned} \tag{2.18}$$

because $t/(c\gamma) < t$.

All that we have to do is to prove that the left hand side of (2.18) is finite for every $t > 0$ and to choose $B \in (0, c_2^{-1})$. The inequality (1.5) then follows with $A \leq Bc_3(1 - c_2\beta)^{-1/q}$.

Since $\omega(\{x : (f\psi\mu)(x) > \lambda\})$ is a decreasing function of λ, it is sufficient to consider only small λ. If $\inf T(f\psi\mu) = \lambda_0 > 0$, then according to the estimate (2.17) with $\mathbb{R}^n$ in place of Q, we have

$$\lambda^q \omega(\{x : T(f\psi\mu)(x) > \lambda\}) \leq c_0^q B^q \|f\|^q_{L^{ps}(\mu)}$$

for every $\lambda \in (0, \lambda_0)$. If $\lambda_0 = 0$, then—taking into account that by (2.10) and (2.12) we have $T(f\psi\mu)(x) \leq T(f\psi\mu)(z)$ for $|x|_\alpha \geq a^2(|z|_\alpha + 2r)$—we can assume that $\lambda = cT(f\psi\mu)(z)$ for some $|z|_\alpha$ large enough, say $|z|_\alpha \geq 2ar$. Then

$$\{x : T(f\psi\mu)(x) > \lambda\} \subset Q^* := Q(0, a^2(|z|_\alpha + 2r)),$$

and since for $x \in Q := Q(0, r), y \in Q^*$ the inequalities

$$|x - y|_\alpha \leq \frac{a(r + a^2(|z|_\alpha + 2r))}{a^{-1}|z|_\alpha - r}|z - x|_\alpha \leq 5a^4|z - x|_\alpha$$

hold, from (2.12) we get

$$\begin{aligned} K(z - x) \leq c5^n a^{4n} K(x - y) \leq& c5^n a^{4n} \omega(Q^*)^{-1} \int_{Q^*} K(x - y)\, d\omega(y) = \\ =& c5^n a^{4n} \omega(Q^*)^{-1} T(\chi_{Q^*}\omega)(x). \end{aligned}$$

This yields

$$\begin{aligned} \|\chi_Q K(z - .)\psi\|_{L^{p's'}(\mu)} \leq& c5^n a^{4n} \omega(Q^*)^{-1} \|\chi_Q T(\chi_{Q^*}\omega)\psi\|_{L^{p's'}(\mu)} \leq \\ \leq& c5^n a^{4n} B\omega(Q^*)^{-1/q}\,, \end{aligned}$$

and so,

$$\begin{aligned} \lambda^q \omega(\{x : T(f\psi\mu)(x) > \lambda\}) \leq& \left(\int_Q K(z - y) f(y)\psi(y)\, d\mu(y) \right)^q \omega(Q^*) \leq \\ \leq& c_0^q \|\chi_Q K(z - .)\psi\|^q_{L^{p's'}(\mu)} \|f\|^q_{L^{ps}(\mu)} \omega(Q^*) \leq \\ \leq& (c_0 c5^n a^{4n} B)^q \|f\|^q_{L^{ps}(\mu)} < \infty\,. \end{aligned}$$

□

3. A generalization and another equivalent condition.

In this section, we shall suppose that the topology on $\mathbb{R}^n$ is given by a quasi-metric ϱ satisfying the inequalities

$$a^{-1}\varrho(x,y)-\varrho(y,z)\le\varrho(x,z)\le a(\varrho(x,y)+\varrho(y,z)) \tag{3.1}$$

with some constant $a\ge 1$ independent of $x,y,z\in\mathbb{R}^n$ and we shall denote the corresponding balls by

$$B=B(x,r)=\{y\in\mathbb{R}^n:\varrho(x,y)\le r\}.$$

We shall assume that ω is a Borel measure satisfying the doubling condition

$$\omega B(x,2r)\le D\omega B(x,r) \tag{3.2}$$

with some constant $D>0$ independent of $x\in\mathbb{R}^n$ and $r>0$ and such that for every $x\in\mathbb{R}^n$

$$r\mapsto\omega(B(x,r)) \text{ is a continuous function and } \omega(\{x\})=0. \tag{3.3}$$

We shall assume that K is a positive measurable function on $\mathbb{R}^n\times\mathbb{R}^n$ and that for every $b>0$ there exists $c>0$ such that

$$\varrho(z,y)\le b\varrho(x,y)\Longrightarrow K(x,y)\le cK(z,y),\ x,y,z\in\mathbb{R}^n. \tag{3.4}$$

Note that e.g. the function $K(x,y)=\varrho(x,y)^{-\gamma},\gamma>0$, satisfies the condition (3.4).

A positive function $w\in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ generates a measure for which we shall use the same symbol, i.e.

$$w(E)=\int_E w(x)\,dx.$$

This measure satisfies the continuity condition (3.3).

We shall consider the operator T defined by

$$T(f\mu)(x)=\int_{\mathbb{R}^n}K(x,y)f(y)\,d\mu(y) \tag{3.5}$$

for measurable functions f and a Borel measure μ.

It is easy to see that Lemma 1 and Theorem 1 can be generalized in the following way:

Lemma 3. *Let Ω be an open non-empty proper subset in $\mathbb{R}^n$. Let $\tau>1$ and $\eta>a\tau$. Then there exist balls $B_k=B(x_k,r_k)$, $k=1,2,\dots$, and a number $\vartheta=\vartheta(n,a,\tau,\eta)$ such that*

$$\Omega=\bigcup_k B_k\,,$$

$$B(x_k,\eta r_k)\setminus\Omega\ne\emptyset,\ k=1,2,\dots,$$

and

$$\sum_k\chi_{Q(x_k,\tau r_k)}\le\vartheta\chi_\Omega\,.$$

Theorem 2. *Let $1 < p \le q < \infty$ and $1 \le s \le p$. Let μ, ω be Borel measures on $\mathbb{R}^n$, μ non-trivial and ω satisfying the conditions* (3.2) *and* (3.3) *and let ψ be a positive measurable function. Assume the operator T defined by* (3.5), *where K satisfies the condition* (3.4). *Then the inequality*

$$\omega(\{x : T(f\psi\mu)(x) > \lambda\})^{1/q} \le \frac{A}{\lambda}\|f\|_{L^{ps}(\mu)}$$

holds for every f and $\lambda > 0$ if and only if the inequality

$$\|\chi_B T(\chi_B\omega)\psi\|_{L^{p's'}(\mu)} \le C\omega(B)^{1/q'} < \infty$$

holds for every $B = B(x, r)$, $x \in \mathbb{R}^n$, $r > 0$.

Moreover, the ratio A/C of the optimal constants is bounded from below and above by positive numbers which do not depend on f, μ, ω and ψ.

In [3] (cf. also [6]), the following characterization of the weak inequality is established:

Theorem 3. *Let $1 < p < q < \infty$ and $1 \le s \le p$. Let ω, ψ, K and T be as in Theorem* 2. *Let v be a positive locally integrable function on $\mathbb{R}^n$. Then the following conditions are equivalent:*

(i) *There exists a constant $c > 0$ such that the inequality*

$$\omega(\{x : T(f\psi\mu)(x) > \lambda\}) \le c\lambda^{-q}\|f\|^q_{L^{ps}(v)}$$

holds for every $\lambda > 0$ and for every f.

(ii) *There exists a constant $c > 0$ such that the inequality*

$$\omega(B)\|\chi_{\mathbb{R}^n\setminus B}K(x,.)\psi v^{-1}\|^q_{L^{p's'}(v)} \le c \tag{3.6}$$

holds for every ball $B = B(x, r)$, $x \in \mathbb{R}^n$, $r > 0$.

A comparison of Theorems 2 and 3 yields

Theorem 4. *Let the assumptions of Theorem* 3 *be fulfilled. Then the following conditions are equivalent:*

(i) *There exists a constant $c > 0$ such that the inequality*

$$\|\chi_B T(\chi_B\omega)\psi\|_{L^{p's'}(v)} \le c\omega(B)^{1/q'} < \infty$$

holds for every ball $B = B(x, r)$, $x \in \mathbb{R}^n$, $r > 0$.

(ii) *There exists a constant $c > 0$ such that the inequality*

$$\omega(B)\|\chi_{\mathbb{R}^n\setminus B}K(x,.)\psi\|^q_{L^{p's'}(v)} \le c$$

holds for every $B = B(x, r)$, $x \in \mathbb{R}^n$, $r > 0$.

The (ii) $\Rightarrow$ (i) part of the proof of Theorem 3 is essentially based on the assumption that p is strictly less than q, the constant c in (3.6) is estimated by a quantity which tends to infinity if $q \to p$. Nevertheless, the condition (3.6) remains meaningful even with $q = p$. A natural question arises: Does Theorem 3 remain valid for $p = q$? We shall give a positive answer in this particular case:

Theorem 5. *Let $1 < p < \infty, 1 \le s \le p$ and $0 < \gamma < \sigma$. Let μ, ω be Borel measures such that μ is non-trivial and there exist positive constants c_1, c_2 such that*

$$c_1 \le \frac{\omega(B(x,r))}{r^\sigma} \le c_2 \tag{3.7}$$

for every $x \in \mathbb{R}^n$ and $r > 0$. Let T be given by (3.5) with $K(x,y) = \varrho(x,y)^{-\gamma}$. Then for every positive measurable function ψ there exists a constant $c > 0$ such that

$$\begin{aligned} \sup \omega(B)^{-1/p'} \|\chi_B T(\chi_B \omega)\psi\|_{L^{p's'}(\mu)} \le \\ \le c \sup \omega(B)^{1/p} \|\chi_{\mathbb{R}^n \setminus B} \varrho(x,.)^{-\gamma} \psi\|_{L^{p's'}(\mu)}, \end{aligned} \tag{3.8}$$

where the supremum on both sides is taken over all $B = B(x,r)$, $x \in \mathbb{R}^n$, $r > 0$.

Proof: For $y \in \mathbb{R}^n$ and $R > 0$, we put $B(y,R) = \bigcup_{k=0}^\infty B_k$, where $B_k = B(y, 2^{-k}R) \setminus B(y, 2^{-k-1}R)$, and so, according to (3.7),

$$\begin{aligned} \int_{B(y,R)} \varrho(y,z)^{-\gamma}\, d\omega(z) = \sum_k \int_{B_k} \varrho(y,z)^{-\gamma}\, d\omega(z) \le \sum_k (2^{-k-1}R)^{-\gamma} \omega(B_k) \le \\ \le (2^\sigma c_2 - c_1) \sum_k 2^{-(k+1)(\sigma-\gamma)} R^{\sigma-\gamma} = c_3 R^{\sigma-\gamma} \end{aligned}$$

with $c_3 = (2^\sigma c_2 - c_1)/(2^{\sigma-\gamma} - 1)$. Thus if $x \in \mathbb{R}^n$, $r > 0$ and $y \in B(x,r)$, we use the inequalities (3.1) to obtain

$$\begin{aligned} T(\chi_{B(x,r)}\omega)(y) = \int_{B(x,r)} \varrho(z,y)^{-\gamma}\, d\omega(z) \le \int_{B(y,2ar)} \varrho(z,y)^{-\gamma}\, d\omega(z) \le \\ \le c_3 (2a)^{\sigma-\gamma} r^{\sigma-\gamma}, \end{aligned}$$

and so,

$$\begin{aligned} \omega(B(x,r))^{-1/p'} \|\chi_{B(x,r)} T(\chi_{B(x,r)}\omega)\psi\|_{L^{p's'}(\mu)} \le \\ \le c_4 r^{\sigma - \sigma/p' - \gamma} \|\chi_{B(x,r)}\psi\|_{L^{p's'}(\mu)} \end{aligned} \tag{3.9}$$

with $c_4 = c_1^{-1/p'} c_3 (2a)^{\sigma-\gamma}$. Now, we choose z such that $\varrho(x,z) = 2ar$. Then $B(x,r) \subset \mathbb{R}^n \setminus B(z,r)$, $\varrho(z,y) \le a(2a+1)r$ for $y \in B(x,r)$, and we have

$$\begin{aligned} r^{\sigma/p-\gamma} \|\chi_{B(x,r)}\psi\|_{L^{p's'}(\mu)} \le \\ \le a^\gamma (2a+1)^\gamma c_1^{1/p} \omega(B(z,r))^{1/p} \|\chi_{\mathbb{R}^n \setminus B(z,r)} \varrho(z,.)^{-\gamma} \psi\|_{L^{p's'}(\mu)}. \end{aligned}$$

The last estimate and (3.9) yield (3.8) with $c = a^\gamma (2a+1)^\gamma c_1^{1/p} c_4$. □

As a particular consequence of Theorems 1 and 4, we can state the following pellucid characterization of the weight functions v for which the Riesz potential

$$I_\gamma f(x) = \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-\gamma}}\, dy, \quad 0 < \gamma < n,$$

is a weak type mapping of $L^p(v)$ in L^p.

Theorem 6. *Let $1 < p < \infty, 0 < \gamma < n$ and let v be a positive locally integrable function on $\mathbb{R}^n$. Then the following conditions are equivalent:*

(i) *There exists a constant $c > 0$ such that the inequality*

$$|\{x \in \mathbb{R}^n : I_\gamma f(x) > \lambda\}| \leq c\lambda^{-p} \int_{\mathbb{R}^n} |f(x)|^p v(x)\, dx$$

holds for every $\lambda > 0$ and for every f.

(ii) *There exists a constant $c > 0$ such that the inequality*

$$\left(\int_{|x-y|>r} v(y)^{1-p'} |x-y|^{(\gamma-n)p'}\, dy\right)^{1/p'} \leq cr^{-n/p}$$

holds for every $x \in \mathbb{R}^n$ and $r > 0$.

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