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Generating real maps on a biordered set

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Abstract. Several authors have defined operational quantities derived from the norm of an operator between Banach spaces. This situation is generalized in this paper and we present a general framework in which we derivate several maps $X \to \mathbb{R}$ from an initial one $X \to \mathbb{R}$, where X is a set endowed with two orders, $\leq$ and $\leq^*$, related by certain conditions. We obtain only three different derivated maps, if the initial map is bounded and monotone.

Keywords: derivated map, biordered set, admissible order

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1. Introduction.

We consider an infinite dimensional Banach space (over the real or the complex numbers), say X. The set of all the closed infinite dimensional subspaces of $X, S(X)$, is ordered by

$$M \leq N \quad \text{if and only if} \quad M \subset N.$$

Also, we can define another order in $S(X)$:

$$M \leq^* N \quad \text{if and only if} \quad M \subset N \quad \text{and} \quad \dim (N/M) < \infty\,.$$

Both orders are related by the two following properties:

(1) If $M \leq^* N$, then $M \leq N$.

(2) If $M \leq N$ and $P \leq^* N$, then $M \cap P \leq^* M$.

If T is a linear and continuous operator from an infinite dimensional Banach space X into a Banach space Y, we consider the map

$$n : S(X) \to \mathbb{R}; \quad n(M) := n(TJ_M) := \|TJ_M\|,$$

where J_M is the injection of M into X and $\|\cdot\|$ denotes the norm. B. Gramsch (1969) (see [SC]) defined the operational quantity

$$in(T) := \inf_{M \leq X} n(TJ_M),$$

which can be used to characterize when an operator T is an upper semi-Fredholm operator (closed range and finite dimensional kernel): $in(T) > 0$. Independently,

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A.A. Sedaev (1970) [SE] and A. Lebow and M. Schechter (1971) [LS] consider the operational quantity
$$i^*n(T) := \inf_{M \leq^* X} n(TJ_M).$$

This quantity verifies that $i^*n(T) = 0$, if and only if T is a compact operator (the image of the closed unit ball of X is relatively compact). With a different definition, i^*n has been considered by H.-O. Tylli [TY]. The equality of both definitions has been showed in [GM2], [MA2]. Finally, M. Schechter (1972) [SC] defined the following operational quantity:

$$sin(T) := \sup_{M \leq X} in(TJ_M) = \sup_{M \leq X} \inf_{N \leq M} n(TJ_N).$$

This quantity verifies: $sin(T) = 0$, if and only if T is a strictly singular operator (if TJ_M is an injection, then M is finite dimensional).

If we consider the set of all the closed infinite codimensional subspaces of Y, $S'(Y)$, where Y is an infinite dimensional Banach space, then we define two orders in $S'(Y)$:

$$U \leq V \quad \text{if and only if} \quad U \supset V;$$
$$U \leq^* V \quad \text{if and only if} \quad U \supset V \quad \text{and} \quad \dim(U/V) < \infty\,.$$

Now we obtain the following properties which relate $\leq$ with $\leq^*$,

(1) If $U \leq^* V$, then $U \leq V$.

(2) If $U \leq V$ and $W \leq^* V$, then $U + W \leq^* U$.

Let T be a linear and continuous operator from a Banach space X into an infinite dimensional Banach space Y. From the map

$$n' : S'(Y) \to \mathbb{R}; \quad n'(U) := n(Q_U T) := \|Q_U T\|,$$

where Q_U denotes the quotient map of Y onto Y/U, L. Weis (1976) [WE] derived the operational quantity
$$in'(T) := \inf_{U \leq 0} n'(Q_U T)$$

which can be used to characterize a class of operators: $in'(T) > 0$ if and only if T is a lower semi-Fredholm operator (closed and finite codimensional range). Independently, A.S. Fajnshtejn and V.S. Shulman (1982) (see [FA]) and J. Zemanek (1983) [ZE] consider the operational quantity

$$i^*n'(T) := \inf_{U \leq^* 0} n'(Q_U T).$$

This quantity verifies that $i^*n'(T) = 0$, if and only if T is a compact operator. A.S. Fajnshtejn [FA] has showed that the quantity i^*n' agrees with the Hausdorff measure of noncompactness, which was introduced by Goldenstein, Gohberg and

Markus (1957) (see [BG]). Finally, L. Weis (1976) [WE] defined the following operational quantity:

$$sin'(T) := \sup_{U \leq 0} in'(Q_U T) = \sup_{U \leq 0} \inf_{V \leq U} n'(Q_V T).$$

This quantity verifies: $sin'(T) = 0$, if and only if T is a strictly cosingular operator (if $Q_U T$ is a surjection, then U is finite codimensional).

If we consider the injection modulus and the surjection modulus, instead of the norm, there can be obtained new operational quantities. If T is a linear and continuous operator, then the injection modulus of T is defined by

$$j(T) := \inf\{\|Tx\| : x \in B_X\},$$

and the surjection modulus of T by

$$q(T) := \sup\{\varepsilon > 0 : \varepsilon B_Y \subset T B_X\},$$

where B_X is the closed unit ball of X. M. Schechter (1972) [SC] considers the following operational quantities:

$$sj(T) := \sup_{M \leq X} j(T J_M),$$
$$s^* j(T) := \sup_{M \leq^* X} j(T J_M).$$

He verifies that $sj(T) = 0$, if and only if T is a strictly singular operator and $s^* j(T) > 0$, if and only if T is an upper semi-Fredholm operator. The author (1989) [MA1], [MA2] has defined the operational quantity

$$isj(T) := \inf_{M \leq X} sj(T J_M) = \inf_{M \leq X} \sup_{N \leq M} j(T J_N)$$

and showed that $isj(T) > 0$, if and only if T is an upper semi-Fredholm operator. The quantities iq, siq and $i^* q$, similarly defined, verify $iq = siq = i^* q = 0$. J. Zemanek (1983) [ZE] defines the following operational quantities:

$$sq'(T) := \sup_{U \leq 0} q(Q_U T),$$
$$s^* q'(T) := \sup_{U \leq^* 0} q(Q_U T),$$

where 0 is the null subspace of Y. They verify that $sq'(T) = 0$, if and only if T is a strictly cosingular operator and $s^* q'(T) > 0$, if and only if T is a lower semi-Fredholm operator. The author (1989) [MA1], [MA2] has defined the operational quantity

$$isq'(T) := \inf_{U \leq 0} sq'(Q_U T) = \inf_{U \leq 0} \sup_{V \leq U} q(Q_V T)$$

and showed that $isq'(T) > 0$, if and only if T is a lower semi-Fredholm operator. The quantities ij', sij' and i^*j', similarly defined, verify $ij = sij = i^*j' = 0$.

It is possible to consider other operational quantities by using inf and sup: $isin, i^*s^*si^*n, \dots$, but there are only three different quantities: in, i^*n, sin. Analogously it occurs with n', j and q' [MA2].

If we consider a space ideal $\mathbb{A}$ (in the sense of A. Pietsch [PI]) and the set $S_{\mathbb{A}}(X)$ (respectively $S'_{\mathbb{A}}(Y)$), defined as the set of all the subspaces M of X (U of Y) such that $M(Y/U)$ does not belong to $\mathbb{A}$, then we can define operational quantities of a similar way as above. This procedure is used in [GM1], [GM3], [MA2] to define classes of operators which generalize the classes of the semi-Fredholm operators, strictly singular operators and strictly cosingular operators.

In this paper, we consider a general situation. Let X be a set endowed with two orders, $\leq$ and $\leq^*$, related by similar conditions of (1) and (2). We show that if $a : X \to \mathbb{R}$ is bounded and monotone, then we obtain only three new maps: ia, sia, i^*a (if a is increasing) or sa, isa, s^*a (if a is decreasing).

2. Generating real maps on an ordered set.

In this paper, $(X, \leq)$ is a (partially) ordered set. We denote $B(X, \mathbb{R})$ the set of bounded maps of X in R. We define the maps i and s on $B(X, \mathbb{R})$ in the following way: for $a \in B(X, \mathbb{R})$ and $x \in X$,

$$ia(x) := \inf_{z \leq x} a(z),$$
$$sa(x) := \sup_{z \leq x} a(z).$$

Note that sa is the infimum of all increasing maps $b \in B(X, \mathbb{R})$ such that $a \leq b$ and ia is the supremum of all decreasing maps $c \in B(X, \mathbb{R})$ such that $c \leq a$. That is, sa is the lower hull of the family $\{b \in B(X, \mathbb{R}) : a \leq b, b \text{ increasing } \}$ and ia is the upper hull of the family $\{c \in B(X, \mathbb{R}) : c \leq a, c \text{ decreasing } \}$ [BO, IV, S5, No. 5].

We can iterate the procedure obtaining many derivated maps from a : $isa, ssa, sissia, \dots$. If a is monotone, we only obtain two different new maps.

We will denote a increasing by $a_\uparrow$ and a decreasing by $a^\downarrow$.

Proposition 1. *Suppose $(X, \leq)$ is an ordered set and $a \in B(X, \mathbb{R})$ is monotone.*

(1) *If $a_\uparrow$, then $ia^\downarrow, sia_\uparrow$, and they are the only different derivated maps which are obtained from a using i and s. Moreover,*

$$ia^\downarrow \leq sia_\uparrow \leq a_\uparrow .$$

(2) *If $a^\downarrow$, then $sa_\uparrow, isa^\downarrow$, and they are the only different derivated maps which are obtained from a using i and s. Moreover,*

$$a^\downarrow \leq isa^\downarrow \leq sa_\uparrow .$$

Proof: We give a proof in several steps. For every a (monotone or not), we obtain that

(1) $$ia^{\downarrow} \leq a \leq sa_{\uparrow}\,.$$

Moreover,

(2) $$(-a)_{\uparrow} \Leftrightarrow a^{\downarrow}; \quad i(-a) = -sa.$$

In the "first generation", we obtain ia and sa. If $a_{\uparrow}$, then $a = sa$, hence

(3) $$a_{\uparrow} \Rightarrow ia^{\downarrow} \leq a = sa_{\uparrow}\,.$$

Analogously

(4) $$a^{\downarrow} \Rightarrow ia = a^{\downarrow} \leq sa_{\uparrow}\,.$$

In the "second generation": If $a_{\uparrow}$, then we obtain iia and sia. Because $ia^{\downarrow}$, by (4), it is $iia = ia$. On the other hand, by (1), it is $ia \leq sia$ and $sia \leq sa = a$. Hence

(5) $$a_{\uparrow} \Rightarrow ia^{\downarrow} \leq sia_{\uparrow} \leq a_{\uparrow}\,.$$

Analogously, by (2),

(6) $$a^{\downarrow} \Rightarrow a^{\downarrow} \leq isa^{\downarrow} \leq sa_{\uparrow}\,.$$

In the "third generation": If $a_{\uparrow}$, then we obtain $isia$ and $ssia$. Because $sia_{\uparrow}$, using (3), it is $ssia = sia$. On the other hand, using (5), it is

$$iis = ia \leq isia \leq ia,$$

hence $ia = isia$. Analogously, by (2), if $a^{\downarrow}$, then $iisa = sa$ and $sisa = sa$. □

3. Generating real maps on a biordered set.

Let $\leq^*$ be another order on X (that is, $(X, \leq^*)$ is an ordered set). If $a \in B(X, \mathbb{R})$ is $*$-monotone ($a_{\uparrow *}$ or $a^{\downarrow *}$), then using i^* and s^* (defined using $\leq^*$ instead of $\leq$), by Proposition 1, we can write

$$\begin{aligned} a_{\uparrow *} &\Rightarrow i^* a^{\downarrow *} \leq s^* i^* a_{\uparrow *} \leq a_{\uparrow *}\,, \\ a^{\downarrow *} &\Rightarrow a^{\downarrow *} \leq i^* s^* a^{\downarrow *} \leq s^* a_{\uparrow *}\,. \end{aligned}$$

In the following results, we consider the case a monotone (for $\leq$), when $\leq^*$ verifies a certain condition related to $\leq$.

If $(X, \leq)$ and $(X, \leq^*)$ are ordered sets, we say that $\leq^*$ is admissible with regard to $\leq$, if

(1) $x \leq^* y \Rightarrow x \leq y$, and moreover,

(2) $y \leq x$ and $z \leq^* x \Rightarrow \exists\, y \cap z$ and $y \cap z \leq^* y$,

$y \cap z$ being the infimum of $\{y, z\}$ for $\leq$. If $\leq^*$ is admissible with regard to $\leq$, then $(X, \leq, \leq^*)$ will be called a biordered set.

Let E be an infinite set. The set

$$\mathcal{P}_\infty(E) := \{A \subset E : A \text{ infinite }\}$$

is a simple example of a biordered set, taking $A \leq B \Leftrightarrow A \subset B$, $A \leq^* B \Leftrightarrow A \subset B$ and $B \setminus A$ finite. Note that $A \leq^* B$, if and only if A belongs to the Fréchet filter on B.

Proposition 2. *Suppose $(X, \leq, \leq^*)$ is a biordered set and $a \in B(X, \mathbb{R})$ is monotone.*

(1) *If $a_\uparrow$, then $i^* a_\uparrow$ is the only derivated map which is obtained using i^* and s^*. Moreover,*
$$ia^\downarrow \leq sia_\uparrow \leq i^* a_\uparrow \leq a_\uparrow \,.$$

(2) *If $a^\downarrow$, then $s^* a^\downarrow$ is the only derivated map which is obtained using i^* and s^*. Moreover,*
$$a^\downarrow \leq s^* a^\downarrow \leq isa^\downarrow \leq sa_\uparrow \,.$$

Proof: We give only the proof of (1). (2) can be obtained analogously.

We have $i^* a_\uparrow$: let $x, y \in X$ with $x \leq y$, and let $\varepsilon > 0$. Then there exists $z \leq^* y$ such that $a(z) < i^* a(y) + \varepsilon$. As $\leq^*$ is admissible with regard to $\leq$, there exists $x \cap z \leq^* x$ and hence

$$i^* a(x) \leq a(x \cap z) \leq a(z) < i^* a(y) + \varepsilon$$

for every $\varepsilon > 0$. Consequently, $i^* a(x) \leq i^* a(y)$.

It is obvious that $ia \leq i^* a \leq s^* a = sa = a$. Moreover, using $i^* a_\uparrow$, we obtain $sia \leq si^* a = i^* a \leq a$.

In the "second generation", using i^* and s^*, we obtain $i^* i^* a$ and $s^* i^* a$. Using Proposition 1, we obtain $i^* i^* a = i^* a$, because $i^* a^\downarrow *$. From $i^* a_\uparrow$ it results $s^* i^* a = i^* a$. ☐

Proposition 3. *Suppose $(X, \leq, \leq^*)$ is a biordered set and $a \in B(X, \mathbb{R})$ is monotone.*

(1) *If $a_\uparrow$, then $i^* a, sia, ia$ are constant on $\{z \in X : z \leq^* x\}$ for every $x \in X$.*

(2) *If $a^\downarrow$, then $s^* a, isa, sa$ are constant on $\{z \in X : z \leq^* x\}$ for every $x \in X$.*

Proof: We give only the proof of (2). (1) can be obtained analogously.

Let $x \in X$ and $z \leq^* x$, hence $z \leq x$. From $s^* a_\uparrow *$, we obtain $s^* a(z) \leq s^* a(x)$. From $s^* a^\downarrow$, we obtain $s^* a(z) \geq s^* a(x)$. Hence $s^* a$ is constant on $\{z \in X : z \leq^* x\}$.

From $sa_\uparrow$, we obtain $sa(z) \leq sa(x)$. On the other hand, for every $\varepsilon > 0$ there exists $y \in X$, with $y \leq x$, such that $a(y) > sa(x) - \varepsilon$. As $\leq^*$ is admissible with regard to $\leq$, there exists $y \cap z$. Hence

$$sa(x) - \varepsilon < a(y) \leq a(y \cap z) \leq sa(z).$$

Consequently $sa(x) \leq sa(z)$ and sa is constant on $\{z \in X : z \leq^* x\}$.

It follows from (1) and $sa_\uparrow$ that isa is constant. $\square$

Propositions 1 and 2 assure us that there is only a finite number of different derivated maps which are obtained using i and s, or i^* and s^*. The following theorem assures the same result when we use i, s, i^* and s^*.

Theorem 4. *Suppose $(X, \leq, \leq^*)$ is a biordered set and $a \in B(X, \mathbb{R})$ is monotone.*

(1) *If $a_\uparrow$, then $ia^\downarrow, sia_\uparrow, i^*a_\uparrow$ are the only different derivated maps obtained from a using i, s, i^* and s^*. Moreover*

$$ia^\downarrow \leq sia_\uparrow \leq i^*a_\uparrow \leq a_\uparrow\,.$$

(2) *If $a^\downarrow$, then $sa_\uparrow, isa^\downarrow, s^*a^\downarrow$ are the only different derivated maps obtained from a using i, s, i^* and s^*. Moreover*

$$a^\downarrow \leq s^*a^\downarrow \leq isa^\downarrow \leq sa_\uparrow\,.$$

Proof: Using Propositions 1, 2 and 3, and the techniques of Propositions 1 and 2, we can see that the generation process ends in a finite number of steps which are represented in the following diagrams:

$$a_\uparrow \left| \begin{array}{l} sa = a \\ s^*a = a \\ i^*a_\uparrow \left| \begin{array}{l} ii^*a = ia \\ \\ si^*a = i^*i^*a = s^*i^*a = i^*a \end{array}\right. \\ \\ ia^\downarrow \left| \begin{array}{l} iia = i^*ia = s^*ia = ia \\ sia_\uparrow \left| \begin{array}{l} isia = ia \\ \\ ssia = i^*sia = s^*sia = sia \end{array}\right. \end{array}\right. \end{array}\right.$$

$$a^\downarrow \left| \begin{array}{l} ia = a \\ i^*a = a \\ s^*a^\downarrow \left| \begin{array}{l} ss^*a = a \\ \\ is^*a = s^*s^*a = i^*s^*a = s^*a \end{array}\right. \\ \\ sa_\uparrow \left| \begin{array}{l} ssa = s^*sa = i^*sa = sa \\ isa_\uparrow \left| \begin{array}{l} sisa = sa \\ \\ iisa = s^*isa = i^*isa = isa \end{array}\right. \end{array}\right. \end{array}\right.$$

For example, using Proposition 3 we obtain $\alpha^*\beta a = \beta a$, with $\alpha, \beta \in \{i, s\}$. $\square$

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