# An existence theorem for a class of nonlinear elliptic optimal control problems

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*Abstract.* We establish the existence of an optimal "state-control" pair for an optimal control problem of Lagrange type, monitored by a nonlinear elliptic partial equation involving nonmonotone nonlinearities.

*Keywords:* Sobolev embedding theorem, Novikov's theorem, Aumann's theorem, pseudomonotone operator, property $(M)$, nonlinear elliptic equation

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## 1. Introduction.

The purpose of this note is to establish the existence of optimal controls for a class of nonlinear elliptic control systems of Lagrange type, with control constraints. In the past, the question of the existence of optimal controls for elliptic systems was addressed primarily for linear and semilinear ones. We refer to the works of Barbu [1, Chapter 3], Lions [6, Chapter 2], Raitum [7] and Zolezzi [10]. Here we employ the Cesari–Rockafellar technique, to establish the existence of optimal trajectories for a large class of strongly nonlinear, elliptic distributed parameter systems.

## 2. Preliminaries.

Let $Z$ be a bounded domain in $\mathbb{R}^n$ with smooth boundary $\partial Z = \Gamma$. We will be considering the following nonlinear elliptic control problem of Lagrange type:

$$(*)\quad \left[\begin{array}{c} J(x,u) = \int_z L(z,\theta(x(z)),u(z))\,dz \to inf = m \\ \text{s.t. } \Sigma_{|\alpha|\le m}(-1)^{|\alpha|} D^\alpha(A_\alpha(z,\eta(x(z)))) + f(z,\theta(x(z)),u(z)) = 0 \text{ on } Z \\ D^\beta x\mid_\Gamma = 0 \text{ for } |\beta| \le m-1, u(z)\in U(z,\theta(x,(z))) \text{ a.e.}, u(\cdot) \text{ measurable.} \end{array}\right]$$

Here $\alpha = (\alpha_1,\dots,\alpha_n)$ is a multi-index of positive integers, $|\alpha| = \Sigma_{i=1}^n \alpha_i$ is the length of the multi-index and $D^\alpha = D_1^{\alpha_1}\dots D_n^{\alpha_n}$ with $D_i = \frac{\partial}{\partial z_i}$. Also $\eta(x) = \{D_x^\alpha(\cdot) : |\alpha| \le m\}$ and $\theta(x) = \{D^\beta x(\cdot) : |\beta| \le m-1\}$.

We will need the following hypotheses on the data of $(*)$:

$\underline{H(A)}$ : $\quad A_\alpha : Z\times\mathbb{R}^{n_m} \to \mathbb{R}(n_m = \frac{(n+m)!}{n!m!})$ are functions s.t.

(1) for every $\eta\in\mathbb{R}^{n_m}, z\to A_\alpha(z,\eta)$ is measurable,

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(2) for every $z \in Z, \eta \to A_\alpha(z,\eta)$ is continuous,
(3) $|A_\alpha(z,\eta)| \le a(z) + b|\eta|^{p-1}$ a.e. with $a(\cdot) \in L_+^q(Z), p, q > 1$ $\frac{1}{p} + \frac{1}{q} = 1$,
(4) if $\eta_1 = \{\eta_\beta : |\beta| \le m-1\}$ and $\eta_2 = \{\eta_\gamma : |\gamma| = m\}$, then

$$\Sigma_{|\gamma|=m} A_\alpha(z,\eta_1,\eta_2) - A_\alpha(z,\eta_1,\eta_2')(\eta_{2,\alpha} - \eta_{2,\alpha}') > 0,$$

(5) there exists $c > 0$ and $k(\cdot) \in L^1(Z)$ s.t.

$$\Sigma_{|\alpha|\le m} A_\alpha(z,\eta_1,\eta_2)\eta_{2,\alpha} \ge c|\eta_2|^p - k(z) \text{ a.e.}$$

$\underline{H(f)}$ : $f : Z \times \mathbb{R}^{n_{m-1}} \times \mathbb{R}^l \to \mathbb{R}(n_{m-1} = \frac{(n+m-1)!}{n!(m-1)!})$ is a function s.t.

(1) for every $(\theta,u) \in \mathbb{R}^{n_{m-1}} \times \mathbb{R}^l, z \to f(z,\theta,u)$ is measurable,
(2) for every $z \in Z, (\theta,u) \to f(z,\theta,u)$ is continuous,
(3) $|f(z,\theta,u)| \le a_1(z) + b(|\theta| + \|u\|)^{p-1}$ a.e. with $a_1(\cdot) \in L_+^q(Z), b_1 > 0$.
(4) $\Sigma_{|\beta|\le m-1}$ $f(z,\theta,u)\theta_\alpha \ge -c_1$ a.e. for all $u \in U(z,\theta)$, with $c_1 > 0$.

$\underline{H(U)}$ : $U : Z \times \mathbb{R}^{n_{m-1}} \to P_f(\mathbb{R}^l) = \{B \subseteq \mathbb{R}^l :$ nonempty, closed $\}$ is a multifunction s.t.

(1) $(z,\theta) \to U(z,\theta)$ is graph measurable; i.e. $GrU = \{(z,\theta,u) \in Z \times \mathbb{R}^{n_{m-1}} \times \mathbb{R}^l : u \in U(z,\theta)\} \epsilon B(Z) \times B(\mathbb{R}^{n_{m-1}}) \times B(\mathbb{R}^l)$, with $B(Z)$ (resp $B(\mathbb{R}^{n_{m-1}})$, $B(\mathbb{R}^l)$) being the Borel $\sigma$-field of $Z$ (resp of $\mathbb{R}^{n_{m-1}}, \mathbb{R}^l$),
(2) for every $z \in Z, \theta \to U(z,\theta)$ is an upper semicontinuous (u.s.c.) multifunction; i.e. for every $V \subseteq \mathbb{R}^l$ open $U(z,V)^+ = \{\theta \in \mathbb{R}^{n_{m-1}} : U(z,\theta) \subseteq V\}$ is open in $\mathbb{R}^{n_{m-1}}$ (see Delahaye–Denel [4]),
(3) $|U(z,\theta)| = \sup\{\|v\| : v \in U(z,\theta)\} \le a_2(z) + b_2\|\theta\|$ a.e. with $a_2(\cdot) \in L_+^p(Z), b_2 > 0$.

$\underline{H(L)}$ : $L : Z \times \mathbb{R}^{n_{m-1}} \times \mathbb{R}^l \to \overline{\mathbb{R}} = \mathbb{R}u\{+\infty\}$ is integrand s.t.

(1) $(z,\theta,u) \to L(z,\theta,u)$ is measurable,
(2) for every $z \in Z, (\theta,u) \to L(z,\theta,u)$ is l.s.c.,
(3) $\psi(z) - r(\|\theta\| + \|u\|) \le L(z,\theta,u)$ a.e., with $\psi(\cdot) \in L^1(Z)$, $r > 0$.

It is well known, even from the theory of finite dimensional systems, that an optimal control need not exist, unless some appropriate convexity hypotheses are present (recall Cesari' property $Q$; see Cesari [3]). So we make the following hypothesis:

$\underline{H_c}$ : $Q(z,\theta) = \{(\nu,\mu) \in \mathbb{R} \times \mathbb{R} : \nu + f(z,\theta,u) = 0, L(z,\theta,u) \le \mu, u \in U(z,\theta)\}$ is convex for every $(z,\theta) \in Z \times \mathbb{R}^{n_{m-1}}$.

Note that this hypothesis is automatically satisfied, if the control $u$ enters linearly in the dynamics of the system, the control constraint multifunction $U(\cdot,\cdot)$ is convex valued and for every $(z,\theta) \in Z \times \mathbb{R}^{n_{m-1}}$ the cost integrand $L(z,\theta,\cdot)$ is convex.

Let $X$ be a reflexive Banach and let $A : X \to X^*$ be an operator. We say that $A(\cdot)$ has the property $(M)$, if $x_n \xrightarrow{w} x$ in $X$, $Ax_n \xrightarrow{w} x^*$ in $X^*$ and $li\overline{m}\langle Ax_n, x_n\rangle_{X^*,X}$, imply that $Ax = x^*$ (see for example Zeidler [9, Definition 27.1, p. 538]).

Finally let us recall two important results from the measure theory which we will need in the proof of our main theorem. We say that a topological space $Y$ belongs in the class $\sigma MK$, if it is representable in the form $Y = \bigcup_{n=1}^{\infty} K_n$, where for each $n \geq 1$, $K_n$ is metrizable compact. The class $\sigma MK$ includes in particular all separable, metrizable locally compact spaces and the duals of separable Banach spaces endowed with the weak* topology. The following theorem is known in the literature as "Novikov's projection theorem" and can be found in Levin [5, Lemma 2, p. 435]. Recall that a Polish space is a complete, separable metrizable space.

**Theorem 2.1.** (Novikov). *If $T$ is a Borel set in a Polish space, $Y \in \sigma MK, \Gamma \subseteq T \times Y$ is Borel and $\Gamma(t) = \{y \in Y : (t, y) \in \Gamma\}$ is closed for every $t \in T$, then the projection $proj_T(\Gamma)$ of the set $\Gamma$ onto $T$ is a Borel set.*

The second measure theoretic result that we will need, is "Aumann's selection theorem". It was first proved by Aumann for Polish spaces and in its present form can be found in Saint–Beuve [8, Theorem 3, p. 119]. Recall that a Souslin space is the continuous image of a Polish space. So a separable Banach space endowed with the weak topology is a Souslin space. Hence a Souslin space is always separable, but need not be metrizable.

**Theorem 2.2** (Aumann). *If $(\Omega, \Sigma)$ is a complete measurable space, $Y$ is a Souslin space and $\Gamma : \Omega \to 2^Y \setminus \{\emptyset\}$ is a multifunction s.t. $Gr\Gamma = \{(w, y) \in \Omega \times Y : y \in \Gamma(w)\} \epsilon \Sigma \times B(Y)$, then there exists a map $\gamma : \Omega \to Y, (\Sigma, B(Y))$-measurable s.t. for all $w \in \Omega, \gamma(w) \in \Gamma(w)$.*

## 3. Main theorem.

In this section, using the hypotheses of Section 2, we state and prove an existence theorem for the Lagrange optimal control problem $(*)$.

**Theorem 3.1.** *If hypotheses $H(A), H(f), H(U), H(L)$ and $H_c$ hold, then $(*)$ admits an optimal admissible "state-control" pair, i.e. there exists a pair $(x, u) \in W_0^{m,p}(Z) \times L^p(Z, \mathbb{R}^l)$ which satisfies the constraints of $(*)$ and $J(x, u) = m$.*

Proof: Let $M : Z \times \mathbb{R}^{n_m} \times \mathbb{R} \to 2^{\mathbb{R}^l}$ be the multifunction defined by

$$M(z, \theta, v) = \{u \in U(z, \theta) : v + f(z, \theta, u) = 0\}.$$

Let $\delta_{M(z,\theta,v)}(\cdot)$ be the indicator function of the set $M(z, \theta, v)$ (i.e. $\delta_{M(z,\theta,v)}(u) = 0$, if $u \in M(z, \theta, v), +\infty$ otherwise) and set $p(z, \theta, v) = inf\{L(z, \theta, u) + \delta_{M(z,\theta,v)}(u) : u \in \mathbb{R}^l\}$. So $p(z, \theta, v)$ measures the minimum cost necessary to produce the "velocity" $v$, given the space-state pair $(z, \theta)$ and using only admissible controls. We will establish some properties of $p(z, \theta, v)$ that we will need in the sequel.

**Claim #1.** $p(\cdot, \cdot, \cdot)$ is Borel measurable.

We need to show that given $\lambda \in \mathbb{R}$, the set

$$H(\lambda) = \{(z, \theta, v) \in Z \times \mathbb{R}^{n_{m-1}} \times \mathbb{R} : p(z, \theta, v) \leq \lambda\}$$

is a Borel set in $Z \times \mathbb{R}^{n_{m-1}} \times \mathbb{R}$. Let $\hat{U}(z,\theta) = U(z,\theta) \times \mathbb{R}$. Because of hypotheses $H(U)\,(1), \hat{U}(\cdot,\cdot)$ is graph measurable. Also let $q : Z \times \mathbb{R}^{n_{m-1}} \times \mathbb{R} \times \mathbb{R}^l \to Z \times \mathbb{R}^{n_{m-1}} \times \mathbb{R}^l \times \mathbb{R}$ be defined by $q(z,\theta,v,u) = (z,\theta,u,v)$. Clearly $q(\cdot,\cdot,\cdot,\cdot)$ is Borel measurable and so $V = q^{-1}(Gr\hat{U}) \in B(Z) \times B(\mathbb{R}^{n_{m-1}}) \times B(\mathbb{R}) \times B(\mathbb{R}^l)$. Also note that since by hypotheses $H(f)$ (1) and (2), $f(z,\theta,u)$ is measurable in $z$, continuous in $(\theta,u)$ (i.e. a Carathéodory function), it is jointly measurable and so we have

$$GrM = \{(z,\theta,v,u) \in V; v + f(z,\theta,u)\} \epsilon B(Z) \times B(\mathbb{R}^{n_{m-1}}) \times B(\mathbb{R}) \times B(\mathbb{R}^l).$$

Hence, $(z,\theta,v,u) \to \delta_{M(z,\theta,v)}(u)$ is a Borel measurable, $\overline{\mathbb{R}}$-valued integrand. This observation combined with hypothesis $H(L)\,(1)$ implies that $(z,\theta,v,u) \to L(z,\theta,u) + \delta_{M(z,\theta,v)}(u)$ is a Borel measurable, $\overline{\mathbb{R}}$-valued integrand. Now, note that since by hypothesis $L(z,\cdot,\cdot)$ is l.s.c. and since $M(z,\theta,v)$ is compact, if it is also nonempty, the infimum involved in the definition of $p(z,\theta,v)$ is attained. So we have

$$\begin{aligned} H(\lambda) =& \{(z,\theta,v) \in Z \times \mathbb{R}^{n_{m-1}} \times \mathbb{R} : p(z,\theta,v) \le \lambda\} = \\ =& proj_{Z\times\mathbb{R}^{n_{m-1}}\times\mathbb{R}}\{(z,\theta,v,u) : L(z,\theta,u) \le \lambda, u \in M(z,\theta,v)\} = \\ =& proj_{Z\times\mathbb{R}^{n_{m-1}}\times\mathbb{R}}\{(z,\theta,v,u) : L(z,\theta,u) + \delta_{M(z,\theta,v)}(u) \le \lambda\}. \end{aligned}$$

From what we have proved above, we have $\hat{H}(\lambda) = \{(z,\theta,v,u) : L(z,\theta,u) + \delta_{M(z,\theta,v)}(u) \le \lambda\} \in B(Z) \times B(\mathbb{R}^{n_{m-1}}) \times B(\mathbb{R}) \times B(\mathbb{R}^l)$. Since $Z \times \mathbb{R}^{n_{m-1}} \times \mathbb{R}$ is a Borel set in the Polish space $\mathbb{R}^n \times \mathbb{R}^{n_{m-1}} \times \mathbb{R}$ and for every $(z,\theta,v) \in Z \times \mathbb{R}^{n_{m-1}} \times \mathbb{R}, \hat{H}(\lambda)(z,\theta,v) = \{u \in \mathbb{R}^l : (z,\theta,v,u) \in \hat{H}(\lambda)\}$ is clearly closed (since $L(z,\theta,\cdot)$ is l.s.c. and $M(z,\theta,v)$ is closed), invoking Novikov's theorem (see Theorem 2.1), we deduce that

$$\begin{gathered} H(\lambda) = proj_{Z\times\mathbb{R}^{n_{m-1}}\times\mathbb{R}}\hat{H}(\lambda) \in B(Z) \times B(\mathbb{R}^{n_{m-1}}) \times B(\mathbb{R}) = B(Z \times \mathbb{R}^{n_{m-1}} \times \mathbb{R}) \\ \Rightarrow p(\cdot,\cdot,\cdot) \text{ is Borel measurable as claimed.} \end{gathered}$$

**Claim #2.** For every $z \in Z, p(z,\cdot,\cdot)$ is l.s.c.

We need to show that given $\lambda \in \mathbb{R}$, the level set

$$R(\lambda) = \{(\theta,v) \in \mathbb{R}^{n_{m-1}} \times \mathbb{R} : p(z,\theta,v) \le \lambda\}$$

is closed. To this end let $\{(\theta_n,v_n)\}_{n\ge1} \subseteq R(\lambda)$ and assume that $(\theta_n,v_n) \to (\theta,v)$ in $\mathbb{R}^{n_{m-1}} \times \mathbb{R}$. Since $p(z,\theta_n,v_n) \le \lambda$, we have that $M(z,\theta_n,v_n) \ne \emptyset$ for every $n \ge 1$. So by Weierstrass' theorem, we can find $u_n \in U(z,\theta_n) n \ge 1$ s.t. $p(z,\theta_n,v_n) = L(z,\theta_n,u_n)$. Using hypothesis $H(U)\,3$ and by passing to a subsequence if necessary, we may assume that $u_n \to u$ in $\mathbb{R}^l$. Then because of hypothesis $H(U)\,(2)$, we have $u \in \overline{\lim} U(z,\theta_{n'}) = \{u' \in \mathbb{R}^l : \underline{\lim}_{n\to\infty} d(u', U(z,\theta_n)) = 0\} \subseteq U(z,\theta)$ (see Delahaye–Denel [4]). In addition because of hypothesis $H(f)\,(2)$, we have $v_n + f(z,\theta_n,u_n) \to v + f(z,\theta,u) \Rightarrow v + f(z,\theta,u) = 0$. Therefore $u \in M(z,\theta,v)$. Furthermore, using hypothesis $H(L)\,(2)$ we have

$$\begin{aligned} & p(z,\theta,v) \le L(z,\theta,u) \le \underline{\lim} L(z,\theta_n,u_n) = \underline{\lim} p(z,\theta_n,v_n) \le \lambda \\ \Rightarrow & (\theta,v) \in R(\lambda) \\ \Rightarrow & R(\lambda) \text{ is closed and so } p(z,\cdot,\cdot) \text{ is l.s.c. as claimed.} \end{aligned}$$

**Claim #3.** For every $(z,\theta)\in Z\times\mathbb{R}^{n_{m-1}}, p(z,\theta,\cdot)$ is convex.

$$\begin{aligned}\text{Note that }\ epip(z,\theta,\cdot) =&\{(v,\mu)\in\mathbb{R}^l\times\mathbb{R}: p(z,\theta,v)\le\mu\} =\\ =&\{(v,\mu)\in\mathbb{R}^l\times\mathbb{R}: L(z,\theta,u)\le\mu, u\in U(z,\theta), v+\\ &+f(z,\theta,u)=0\} =\\ =&Q(z,\theta).\end{aligned}$$

But the latter is by hypothesis $H_c$ convex. So the epigraph of $p(z,\theta,\cdot)$ is convex, hence the function is convex as claimed.

Now let $\{(x_n,u_n)\}_{n\ge1}\subseteq W_0^{m,p}(Z)\times L^p(Z,\mathbb{R}^l)$ be a minimizing sequence for our problem $(*)$. Let $\hat{A}: W_0^{m,p}(Z)\to W^{-m,q}(Z)=W_0^{m,p}(Z)^*$be the nonlinear operator corresponding to the elliptic partial differential operator via Dirichlet form

$$a(x,y)=\Sigma_{|\alpha|\le m}\int_Z A_\alpha(z,\eta(x(z)))D^\alpha y(z)\,dz,\ x,y\in W_0^{m,p}(Z),$$

i.e. $\alpha(x,y)=\langle\hat{A}x,y\rangle$, where by $\langle\cdot,\cdot\rangle$ we denote the duality brackets for the pair $(W_0^{m,p}(Z),W^{-m,q}(Z))$. Also let $\hat{F}: L^p(Z,\mathbb{R}^{n_{m-1}})\times L^p(Z,\mathbb{R}^l)\to L^q(Z)$ be the Nemitsky operator corresponding to the function $f(\cdot,\cdot,\cdot)\cdot$, i.e. $\hat{F}(x,u)(z)=f(z,\theta(x(z)),u(z))$. Using hypothesis $H(A)\,(5)$ and $H(f)\,(4)$ and recalling (see for example Zeidler [9, p. 1033]) that $(\Sigma_{|\gamma|=m}\ \|D^\alpha x\|^p)^{\frac{1}{p}}$ is an equivalent norm on $W_0^{m,p}(Z)$, we get for every $n\ge1$

$$0=\langle\hat{A}(x_n),x_n\rangle+\langle\hat{F}(x_n,u_n),x_n\rangle\ge\hat{C}\|x_n\|^p_{W_0^{m,p}(Z)}-\|k\|_{L^d(Z)}-\hat{c}_1,\ \text{with } \hat{c},\hat{c}_1>0$$
$$\Rightarrow\|x_n\|^p_{W_0^{m,p}(Z)}\le\frac{1}{\hat{c}}\left[\|k\|_{L^d(Z)}+\hat{c}_1\right].$$

Therefore $\{x_n\}_{n\ge1}$ is bounded in $W_0^{m,p}(Z)$. Since the latter is a reflexive separable Banach space, by passing to a subsequence if necessary, we may assume that $x_n\xrightarrow{w}x$ in $W_0^{m,p}(Z)$. But from the Sobolev embedding theorem, we know that $W_0^{m,p}(Z)\hookrightarrow W_0^{m-1,p}(Z)$ compactly. Hence we get that $x_n\xrightarrow{s}x$ in $W_0^{m-1,p}(Z)$. Also from hypotheses $H(f)\,(3)$ and $H(U)\,(3)$ we see that there exists $M_1>0$ s.t. $\|\hat{F}(x_n,u_n)\|_{L^q(Z)}\le M_1$ for all $n\ge1$. Since $\hat{A}(x_n)+\hat{F}(x_n,u_n)=0$, $n\ge1$, we see that $\{\hat{A}(x_n)\}_{n\ge1}$ is bounded in $L^q(Z)$. Hence by passing to a subsequence if necessary, we may assume that $\hat{A}(x_n)\xrightarrow{w}y$ in $L^q(Z)$. Furthermore, note that since $\hat{A}(x_n)\in L^q(Z)$, we have $\langle\hat{A}(x_n),x_n\rangle=(\hat{A}(x_n),x_n)_{L^q(Z),L^p(Z)}$ and also $\hat{A}(x_n)\xrightarrow{w}y$ in $L^q(Z)$, while $x_n\xrightarrow{s}x$ in $L^p(Z)$ (because $W_0^{m,p}(Z)\hookrightarrow L^p(Z)$ compactly by Sobolev's embedding theorem). Hence $\lim_{n\to\infty}\langle\hat{A}(x_n),x_n\rangle=\langle y,x\rangle$. But from Theorem 1 of Browder [2], we know that $\hat{A}$ is a pseudomonotone operator, in particular, it has property $(M)$ (see Zeidler [9, Proposition 27.7, p. 588]). Hence $y=\hat{A}(x)$; i.e. $\hat{A}(x_n)\xrightarrow{w}\hat{A}(x)$ in $L^q(Z)$. Since $x_n\xrightarrow{s}x$ in $W_0^{m-1,p}(Z)$, we may assume that $\theta(x_n)(z) \to \theta(x)(z)$ a.e.. So because of the claims #1, #2 and #3 and using Theorem 10.8 i, p. 352 in Cesari [3], we have

$$\int_Z p(z, \theta(x(z)), \hat{A}(x)(z))\, dz \leq$$

$$\leq \underline{\lim} \int_Z p(z, \theta(x_n(z)), \hat{A}(x_n)(z))\, dz \leq$$

$$\leq \underline{\lim} \int_Z L(z, \theta(x_n(z)), u_n(z))\, dz = m < \infty\,.$$

Therefore $p(z, \theta(x(z)), u(z))$ is finite for almost all $z \in Z$. By redefining the function on a Lebesgue null set, we can say that $p(z, \theta(x(z)), u(z))$ is finite for every $z \in Z$. Hence for every $z \in Z, M(z, \theta(x(z)), \hat{A}(x)(z)) \neq \emptyset$. Let $E(z) = \{u \in M(z, \theta(x(z)), \hat{A}(x)(z)) : L(z, \theta(x(z)), 0, u) = p(z, \theta(x(z)), \hat{A}(x)(z))\}$. Recalling that $M(\cdot, \cdot, \cdot)$ is graph measurable and since $L(\cdot, \cdot, \cdot)$ and $p(\cdot, \cdot, \cdot)$ are measurable (see hypothesis $H(L)$ (1) and the claim #1, respectively), we get that $GrE \in \mathcal{L}(Z) \times B(\mathbb{R}^l)$, with $\mathcal{L}(Z)$ being the Lebesgue completion of the Borel $\sigma$-field $B(Z)$. Apply Aumann's selection theorem (see Theorem 2.2), to get $u : Z \to \mathbb{R}^l$ Lebesgue measurable s.t. $u(z) \in E(z)$ for all $z \in Z$. Then we have

$$\int_Z p(z, \theta(x(z)), \hat{A}(x)(z))\, dz = \int_Z L(z, \theta(x(z)), u(z))\, dz \leq m$$

$$\Sigma_{|\alpha| \leq m} (-1)^{|\alpha|} D^\alpha (A_\alpha(z, \eta(x(z)))) + f(z, \theta(x(z)), u(z)) = 0 \text{ a.e. on } Z$$

$$D^\beta x(z) \mid_\Gamma = 0 \ |\beta| \leq m-1, u(z) \in U(z, \theta(x(z))) \text{ a.e.}$$

$$u(\cdot) \text{ is measurable.}$$

Thus $(x, u) \in W_0^{m,p}(Z) \times L^p(Z, \mathbb{R}^l)$ is an admissible state-control pair and so we have $J(x, u) = m$. Therefore $(x, u)$ is the desired solution of $(*)$. $\square$

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