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On the covering dimension of the fixed point set of certain multifunctions

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Abstract. We study the covering dimension of the fixed point set of lower semicontinuous multifunctions of which many values can be non-closed or non-convex. An application to variational inequalities is presented.

Keywords: multifunction, fixed point, covering dimension, variational inequality

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Introduction.

In [4] (see Theorem 2), B. Ricceri has established a fixed point theorem for lower semicontinuous multifunctions that are allowed to have many non-closed or non-convex values. We recall here its statement.

Theorem A. *Let $(U, \|\cdot\|_U)$ be a Banach space and $X \subseteq U$ a non-empty set. Let τ be a topology on X, weaker than the norm topology, such that (X,τ) is compact and Hausdorff. Let C be a countable subset of X and Z another subset of X with* $\dim_{(X,\tau)}(Z) \leq 0$. *Let F be a non-empty valued $(\tau, \|\cdot\|_U)$-lower semicontinuous multifunction from X into U such that $F(x)$ is $\|\cdot\|_U$-closed for every $x \in X \setminus C$, $(\overline{F(x)})_{\|\cdot\|_U}$ is convex for every $x \in X \setminus Z$ and $(\overline{\mathrm{conv}(F(x))})_{\|\cdot\|_U} \subseteq X$. Then*

- (α) $\mathrm{Fix}(F) \neq \phi$.
- (β) *If for every $x \in \mathrm{Fix}(F)$, x is a $\|\cdot\|_U$-accumulation point of $F(x)$, then* $\mathrm{Fix}(F)$ *is uncountable.*

In the present note, first, we want to establish a result (see Theorem 2.1 below) which improves (β) of Theorem A, showing that, under some mild additional assumptions, the covering dimension of $\mathrm{Fix}(F)$ is greater or equal to 1. Afterwards, we will apply this result to the solution set of some variational inequalities (see Theorem 2.2 below).

For another result concerning the covering dimension of the set of fixed points of a multifunction, see [5].

1. Notation.

Let X, Y be two non-empty sets. A multifunction F from X into Y (briefly, $F : X \to 2^Y$) is a function from X into the family of all subsets of Y. If X, Y are two topological spaces, F is said to be lower semicontinuous (respectively, upper semicontinuous) in X, if the set $F^-(\Omega) = \{x \in X : F(x) \cap \Omega \neq \phi\}$ is open (closed) for every open (closed) subset Ω of Y. If $X = Y$, a point $x \in X$ is said to be a fixed point of F, if $x \in F(x)$. We denote by $\mathrm{Fix}(F)$ the set of all fixed points of F.

Now, let (M, d) be a metric space, $x_0 \in M, r > 0, X, Y$ two non-empty subsets of M. We put

$$\begin{aligned} B_d(x_0, r)0 &= \{x \in M : d(x_0, x) < r\}; \\ \overline{B}_d(x_0, r)0 &= \{x \in M : d(x_0, x) \leq r\}; \\ d(x_0, X) &= \inf_{x \in X} d(x_0, x); \\ d^*(X, Y) &= \sup_{x \in X} d(x, Y); \\ d_H(X < Y) &= \max(d^*(X, Y), d^*(Y, X)) \quad \text{(Hausdorff metric)}; \\ \mathrm{diam}_d(X) &= \sup_{x,y \in x} d(x, y). \end{aligned}$$

Moreover, given a normal topological space (X, τ), we denote by $\dim_\tau(X)$ the covering dimension of X (see [2, Definition 1.6.7]).

While, if S is a subset of X, $\dim_{(X,\tau)}(S) \leq 0$ means that $\dim_\tau(T) \leq 0$ for every set $T \subseteq S$ which is closed in X.

2. Results.

Our main result is the following

Theorem 2.1. *Let U, X, τ, C, Z, F be as in Theorem A with, moreover, (X, τ) metrizable and $\dim_\tau(Z) \leq 0$. Suppose also that $\mathrm{Fix}(F)$ is τ-closed and that, for every $x \in \mathrm{Fix}(F)$, one has $F(x) \neq \{x\}$.*

Then, $\dim_\tau(\mathrm{Fix}(F)) \geq 1$.

Before proving this theorem, we need some preliminary results.

Lemma 2.1. *Let X be a topological space, (Y, d) a metric space, $F : X \to 2^Y$ a bounded-valued lower semicontinuous multifunction. For every $x \in X$, put $\alpha(x) = \mathrm{diam}_d(\mathrm{Fix}(F))$.*

Then, the real function $x \to \alpha(x)$ is lower semicontinuous in X.

Proof: Let us show the lower semicontinuity at a point $\bar{x} \in X$. Having chosen $\varepsilon > 0$, let us take $\bar{y}, \bar{z} \in F(x)$ such that $d(\bar{y}, \bar{z}) > \alpha(\bar{x}) - \varepsilon$.

Since the function $(\xi, \zeta) \to d(\xi, \zeta)$ is continuous in $Y \times X$, there exists $\varrho > 0$ such that if $y \in B_d(\bar{y}, \varrho)$, then $d(y, z) > \alpha(\bar{x}) - \varepsilon$. Since F is lower semicontinuous at $\bar{x}$, there exist two neighbourhoods of $\bar{x}$, say U_1 and U_2, such that, if $x \in U_1$ then $F(x) \cap B_d(\bar{y}, \varrho) \neq \phi$ and, if $x \in U_2$ then $F(x) \cap B_d(\bar{z}, \varrho) \neq \phi$. Of course, if $x \in U_1 \cap U_2$, then $\alpha(x) > \alpha(\bar{x}) - \varepsilon$, which proves our thesis. □

Lemma 2.2. *Let (X, d_1) be a compact metric space and let d_2 be another metric on X such that the d_2-topology is stronger than the d_1-topology. Moreover, let $F : X \to 2^X$ be a (d_1, d_2)-lower semicontinuous multifunction and $\varepsilon > 0$. For every $x \in X$, put $\Phi(x) = F(x) \setminus \overline{B}_{d_1}(x, \varepsilon)$.*

Then, the multifunction Φ is (d_1, d_2)-lower semicontinuous.

Proof: Let Ω be a d_2-open subset of X, $x_0 \in \Phi^-(\Omega), y_0 \in \Phi(x_0) \cap \Omega$. Choose $\varrho \in]\varepsilon, d_1(y_0, x_0)[$. We claim that

$$\text{(1)} \qquad \text{there exists } \delta > 0 \text{ such that } \overline{B}_{d_1}(x_0, \varrho) \cap \overline{B}_{d_2}(y_0, \delta) = \phi.$$

In fact, if (1) is not true, there exists a sequence $\{z_n\}_{n\in\mathbb{N}}$ in X, with $\lim_{n\to\infty} d_2(z_n, y_0) = 0$, such that for every $n \in \mathbb{N}$, we have $d_1(z_n, x_0) \leq \varrho$. Since (X, d_1) is compact and the d_2-topology is stronger than the d_1-topology, we can find a subsequence of $\{z_n\}_{n\in\mathbb{N}}$ d_1-converging to y_0. So, $d_1(x_0, y_0) \leq \varrho$, that is absurd. Thus, let $\delta > 0$ be such that (1) holds. Of course, we can assume that $\overline{B}_{d_2}(y_0, \delta) \subseteq \Omega$. Now, let V be a d_1-neighbourhood of x_0 such that for every $x \in V$, we have $F(x) \cap B_{d_2}(y_0, \delta) \neq \phi$. Let $x \in V \cap B_{d_1}(x_0, \varrho - \varepsilon)$, and choose $y^* \in F(x) \cap B_{d_2}(y_0, \delta)$. Observe that $y^* \notin \overline{B}_{d_1}(x, \varepsilon)$. Indeed, otherwise, we would have $d_1(y^*, x_0) \leq d_1(y^*, x) + d_1(x, x_0) < \varrho$, and so $y^* \in \overline{B}_{d_1}(x_0, \varrho)$, against (1). Hence, $y^* \in \Phi(x) \cap \Omega$, that proves our thesis. $\square$

We also recall the two following simple facts.

Lemma 2.3. *Let X, Y be two topological spaces and K a closed subset of X. Let $F : X \to 2^Y, \Phi : K \to 2^Y$ be two lower semicontinuous multifunctions such that, for every $x \in K$, one has $\Phi(x) \subseteq F(x)$. Let G be the multifunction from X into Y defined by putting*

$$G(x) = \begin{cases} F(x) & \text{if } x \in X \setminus K, \\ \Phi(x) & \text{if } x \in K. \end{cases}$$

Then, the multifunction G is lower semicontinuous in X.

Lemma 2.4. *Let X, Y be two topological spaces. Given a multifunction $F : X \to 2^Y$, define a multifunction $\overline{F}$ in X by putting, for every $x \in X$, $\overline{F}(x) = \overline{F(x)}$. Then, the multifunction $\overline{F}$ is lower semicontinuous in X if and only if the multifunction F is lower semicontinuous in X.*

Proof of Theorem 2.1: First, observe that by Theorem A, $\mathrm{Fix}(F)$ is non-empty. Let d be a metric on X inducing the topology τ. Arguing by contradiction, suppose that $\dim_\tau(\mathrm{Fix}(F)) = 0$. Let $\overline{\alpha} = \inf_{x\in\mathrm{Fix}(F)} \mathrm{diam}(F(x))$. By our assumptions and by Lemma 2.1, we have $\overline{\alpha} > 0$. Let $\tilde{F}$ be the multifunction from X into X defined by putting

$$\tilde{F}(x) = \begin{cases} F(x) & \text{if } x \in X \setminus \mathrm{Fix}(F), \\ \overline{\left(F(x) \setminus \overline{B}_d\left(x, \frac{\overline{\alpha}}{3}\right)\right)}_{\|\cdot\|_U} & \text{if } x \in \mathrm{Fix}(F). \end{cases}$$

It is easy to see that $\tilde{F}(x) \neq \phi$ for every $x \in X$. Observe also that if $x \in X \setminus C$, $\tilde{F}(x)$ is $\|\cdot\|_U$-closed. Now, let us check that $\tilde{F}$ is $(\tau, \|\cdot\|_U)$-lower semicontinuous in X. That is, by Lemma 2.4, we have to prove that $(\overline{\tilde{F}})_{\|\cdot\|_U}$ is so. To this end, observe that, by Lemma 2.2, the multifunction $x \to \overline{F(x) \setminus \overline{B}_d\left(x, \frac{\alpha}{3}\right)}$ is $(\tau, \|\centerdot\|_U)$-lower semicontinuous in X and hence, so is $x \to \overline{\left(F(x) \setminus B_d\left(x, \frac{\alpha}{3}\right)\right)}_{\|\centerdot\|_U}$. Then, since the multifunction $(\overline{F})_{\|\centerdot\|_U}$ is $(\tau, \|\centerdot\|_U)$-lower semicontinuous in X and $(\overline{\tilde{F}(x)})_{\|\centerdot\|_U} \subseteq (\overline{F(x)})_{\|\centerdot\|_U}$, the lower semicontinuity of $(\overline{\tilde{F}})_{\|\centerdot\|_U}$ follows directly by Lemma 2.3.

Now observe that, by our assumptions and Corollary 1.3.4 of [2] we have $\dim_\tau(Z \cup \mathrm{Fix}(F)) = 0$. Of course, for every $x \in X \setminus (Z \cup \mathrm{Fix}(F))$, $(\overline{\tilde{F}(x)})_{\|\centerdot\|_U}$ is convex. Moreover, it is clear that $(\overline{\mathrm{conv}(\tilde{F}(x))})_{\|\centerdot\|_U} \subseteq (\overline{\mathrm{conv}(F(x))})_{\|\centerdot\|_U} \subseteq X$. Then, it is possible to apply Theorem A to $\tilde{F}$. By this result, we have $\mathrm{Fix}(\tilde{F}) \neq \phi$. Let $x^* \in \mathrm{Fix}(\tilde{F})$. Of course, from the definition of $\tilde{F}$, it follows that $x^* \in \mathrm{Fix}(F)$. Then, there exists a sequence $\{y_n\}_{n\in\mathbb{N}}$ in X such that $\lim_{n\to\infty} \|y_n - x^*\|_U = 0$ and, for every $n \in \mathbb{N}$, $y_n \in F(x^*) \setminus B_d\left(x^*, \frac{\alpha}{3}\right)$. By this latter relation, it follows that the sequence $\{y_n\}_{n\in\mathbb{N}}$ does not converge to x^* with respect to the topology τ, against the fact that τ is weaker than the $\|\centerdot\|_U$-topology. This contradiction concludes the proof. □

If $\mathrm{Fix}(F)$ is not closed, in general, Theorem 2.1 does not hold. We show this fact by means of the following simple

Example 2.1. Take $\mathbb{R}$ with the usual topology. Define in $[0,1]$ the following multifunction

$$F(x) = \begin{cases} [0,1] & \text{if } \ x \in [0,1] \setminus \mathbb{Q}, \\ [0,1] \setminus \{x\} & \text{if } \ x \in [0,1] \cap \mathbb{Q}. \end{cases}$$

Of course, F satisfies the hypotheses of Theorem 2.1, except that requiring $\mathrm{Fix}(F)$ be closed. In fact, $\mathrm{Fix}(F)$ (which is $[0,1] \setminus \mathbb{Q}$) has the covering dimension zero.

As an application of Theorem 2.1, we will establish a theorem on the covering dimension of the solution set of a variational inequality. In what follows, we adopt, as usual, the convention $\inf(\phi) = +\infty$.

Let $(U, \|\centerdot\|_U)$ be a Banach space and X a non-empty, weakly compact and convex subset of U. In the sequel, we denote by τ the relativization to X of the weak topology on U. We also assume that (X, τ) is metrizable. Given an operator $A : X \to U^*$ (U^* being the dual space of U), consider the problem:

$$\text{(V.I.)} \qquad \text{find } x \in X \text{ such that } \ \langle A(x), x - y\rangle \le 0 \quad \text{ for every } \ y \in X.$$

For every $x, y \in X, \varepsilon > 0$, put

$$\begin{aligned} F(x) =& \{z \in X \langle A(x), z\rangle = \min\langle A(x), y\rangle\} \ \text{ and} \\ \Omega(\varepsilon, x) =& \{y \in X : d(y, F(x)) < \varepsilon\}, \end{aligned}$$

where d is the metric induced by the $\|\centerdot\|_U$-norm.

Finally, put

$$S = \{x \in X : x \text{ is a solution of (V.I.)}\}.$$

We establish the following

Theorem 2.2. *Suppose that:*

(i) *the real function $(x, y) \to \langle A(x), y\rangle$ is weakly continuous in $X \times X$;*
(ii) *for every $\varepsilon > 0$, one has $\inf_{x\in X}(\inf_{y\in X\setminus\Omega(\varepsilon,x)}\langle A(x), y\rangle - \min_{y\in X}\langle A(x), y\rangle) > 0$;*
(iii) *if $x \in X$ is such that $\langle A(x), x\rangle = \min_{y\in X}\langle A(x), y\rangle$, then there exists $z \in X \setminus \{x\}$ such that $\langle A(x), z\rangle = \min_{y\in X}\langle A(x), y\rangle$.*

Under such hypotheses, one has $\dim_\tau(S) \geq 1$.

Proof: Observe that $S = \mathrm{Fix}(F)$. Plainly, the multifunction F is non-empty closed convex-valued. Moreover, (i) assures that $\mathrm{Fix}(F)$ is τ-closed. Let us check now that F is $(\tau, \|\cdot\|_U)$-lower semicontinuous. To this end, for every $x \in X$ and $n \in \mathbb{N}$, put:

$$F_n(x) = \left\{z \in X : \langle A(x), z\rangle \; < \; \min_{y\in X}\langle A(x), y\rangle + \frac{1}{n}\right\}$$

and prove that

$$\lim_{n\to\infty} \sup_{x\in X} d_H(F_n(x), F(x)) = 0; \tag{2}$$

(3) for every $n \in \mathbb{N}$, the multifunction F_n is $(\tau, \|\cdot\|_U)$-lower semicontinuous.

Observe that (2) and (3) prove our claim, because the uniform limit, with respect to the Hausdorff metric, of a sequence of lower semicontinuous multifunctions, is lower semicontinuous (see Proposition 1.1 of [3]). First, observe that $F(x) = \bigcap_{n\in\mathbb{N}} F_n(x)$. So, $d^*(F(x), F_n(x)) = 0$ for every $x \in X$ and $n \in \mathbb{N}$. Now, fix $\varepsilon > 0$. By (ii), there exists $\alpha_\varepsilon > 0$ such that

$$\inf_{y\in X\setminus\Omega(\varepsilon,x)} \langle A(x), y\rangle \; > \; \min_{y\in X} \langle A(x), y\rangle + \alpha_\varepsilon \quad \text{for every} \quad x \in X.$$

Then, having chosen $\nu \in \mathbb{N}$ such that $\frac{1}{\nu} < \alpha_\varepsilon$, let $n > \nu, x \in X, y \in F_n(x)$ be fixed. We have:

$$\begin{aligned}\langle A(x), y\rangle \; &< \; \min_{z\in X}\langle A(x), z\rangle + \frac{1}{n} \; < \; \min_{z\in X}\langle A(x), z\rangle + \alpha_\varepsilon < \\ &< \; \inf_{z\in X\setminus(\varepsilon,x)} \langle A(x), z\rangle\,.\end{aligned}$$

Consequently, $d(y, F(x)) < \varepsilon$, that proves (2). Now, let us fix $n \in \mathbb{N}$ and $y \in X$. By (i) and Theorem 1, p. 67, of [1], it follows that the function $x \to \langle A(x), y\rangle - \min_{z\in X}\langle A(x), z\rangle$ is upper semicontinuous. Hence, $F_n^-(y)$ is open, that proves (3). Then, our conclusion follows from Theorem 2.1. $\square$

Example 2.2. By means of the present example, we show that the conclusion of Theorem 2.2 is not true, in general, if the condition (ii) is not satisfied. Take as U the Euclidean space $\mathbb{R}$, as X the compact interval $[0,1]$, as A the continuous operator defined by $A(x) = x$ for every $x \in [0,1]$. So, the problem (V.I.) in this case is:

$$\text{(4)} \qquad \text{find } x \in [0,1] \text{ such that } x(x-y) \leq 0 \text{ for every } y \in [0,1].$$

Evidently, $0 \in S$, moreover, observe that if $\bar{x} \in [0,1]$ is a solution, since $\bar{x} \geq 0$, we have by (4) $\bar{x} - y \leq 0$ for every $y \in [0,1]$, so, $\bar{x} = 0$. Then $S = \{0\}$, and $\dim(S) = 0$. Observe that $F(0) = [0,1]$ and $F(x) = \{0\}$ for every $x \in]0,1]$, so, (iii) is satisfied. Finally, observe that, if $\varepsilon > 0$ is fixed, we have $\Omega(\varepsilon, 0) = [0,1], \Omega(\varepsilon, x) = [0, \varepsilon[$ if $x \in [0,1]$. So, we have

$$\inf_{y \in X \setminus \Omega(\varepsilon,x)} \langle A(x), y\rangle - \min_{y \in X} \langle A(x), y\rangle = \begin{cases} +\infty & \text{if } x = 0, \\ \varepsilon x & \text{if } x \in]0,1]. \end{cases}$$

Hence

$$\inf_{x \in X} \Big(\inf_{y \in X \setminus \Omega(\varepsilon,x)} \langle A(x), y\rangle - \min_{y \in X} \langle A(x), y\rangle \Big) = \inf_{x \in]0,1]} \varepsilon x = 0;$$

so, the condition (i) is not satisfied.

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