Forcing in the alternative set theory II

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Abstract. By the technique of forcing, some new independence results are proved for the alternative set theory (AST) and similar weak theories: The scheme of choice is independent both of AST and of second order arithmetic, axiom of constructibility is independent of AST plus schemes of choice.

Keywords: alternative set theory, second order arithmetic, forcing, schemes of choice, axiom of constructibility, degrees of constructibility

Classification: Primary 03E70; Secondary 03E25, 03E35, 03E45

We prove some new independence results by the method of forcing in the alternative set theory (AST) and the second order arithmetic. This paper is a direct continuation of [Sg], where the technique of forcing is developed. We use all definitions and notations from this paper, also the references to Sections 1 and 4 refer to the same.

Here we use the developed techniques to prove the following concrete results (numbered by the numbers of the corresponding sections):

(5) Every countable model of TC+(A4)+(A5)+(A8)+(ADC) can be extended into a model of AST by adding new classes.

(6) Every countable model of $\overline{\text{AST}} + \overline{(Q)}$ restricted to classes can be extended to a model of AST with a new (larger) type of well-orderings.

(7) The axiom of constructibility is independent of AST plus the strong scheme of choice plus the scheme of dependent choices.

(8) The scheme of choice is independent of A_2 (the second order arithmetic—without the scheme of choice, of course).

(9) The scheme of choice is independent of AST.

The statements of (5) and (6) are proved by constructing a generic extension with the systems of conditions used in the classical case for collapsing cardinals.

In the cases (7) and (9), the relative consistency of schemes in question was established in [S 1985], here we prove the consistency of their negations.

The statement of (7) is proved by a construction of a symmetric extension, which is essentially a special permutation model.

The proofs of (8) and (9) are based on the following idea: we construct a symmetric extension with a special structure of degrees of constructibility—there exists ω minimal degrees, all their finite joins and no other degrees in the extension. In the classical case of ZF, a convenient technique was developed by Sacks (the construction of a minimal degree) and by Adamowicz (the general structure of degrees, see [A]). For the proof of (9), this technique is further refined, because no countable classes can be added.

The result of (8) is already known due to A. Lévy (see [L]), but our proof works in A_3, while the old one uses cardinals up to $\aleph_\omega$, which needs a much stronger theory.

5. Consistency of the axiom of cardinalities.

We are going to prove that every (countable) model of TC+(A4)+(A5)+(A8) +(ADC) can be extended into a model of the AST. In this result, it is substantial that sets, FN and the predicate $\in$ is absolute. If we want to prove only the consistency, we have a stronger result—it was proved in [S 1982] that AST is consistent relatively to an even weaker theory.

The idea of this proof is to add a well-ordering of V of the type Ω by forcing. To achieve this goal, we take as conditions all countable well-orderings and code them in such a way that the ordering of conditions is given by the natural ordering of well-orderings (Definition 5.2). By Lemma 5.3, this system has all required properties, moreover, it is codable, hence we can work in the theory TC.

We work in the theory TC+(A4)+(A5)+(A8)+(ADC).

Definition 5.1.

(i) Let R be well-ordering. Then

$$\pi_R =_{\text{df}} R \cup V \times (V \setminus \operatorname{dom}(R));$$

(ii) $\mathcal{P} =_{\text{df}} \{\pi_R; R \preccurlyeq \mathrm{FN} \;\&\; \mathrm{We}(R)\}$.

Lemma 5.2.

(i) $\pi_R \subseteq \pi_S \Leftrightarrow S$ *is a segment of* R.

(ii) *The system* $\mathcal{P}$ *satisfies the requirement from Definition* 2.1.

(iii) *The system* $\mathcal{P}$ *is closed under countable decreasing intersections.*

(iv) $\pi_R \Vdash x \in \Gamma \Leftrightarrow x \in (((\operatorname{dom}(R) \times V) \setminus R^{-1}) \cup \mathrm{Id})$,
$\pi_R \Vdash x \notin \Gamma \Leftrightarrow x \in ((V \times \operatorname{dom}(R)) \setminus R)$,

(v) $P \Vdash \mathrm{We}(V, \Gamma)$.

Proof: **(i)–(iv)** follows easily from Definition 5.1 (in (iii) we use the fact that a countable union of countable classes is countable, which is a consequence of (ADC)).

(v) For every R and x, y there exists a prolongation T of R such that $\langle x, y\rangle \in T$ or $\langle y, x\rangle \in T$. By (iv) for $x \neq y$, we have

$$\pi \Vdash \langle x, y\rangle \in \Gamma \Leftrightarrow \pi \Vdash \langle y, x\rangle \notin \Gamma.$$

Thus we have

$$P \Vdash (\;\Gamma \text{ is a linear ordering}) \;\;\&\;\; \operatorname{dom}(\Gamma) = V.$$

Let us suppose that for some $\mathcal{D}$ there holds

$$\pi_R \Vdash \mathcal{D} \neq \emptyset.$$

We can suppose that for some $x \in \operatorname{dom}(R)$ there holds

$$\pi_R \Vdash x \in \mathcal{D},$$

because this statement holds for almost every prolongation of R. Thus we have

$$\pi_R \Vdash \mathcal{D} \cap \operatorname{dom}(\breve{R}) \neq \emptyset$$

and, since R is a well-ordering,

$$\pi_R \Vdash \mathcal{D} \text{ has a } \breve{R} \text{ minimum.}$$

By (iv) we have

$$\pi_R \Vdash \breve{R} \text{ is a segment of } \Gamma.$$

Thus the $\breve{R}$-minimum of $\mathcal{D}$ is also the Γ-minimum of $\mathcal{D}$ and we have

$$\pi_R \Vdash \mathcal{D} \text{ has a } \Gamma\text{-minimum.}$$

We proved $P \Vdash \mathrm{We}(V, \Gamma)$. □

Metatheorem 5.3. *Let* $\mathfrak{M}$ *be a countable model of* TC+(A4)+(A5)+(A8)+(ADC). *Then there exists* $\mathfrak{N} \supseteq \mathfrak{M}$ *with sets, FN and* $\in$ *absolute relatively to* $\mathfrak{M}$ *such that* $\mathfrak{N} \models$ *AST.*

Demonstration: Let $\mathfrak{N} = \mathfrak{M}[G, \mathfrak{R}]$ for some G generic. By Metatheorem 3.8 it is sufficient to prove $P \Vdash$ AST in the theory TC+(A4)+(A5)+(A8)+(ADC). By Theorem 2.9 there holds $P \Vdash$ TC+(A4)+(A8). $P \Vdash$ (A5) holds by Lemma 2.13. The system of conditions $\mathcal{P}$ is codable, and thus, according to the remark following the Lemma 2.13, the axiom (ADC) instead of (SDC) is sufficient for this proof.

$P \Vdash$ (A6) follows from Lemma 5.2 (v). To prove $P \Vdash$ (A7), it is sufficient to prove that

$$P \Vdash \Gamma \text{ has a type at most } \Omega.$$

Let x be given. We are going to prove

$$P \Vdash \Gamma''\{x\} \preccurlyeq \mathrm{FN}\,.$$

For almost every π_R there holds $x \in \operatorname{dom}(R)$. From

$$\pi_R \Vdash \breve{R} \text{ is a segment of } \Gamma$$

it follows that

$$\pi_R \Vdash \Gamma''\{x\} = \breve{R}''\{x\} \preccurlyeq \mathrm{FN}$$

for almost every π_R. □

6. New types of well-orderings.

In this section, we work with the system of all well-orderings that are semisets. The conditions are similar to those in the Section 5. We work in the theory $\overline{\mathrm{AST}} + \overline{(Q)}$.

Definition 6.1.

(i) Let R be a well-ordering. Then

$$\pi_R =_{\mathrm{df}} R \cup V \times (V \setminus \mathrm{dom}\,(R)),$$

(ii) $\mathcal{P} =_{\mathrm{df}} \{\pi_R; \mathrm{Sms}(R)\ \&\ \mathrm{We}(R)\}$.

It can be easily checked that Lemma 5.2 holds also for this system of conditions.

Lemma 6.2. $(\forall R)(\forall \exists \varrho \subseteq P)(\varrho \Vdash \breve{R} \overset{\sim}{\prec} \Gamma)$.

Demonstration: Let R and $\pi = \pi_T$ be given. From $\mathrm{Sms}(T)$, it follows that there exists U such that $R \cong U, \mathrm{dom}\,(U) \cap \mathrm{dom}\,(T) = \emptyset$ and $\mathrm{Sms}(U)$. Now we take $\varrho = \pi_W$, where $W = T \cup U \cup (\mathrm{dom}\,(T) \times \mathrm{dom}\,(U))$ is a concatenation of these orderings. □

Metatheorem 6.3. *Let $\mathfrak{M}$ be a countable model of $\overline{\mathrm{AST}}$ + (SDC). Then there exists a model $\mathfrak{N} \supseteq \mathfrak{M}$ of $\overline{\mathrm{AST}} + \overline{(Q)}$ with sets and $\in$ absolute relatively to $\mathfrak{M}$ and a well-ordering $R \in \mathfrak{N}$ such that for no $S \in \mathfrak{M}$ there holds $R \overset{\sim}{\preccurlyeq} S$.*

Demonstration: Take $R = G$ for some generic G and $\mathfrak{N} = \mathfrak{L}(\langle Q, G\rangle)$, where Q is such that $\mathfrak{M} \models (\forall \mathcal{X})\mathcal{X} \in \mathfrak{L}(Q)$. The assertion follows from the fact that $\mathfrak{N} \supseteq \mathfrak{M}[G, \mathfrak{R}]$ and the relation $\overset{\sim}{\preccurlyeq}$ is absolute. □

Note that this process can be iterated and a very interesting structure of well-orderings can be obtained.

7. Independence of the axiom of constructibility.

We work in the theory AST + $(\forall X)X \in \mathcal{L}(Q)$ (Q is fixed for the whole section). The system of conditions will be the same as in the Section 5—the system of all countable well-orderings. Now we are interested in the forcing with $[\mathcal{P}_{\mathcal{A}}, \mathfrak{G}_{\mathcal{A}}]$, where $\mathcal{A} = \{n{\cdot}\omega{\cdot}\omega{+}m{\cdot}\omega; n, m \in \mathrm{FN}\}$, $A = \omega{\cdot}\omega{\cdot}\omega$ ($n{\cdot}\omega{\cdot}\omega$, etc. means the corresponding ordinals coded as cuts on Ω, i.e. as $\alpha \cap \Omega$). This system is codable.

From the Section 4 we know that in such system, the axiom of constructibility does not hold. The proof of the scheme of choice is based on the fact that if there exists a $\mathcal{D}$ satisfying some formula with parameters from $\mathfrak{G}_\alpha$, then such a $\mathcal{D}$ must exists even in $\mathfrak{G}_{\alpha+\omega}$, i.e. in $\mathcal{L}(\langle\breve{Q}, \Gamma_{\alpha+\omega}\rangle)$, but the scheme of choice does hold there. The idea of the proof of the scheme of dependent choices is similar, but we must iterate this process ω-times and a suitable class is found in $\mathcal{L}(\langle\breve{Q}, \Gamma_{\alpha+\omega\cdot\omega}\rangle)$, the scheme holds there again.

Definition 7.1.

(i) Let R be a well-ordering. Then $\pi_R =_{\mathrm{df}} R \cup V \times (V \setminus \mathrm{dom}\,(R))$;
(ii) $\mathcal{P} =_{\mathrm{df}} \{\pi_R; R \preccurlyeq \mathrm{FN}\ \&\ \mathrm{We}(R)\}$.

Lemma 7.2. *Let $\pi_R, \pi_T \in \mathcal{P}$. Then there exists an automorphism $\mathcal{F}$ of the system $\mathcal{P}$ and $\sigma \subseteq \mathcal{F}(\pi_R)$ such that $\sigma \subseteq \pi_T$.*

PROOF: Let F be the maximal isomorphism between cuts of orderings R and T (i.e. either an isomorphism between some cut of R and domain of T or an isomorphism between domain of R and some cut of T). Let us extend F to a permutation of V (this is possible, because $F \prec V$). Let us define $\mathcal{F}$ in the following way:

$$\mathcal{F}(\pi) = \{\langle F(x), F(y)\rangle; \langle x, y\rangle \in \pi\}.$$

$\mathcal{F}$ is an automorphism of $\mathcal{P}$, moreover, there holds either $\mathcal{F}(\pi_R) \subseteq \pi_T$ or $\pi_T \subseteq \mathcal{F}(\pi_R)$. In the former case, we take $\sigma = \mathcal{F}(\pi_R)$, in the latter one, we take $\sigma = \pi_T$. □

Lemma 7.3. *Let $\varphi \in \mathrm{FL}, \mathcal{D}_1, \dots, \mathcal{D}_p \in \mathfrak{G}_\alpha$. Then*

$$P \Vdash (\exists\, \mathcal{D})\varphi(\mathcal{D}, \mathcal{D}_1, \dots, \mathcal{D}_p) \Rightarrow (\exists\, \mathcal{D} \in \mathcal{L}(\langle \breve{Q}, \Gamma_{\alpha+\omega}\rangle))\varphi(\mathcal{D}, \mathcal{D}_1, \dots, \mathcal{D}_p).$$

PROOF: Let π and $\mathcal{D} \in \mathfrak{G}_\beta$ be such that

$$\pi \Vdash \varphi(\mathcal{D}, \mathcal{D}_1, \dots, \mathcal{D}_p).$$

We will find $\pi_1 \subseteq \pi$ and $\mathcal{E} \in \mathfrak{G}_{\alpha+\omega}$ such that

$$\pi_1 \Vdash \varphi(\mathcal{E}, \mathcal{D}_1, \dots, \mathcal{D}_p).$$

Let us take a permutation G of A such that

$$G \restriction \alpha = \mathrm{Id},\ G''\beta \subseteq \alpha + \omega,\ G \restriction A \setminus (\beta + \omega) = \mathrm{Id}$$

(there exists such permutation, because both α and β are countable). Let us take the corresponding symmetric automorphism $\mathcal{F}_G$. Let $\varrho = \mathcal{F}_G(\pi)$. We have $\mathcal{F}_G \restriction \mathcal{P}_\alpha = \mathrm{Id}, \varrho/\alpha = \pi/\alpha, \overline{\mathcal{F}}_G(\mathcal{D}) \in \mathfrak{G}_{\alpha+\omega}$.

Let $\varrho \in \mathcal{P}_\gamma$. For every $x \in \gamma \setminus \alpha$, there exists by Lemma 7.2 an automorphism $\mathcal{F}_x$ of $\mathcal{P}$ and $\sigma_x \subseteq \pi''\{x\}$ such that $\sigma_x \subseteq \mathcal{F}_x(\varrho''\{x\})$. Let us define $\mathcal{F}$ in the following way

$$\mathcal{F}(\pi)''\{x\} = \pi''\{x\}, \quad \text{if } x \notin \gamma \setminus \alpha,$$
$$\mathcal{F}(\pi)''\{x\} = \mathcal{F}_x(\pi''\{x\}), \quad \text{if } x \in \gamma \setminus \alpha.$$

$\mathcal{F}$ is a symmetric automorphism of $\mathcal{P}_\mathcal{A}$, moreover, for every β and $\mathcal{X} \in \mathfrak{G}_\beta$ we have $\overline{\mathcal{F}}(\mathcal{X}) \in \mathfrak{G}_\beta$. By Lemma 4.6 we have

$$\mathcal{F}(\mathcal{F}_G(\pi)) \Vdash \varphi(\overline{\mathcal{F}}(\overline{\mathcal{F}}_G(\mathcal{D})), \mathcal{D}_1, \dots, \mathcal{D}_p)$$

and thus

$$\mathcal{F}(\varrho) \Vdash \varphi(\overline{\mathcal{F}}(\overline{\mathcal{F}}_G(\mathcal{D})), \mathcal{D}_1, \dots, \mathcal{D}_p).$$

Now it is sufficient to take

$$\mathcal{E} = \overline{\mathcal{F}}(\overline{\mathcal{F}}_G(\mathcal{D})),$$
$$\pi_1''\{x\} = \pi''\{x\} \quad \text{if } x \notin \gamma \setminus \alpha,$$
$$\pi_1''\{x\} = \sigma_x \quad \text{if } x \in \gamma \setminus \alpha.$$

By Metatheorem 3.6, $\mathcal{E} \in \mathfrak{G}_{\alpha+\omega}$ implies that $\mathcal{E} \in \mathcal{L}(\langle \breve{Q}, \Gamma_{\alpha+\omega}\rangle)$ is forced. □

Metatheorem 7.4. Con (AST) $\Rightarrow$ Con (AST + (SSC) + (SDC)+ $\neg(Q)$).

Demonstration: We will prove that $P \Vdash$ AST + (SSC) + (SDC)+ $\neg(Q)$. $P \Vdash$ AST follows from the Section 2, the negation of the axiom of constructibility follows by Corollary of Theorem 4.10.

(SSC) Let a formula $\Phi(x, \mathcal{D})$ with constants from $\mathfrak{G}_\alpha$ be given. By Lemma 7.3 and by (SSC) in $\mathcal{L}(\langle \check{Q}, \Gamma_{\alpha+\omega}\rangle)$ we have

$$\begin{aligned}
&\pi \Vdash (\forall x)(\exists \mathcal{D})\Phi(x, \mathcal{D}) \Rightarrow \\
&\Rightarrow (\forall x)(\pi \Vdash (\exists \mathcal{D})\Phi(x, \mathcal{D})) \Rightarrow \\
&\Rightarrow (\forall x)(\pi \Vdash (\exists \mathcal{D} \in \mathcal{L}(\langle \check{Q}, \Gamma_{\alpha+\omega}\rangle))\Phi(x, \mathcal{D})) \Rightarrow \\
&\Rightarrow \pi \Vdash (\forall x)(\exists \mathcal{D} \in \mathcal{L}(\langle \check{Q}, \Gamma_{\alpha+\omega}\rangle))\Phi(x, \mathcal{D}) \Rightarrow \\
&\Rightarrow \pi \Vdash (\exists \mathcal{E})(\forall x)\Phi(x, \mathcal{E}''\{x\}),
\end{aligned}$$

thus $P \Vdash ((\forall x)(\exists \mathcal{D})\Phi(x, \mathcal{D}) \Rightarrow (\exists \mathcal{E})(\forall x)\Phi(x, \mathcal{E}''\{x\}))$.

(SDC) Let a formula $\Phi(\mathcal{D}, \mathcal{E})$ with constants from $\mathfrak{G}_\alpha$ and $\mathcal{D}_0 \in \mathfrak{G}_\alpha$ be given. Let $\pi \Vdash (\forall \mathcal{D})(\exists \mathcal{E})\Phi(\mathcal{D}, \mathcal{E})$. Then by Lemma 7.3 we have

$$\pi \Vdash (\forall \mathcal{D} \in \mathcal{L}(\langle \check{Q}, \Gamma_{\beta+n\cdot\omega}\rangle))(\exists \mathcal{E} \in \mathcal{L}(\langle \check{Q}, \Gamma_{\beta+(n+1)\cdot\omega}\rangle))\Phi(\mathcal{D}, \mathcal{E}).$$

(Note that the system $\{\Gamma_{\beta+n\cdot\omega};\, n \in \mathrm{FN}\}$ is codable in $\mathfrak{G}_{\beta+\omega\cdot\omega}$, so the formula above is correct.) By (SDC) in $\mathcal{L}(\langle \check{Q}, \Gamma_{\beta+\omega\cdot\omega}\rangle)$ we get

$$\pi \Vdash (\exists \mathcal{D}_1)(\mathcal{D}_1''\{0\} = \mathcal{D}_0 \ \&\ (\forall n)\Phi(\mathcal{D}_1''\{n\}, \mathcal{D}_1''\{n+1\})).$$

$\square$

8. Independence of the scheme of choice in second order arithmetic.

Instead of the second order arithmetic we take the theory TC + We$(N, \leq)$ + (A8), which is with the second order arithmetic mutually interpretable (if we extend standard interpretations between Peano arithmetic and the theory of finite sets to classes). In this theory we have FN $= N$.

We will work in the stronger theory $\overline{\mathrm{TC}}$ + We$(N, \leq)$ + (A8) +$\overline{(Q)}$ which is by Metatheorem 3.1 consistent relatively to the third order arithmetic.

During the proof we also use the fact that the axiom of choice holds in this theory (this is proved e.g. in [V] by the construction of set-theoretically definable one to one map between N and V). The constants N and $\leq$ are defined by set-theoretical formulae, so we can write N instead of $\check{N}$ and similarly for $\leq$.

In the first definition, we define some technical notions: $p^\frown 0$ resp. $p^\frown 1$ is the concatenation of the (finite) sequence p with a one-element sequence with the element 0 resp. 1. π_p is the largest subclass of π that has no element incomparable with p. $\mathrm{Tr}(\pi)$ is the smallest tree containing π. $\mathrm{Sp}(\pi)$ is the class of vertices, where π splits. π^i is the level of the i-th splitting in the tree, $\pi^{(i)}$ is the next level. Note that these notions are defined (and used) not only for conditions, but for any $\pi \subseteq P$.

Definition 8.1.

(i) $P =_{\mathrm{df}} \{p : i \to 2; i \in N\}$,

(ii) $p\widehat{\ }0 =_{\mathrm{df}} p \cup \{\langle 0, \mathrm{dom}\,(p)\rangle\}$, $p\widehat{\ }1 =_{\mathrm{df}} p \cup \{\langle 1, \mathrm{dom}\,(p)\rangle\}$,

(iii) Let $\pi \subseteq P, p \in \pi$. Then

$$\pi_p =_{\mathrm{df}} \{q \in \pi; q \subseteq p \vee p \subseteq q\},$$
$$\mathrm{Tr}(\pi) =_{\mathrm{df}} \{q \in P; (\exists\, p \in \pi)(q \subseteq p)\}.$$

(iv) Let $\pi = \mathrm{Tr}(\pi)$ (i.e. π is a tree). Then

$$\mathrm{Sp}(\pi) =_{\mathrm{df}} \{p; p\widehat{\ }0, p\widehat{\ }1 \in \pi\},$$
$$\pi^i =_{\mathrm{df}} \{p \in \mathrm{Sp}(\pi); \{q \in \mathrm{Sp}(\pi); q \subset p\} \approx i\},$$
$$\pi^{(i)} =_{\mathrm{df}} \{p\widehat{\ }0, p\widehat{\ }1; p \in \pi_i\}.$$

Let us define the system $\mathcal{P}$ as follows:

Definition 8.2. $\pi \subseteq P$ is a condition, if both (a) and (b) hold:

(a) $\pi = \mathrm{Tr}(\pi)$,

(b) $\mathrm{Sp}(\pi)$ is cofinal in π.

Observation.

(i) $P \in \mathcal{P}$.

(ii) Let $\pi \in \mathcal{P}, p \in \pi$. Then $\pi_p \in \mathcal{P}$.

(iii) The system $\mathcal{P}$ is a system of conditions (Definition 2.1).

(iv) The condition (b) also forces that $\mathrm{Sp}(\pi)$ has no last element for $\pi \in \mathcal{P}$.

We are interested in the system of conditions $\mathcal{P}_{\mathcal{A}}$, where $\mathcal{A} = \{a; a \subseteq N\}$. Elements of $\mathcal{P}_a, a = \{a_1, \ldots, a_m\}$, have by definition the form

$$\pi_{a_1} \times \{a_1\} \cup \cdots \cup \pi_{a_m} \times \{a_m\}, \quad \text{where} \quad \pi_{a_i} \in \mathcal{P}.$$

When we work with the system $\mathcal{P}_a$, we identify this class with a code for the m-tuple $\langle \pi_{a_1}, \ldots, \pi_{a_m}\rangle$. Thus we will simplify our notation by writing

$$\langle \pi_{a_1}, \ldots, \pi_{a_m}\rangle \in \mathcal{P}_a.$$

The variables a, b denote elements of $\mathcal{A}$.

In the following definition, we want to define some analogy of the levels $\pi^{(i)}$ also in conditions from $\mathcal{P}_a$.

Definition 8.3. Let $\pi = \langle \pi_{a_1}, \ldots, \pi_{a_m}\rangle \subseteq P_a$.

(i) $p \varepsilon \pi \Leftrightarrow_{\mathrm{df}} p = \{\langle p_1, a_1\rangle, \ldots, \langle p_m, a_m\rangle\} \,\&\, p_1 \in \pi_{a_1}, \ldots, p_m \in \pi_{a_m}$. We will write (similarly as for π) $p = \langle p_1, \ldots, p_m\rangle$.

(ii) Let $p \varepsilon \pi$. $\pi_p =_{\mathrm{df}} \langle (\pi_{a_1})_{p_1}, \ldots, (\pi_{a_m})_{p_m}\rangle$.

(iii) Let $i \in N$. Then

$$\pi^{(i)} =_{\mathrm{df}} \{p = \langle p_1, \ldots, p_m\rangle;\ p_1 \in \pi^{(i)}_{a_1}, \ldots, p_m \in \pi^{(i)}_{a_m}\}.$$

Observation. **(i)** Let $\pi \in \mathcal{P}_a, p \varepsilon \pi$. Then $\pi_p \in \mathcal{P}_a$.

(ii) The systems $\mathcal{P}_a$ and $\mathcal{P}_{\mathcal{A}}$ are systems of conditions.

The following theorem summarizes results from the Sections 2 to 4 for our system of conditions.

Theorem 8.4. *Let $\mathfrak{G} = \mathfrak{G}_a$ or $\mathfrak{G} = \mathfrak{G}_{\mathcal{A}}$. Then $P[\mathfrak{G}] \Vdash \mathrm{TC} + \mathrm{We}(N, \leq) + (\mathrm{A8})$.*

Now we will work with $\mathcal{P}_a$ and $\mathfrak{R}_a$. First we prove a technical lemma. The point (i) asserts that $\varrho^{(i)}$ may be thought as a level of the condition ϱ. The point (ii) asserts that π is in a sense a disjoint union of π_p for p from a given level of π.

Lemma 8.5.

(i) *Let $i \in N$. Then $\varrho = \bigcup\{\varrho_p; p \in \varrho^{(i)}\}$.*

(ii) *Let $i \in N$, let φ be a formula such that $(\forall\, p \in \pi^{(i)})(\pi_p \Vdash \varphi)$. Then $\pi \Vdash \varphi$.*

Proof: **(i)** For the system $\mathcal{P}$ it is trivial. For $\mathcal{P}_a$ and given $q \varepsilon \varrho$ we can find $p_n \in (\varrho''\{n\})^{(i)}$ such that $q_n \in (\varrho''\{n\})_{p_n}$, and take $p = \langle p_1, \dots, p_m \rangle$.

(ii) Let $\varrho \subseteq \pi$ be given. From (i) it follows that there exists $p \in \pi^{(i)}$ such that $p \varepsilon \varrho$. For such a p we have $\varrho_p \subseteq \pi_p$ and thus $\varrho_p \Vdash \varphi$. $\square$

Now we are going to prove that if $\pi \Vdash \mathrm{rank}(\mathcal{D}) = a$ holds for some $\mathcal{D}, \mathcal{E} \in \mathfrak{R}_a$ and $\pi \in \mathcal{P}_a$, then $\varrho \Vdash \mathcal{E} \in \mathcal{L}(\langle \check{X}, \mathcal{D} \rangle)$ for some $\varrho \subseteq \pi$ and X. The condition ϱ will be constructed step by step by reducing π in such a way that from the i-th step of the construction the level $\varrho^{(i)}$ is left unchanged. This guarantees that the result of the construction is a condition.

In the i-th step we need to guarantee that from $\mathcal{D}$ we can find which element of $\varrho^{(i)}$ is in Γ_a and that from this fact we can find whether z_i is in $\mathcal{E}$ ($z_i, i \in N$, being an enumeration of all sets). This will be done in the following way: we will go through all pairs p, q from the i-th level (there are finitely many ones) and for each pair we will find x such that for a suitable reduction of the condition we can recognize whether p or q is in Γ_a from the fact whether $x \in \mathcal{D}$ or $x \notin \mathcal{D}$, and for each p we will find another reduction of the condition such that it decides the issue of z_i.

At first in Lemma 8.6, we will construct such an x and a reduced condition for given p and q (here we use the assumption about $\mathrm{rank}(\mathcal{D})$), then in Lemma 8.7 we will construct ϱ and prove that it has the required properties.

Let $\mathcal{D}, \mathcal{E} \in \mathfrak{R}_a$ be given.

Notation. Let $z_i, i \in N$ be an enumeration of all sets. Let us take

$$\begin{aligned}\Phi(\varrho, i, p, q, x, \sigma) \Leftrightarrow_{\mathrm{df}} {} & [(p, q \in \varrho^{(i)} \,\&\, p \neq q) \Rightarrow \\ & \Rightarrow (\sigma \subseteq \varrho \,\&\, \sigma^{(i)} = \varrho^{(i)} \,\&\, (\sigma_p \Vdash z_i \in \mathcal{E} \vee \sigma_p \Vdash z_i \notin \mathcal{E}) \,\&\, \\ & \quad \& \, ((\sigma_p \Vdash x \in \mathcal{D} \,\&\, \sigma_q \Vdash x \notin \mathcal{D}) \vee (\sigma_q \Vdash x \in \mathcal{D} \,\&\, \sigma_p \Vdash x \notin \mathcal{D})))].\end{aligned}$$

Lemma 8.6. *Let $\varrho \Vdash \operatorname{rank}(\mathcal{D}) = a$. Then $(\forall\, i, p, q)(\exists\, x, \sigma)\Phi(\varrho, i, p, q, x, \sigma)$.*

Proof: Let $p, q \in \varrho^{(i)}, p \neq q$. Let $p = \langle p_{a_1}, \dots, p_{a_k}\rangle, q = \langle q_{a_1}, \dots, q_{a_k}\rangle, a = \{a_1, \dots, a_k\}$. Let $n \in a$ be such that $p_n \neq q_n$. By the definition of rank($\mathcal{D}$) the following holds:

$$(\exists\, x)(\exists\, \varrho_1, \varrho_2 \subseteq \varrho)(\varrho_1 =_n \varrho_2 \,\&\, \varrho_1, \varrho_2 \subseteq \varrho_p \,\&\, \varrho_1 \Vdash x \in \mathcal{D} \,\&\, \varrho_2 \Vdash x \notin \mathcal{D}).$$

For such an x

$$(\exists\, \varrho_3)((\forall\, m \neq n)(\varrho_3''\{m\} \subseteq \varrho_1''\{m\}) \,\&\, \varrho_3''\{n\} \subseteq (\varrho''\{n\})_{q_n} \,\& \\ \&\, (\varrho_3 \Vdash x \in \mathcal{D} \vee \varrho_3 \Vdash x \notin \mathcal{D})).$$

First, let us define a condition τ in the following way:
Let $\tau^{(i)} = \varrho^{(i)}$; for $r \in (\varrho''\{m\})^{(i)}$, we distinguish three cases:

- for $r \neq p_m, r \neq q_m$ let $(\tau''\{m\})_r = (\varrho''\{m\})_r$;
- for $m \neq n, r = p_m = q_m$ we take $(\tau''\{m\}) = \varrho_3''\{m\}$;
- for $r \in \{p_n, q_n\}$ we distinguish two cases:

  if $\tau \Vdash x \in \mathcal{D}$, then $(\tau''\{n\})_{p_n} = \varrho_3''\{n\}$ and $(\tau''\{n\})_{q_n} = \varrho_2''\{n\}$;
  otherwise $(\tau''\{n\})_{q_n} = \varrho_1''\{n\}$ and $(\tau''\{n\})_{q_n} = \varrho_3''\{n\}$.

The condition τ satisfies all requirements except that one concerning z_i. We obtain a required $\sigma \subseteq \tau$ by another reduction. By the definition of forcing, there exists a condition $\tau' \subseteq \tau_p$ such that $(\tau_p' \Vdash z_i \in \mathcal{E} \vee \tau_p' \Vdash z_i \notin \mathcal{E})$. Now let σ be such that $\sigma^{(i)} = \tau^{(i)} = \varrho^{(i)}, \sigma_p = \tau'$ and $(\tau''\{m\})_r = (\varrho''\{m\})_r$ for $r \neq p_m$. The condition σ satisfies all requirements. $\square$

Lemma 8.7. *Let $\pi \Vdash \operatorname{rank}(\mathcal{D}) = a$. Then there exists $\varrho \subseteq \pi$ and X such that $\varrho \Vdash \mathcal{E} \in \mathcal{L}(\langle \check{X}, \mathcal{D}\rangle)$.*

Proof: By Lemma 8.6 and the axiom of constructibility , it is possible (see [S 1985]) to construct a formula $\Psi(\varrho, i, p, q, x, \sigma)$ such that

$$(\Psi(\varrho, i, p, q, x, \sigma) \Rightarrow \Phi(\varrho, i, p, q, x, \sigma)) \,\& \\ \&\, (\forall\, \varrho \subseteq \pi)(\forall\, i)(\exists !\langle x, \sigma\rangle)\Psi(\varrho, i, p, q, x, \sigma).$$

Let $\pi \Vdash \operatorname{rank}(\mathcal{D}) = a$. We will find $\varrho \subseteq \pi$ and X such that $\varrho \Vdash \mathcal{E} \in \mathcal{L}(\langle \check{X}, \mathcal{D}\rangle)$. We will construct by induction a sequence of the conditions $\pi_i (i \in N)$ such that

(a) $\pi_{i+1} \subseteq \pi_i,\ \pi_0 = \pi$,
(b) $\pi_{i+1}^{(i)} = \pi_i^{(i)}$,
(c) for $p \in \pi_i^{(i)}$, there holds $(\pi_{i+1})_p \Vdash z_i \in \mathcal{E} \vee (\pi_{i+1})_p \Vdash z_i \notin \mathcal{E}$.

Simultaneously, we will construct a system of sets y_p for $p \in \pi_i^{(i)}$ such that

(d) $(\pi_{i+1})_p \Vdash y_p \subseteq \chi_{\mathcal{D}}$,
(e) $(\forall\, p_1, p_2 \in \pi_i^{(i)})(p_1 \neq p_2 \Rightarrow \neg \operatorname{Fnc}(y_{p_1} \cup y_{p_2}))$.

Construction. Let us fix a well-ordering $\leq$ of V which is an Sd-class. Let $\pi_0 = \pi$.

Let π_i be given. Let $\{p_1, q_1\}, \ldots, \{p_k, q_k\}$ be a sequence of all pairs of distinct elements of $\pi_i^{(i)}$ ordered by $\leq$.

First, we will construct the sequences $\sigma_0, \ldots, \sigma_k$ and $x_1, \ldots, x_k$ of conditions and sets: Let $\sigma_0 = \pi_i$, let $\langle x_j, \sigma_j\rangle$ be the unique pair such that $\Psi(\sigma_{j-1}, i, p_j, q_j, x_j, \sigma_j)$. Finally, let $\pi_{i+1} = \sigma_k$.

Now we will construct y_p for $p \in \pi_i^{(i)}$. We go through all $j \leq k$ such that $p \in \{p_j, q_j\}$ and for each such j we add one element into y_p, namely

$$\begin{aligned} &\langle 0, x_j\rangle, \quad \text{if} \quad (\sigma_j)_p \Vdash x_j \notin \mathcal{D}, \\ &\langle 1, x_j\rangle, \quad \text{if} \quad (\sigma_j)_p \Vdash x_j \in \mathcal{D}. \end{aligned}$$

There are no other elements in y_p.

Correctness of the construction.

(a) follows from the fact that the sequence $\sigma_0, \ldots, \sigma_k$ is decreasing.

(b) and **(c)** follow by the definition of the formula Φ.

(d) follows from the construction of y_p and from $\pi_{i+1} = \sigma_k \subseteq \sigma_j$.

(e) follows from the fact that for each pair $\langle p_j, q_j\rangle$ we put the element $\langle 0, x_j\rangle$ into one of the sets y_{p_j}, y_{q_j} and the element $\langle 1, x_j\rangle$ into the other one, in the construction we go through all pairs of distinct $p, q \in \pi_i^{(i)}$.

Let $\varrho = \bigcap\{\pi_n, n \in N\}$. We will prove that ϱ is a condition. Trivially $\varrho''\{n\}$ is a tree. From (b) it follows that $\pi_i^{(i)} = \pi_k^{(i)}$ for $k > i$, thus $\varrho^{(i)} = \pi_i^{(i)}$ and $\mathrm{Sp}(\varrho''\{n\})$ is cofinal in $\varrho''\{n\}$.

Definition of X and conclusion of the proof.

Let
$$X = \{\langle z_i, y_p\rangle; i \in N \,\&\, p \in \pi_i^{(i)} \,\&\, (\pi_{i+1})_p \Vdash z_i \in \mathcal{E}\}.$$

By the theorem on the construction by induction, the definition of ϱ and F is correct, because all steps are unambiguous. By the conditions (c), (d) and (e), we have (using Lemma 8.5 (ii))

$$\pi_{i+1} \Vdash (z_i \in \mathcal{E} \Leftrightarrow (\exists\, y \subseteq \chi_{\mathcal{D}})\langle z_i, y\rangle \in \check{X}),$$

and thus (because z_i is an enumeration of all sets and $\varrho \subseteq \pi_i$ for every $i \in N$)

$$\varrho \Vdash \mathcal{E} = \{z; (\exists\, y \subseteq \chi_{\mathcal{D}})\langle z, y\rangle \in \check{X}\} \in \mathcal{L}(\langle \check{X}, \mathcal{D}\rangle),$$

and Lemma 8.7 is proved. □

Now we can return to the forcing with $[\mathcal{P}_{\mathcal{A}}, \mathfrak{G}_{\mathcal{A}}]$. We define the notion of an $\mathcal{L}$-independent class (more precisely, a class coding an $\mathcal{L}$-independent system of classes). We will use it in the extension, though the definition contains the formula $\times \in \mathfrak{M}$, but this is possible by Metatheorem 3.7.

Definition 8.8. Let a class $A, \mathrm{dom}\,(A) \subseteq N$ be given. We say that A is $\mathcal{L}$-independent iff

$$(\forall\, n, m \in \mathrm{dom}\,(A))(n \neq m \Rightarrow (\forall\, X \in \mathfrak{M})(A''\{m\} \notin \mathcal{L}(\langle X, A''\{n\}\rangle))).$$

Lemma 8.9. *$P \Vdash (\forall\, \mathcal{D})$ ($\mathcal{D}$ is $\mathcal{L}$-independent $\Rightarrow \mathrm{dom}\,(\mathcal{D}) \preccurlyeq 2^{\mathrm{rank}(\mathcal{D})}$).*

PROOF: Let us suppose that for some $\mathcal{D}$ and π we have $\pi \Vdash (2^{\mathrm{rank}(\mathcal{D})} \prec \mathrm{dom}\,(\mathcal{D}))$. Then for some $m, n \in \mathrm{dom}\,(\mathcal{D}), m \neq n$ and $\varrho \subseteq \pi$, we have $\varrho \Vdash \mathrm{rank}(\mathcal{D}''\{m\}) = \mathrm{rank}(\mathcal{D}''\{n\})$ and by Lemma 8.7, Theorem 4.12 and Lemma 4.9 (the results from the Section 4 are necessary, because we use a system of conditions different from Lemma 8.7) we have $\varrho \Vdash$ ($\mathcal{D}$ is not $\mathcal{L}$-independent). □

Metatheorem 8.10. *Let A_2 be the second order arithmetic without the scheme of choice. Then*

$$\mathrm{Con}(A_3) \Rightarrow \mathrm{Con}(A_2 + \neg(\textit{scheme of choice})).$$

DEMONSTRATION: We will prove (in $\overline{\mathrm{TC}} + \mathrm{We}(N, \leq) + (\mathrm{A8}) + \overline{(Q)}$) that $P \Vdash \mathrm{TC} + \mathrm{We}(N, \leq) + (\mathrm{A8}) + \neg(\mathrm{SC})$. The rest follows from Metatheorem 3.2 and the mutual interpretability of A_2 and $\mathrm{TC} + \mathrm{We}(N, \leq) + (\mathrm{A8})$ (in the standard interpretation the validity of the scheme of choice is left unchanged).

It remains to prove that $P \Vdash \neg(\mathrm{SC})$. We have

$$P \Vdash (\forall\, n)(\exists\, \mathcal{D})(\mathcal{D} \text{ is } \mathcal{L}\text{-independent } \ \& \ \mathrm{dom}\,(\mathcal{D}) \approx n)$$

(we can put $\mathcal{D}$ equal to $\bigcup\{\Gamma_i \times \{i\}; i < n\}$, which is in $\mathfrak{G}_n$, and use Lemma 4.10). Let us suppose that

$$P \Vdash (\exists\, \mathcal{E})(\forall\, n)(\mathcal{E}''\{n\} \text{ is } \mathcal{L}\text{-independent } \ \& \ \mathrm{dom}\,(\mathcal{E}''\{n\}) \approx n).$$

By Lemma 8.9 we have

$$P \Vdash (\exists\, \mathcal{E})(\forall\, m)(m \preccurlyeq \mathrm{rank}(\mathcal{E})),$$

a contradiction. □

9. Independence of the scheme of choice in the AST.

The proof and the system of conditions will be similar as in the previous section, but the conditions will be trees of depth Ω rather than ω—this ensures that the system is closed under countable monotonous intersections. The change of the system of conditions forces some technical problems but the idea and main line of the proof remain. We omit some proofs that are essentially the same as in the previous section.

We work in the theory $\overline{\mathrm{AST}} + \overline{(Q)}$, which is by Metatheorem 3.2 consistent relatively to the fourth-order arithmetic. Let us suppose that the mappings $F : \alpha \cap \Omega \to 2, \alpha \in \Omega$, are coded by sets (e.g. by their prolongations). In the sequel, we will identify them with their codes, we will use variables p, q for them.

The following notions have a similar meaning as in Section 8.

Definition 9.1.

(i) $P =_{\mathrm{df}} \{p : \alpha \cap \Omega \to 2; \alpha \in \Omega\}$.

(ii) Let $\mathrm{dom}\,(p) = \alpha \cap \Omega, \alpha \in \Omega$. Then

$$p^\frown 0 =_{\mathrm{df}} p \cup \{\langle 0, \alpha\rangle\}, \quad p^\frown 1 =_{\mathrm{df}} p \cup \{\langle 1, \alpha\rangle\}.$$

(iii) Let $\pi \subseteq P, p \in \pi$. Then

$$\begin{aligned} \pi_p &=_{\mathrm{df}} \{q \in \pi; q \subseteq p \vee p \subseteq q\}, \\ \mathrm{Tr}(\pi) &=_{\mathrm{df}} \{q \in P; (\exists\, p \in \pi)(q \subseteq p)\}. \end{aligned}$$

(iv) Let $\pi = \mathrm{Tr}(\pi)$ (i.e. π is a tree). Then

$$\mathrm{Sp}(\pi) =_{\mathrm{df}} \{p; p^\frown 0, p^\frown 1 \in \pi\}.$$

Let us define the system of conditions $\mathcal{P}$:

Definition 9.2. $\pi \subseteq P$ is a condition, if (a) to (c) hold:

(a) $\pi = \mathrm{Tr}(\pi)$,

(b) $\mathrm{Sp}(\pi)$ is cofinal in π,

(c) $\mathrm{Sp}(\pi)$ is closed under countable monotonous unions.

Observation. **(i)** $P \in \mathcal{P}$.

(ii) Let $\pi \in \mathcal{P}$. Then $p \in \pi \Rightarrow \pi_p \in \mathcal{P}$.

(iii) $\mathcal{P}$ is a system of conditions.

Lemma 9.3. *The system $\mathcal{P}$ is closed under countable monotonous intersections.*

Proof: Let $\pi = \bigcap\{\pi_n; n \in \mathrm{FN}\}, \pi_{n+1} \subseteq \pi_n$. We will verify (a)–(c) from Definition 9.2.

(a) is trivial.

We have $\mathrm{Sp}(\pi) = \bigcap\{\mathrm{Sp}(\pi_n); n \in \mathrm{FN}\}$, $\mathrm{Sp}(\pi_{n+1}) \subseteq \mathrm{Sp}(\pi_n)$. From this, (c) follows.

(b): Let $p \in \pi$. By induction we can construct a sequence $\{p_n; n \in \mathrm{FN}\}$ such that $p_0 = p, p_n \in \mathrm{Sp}(\pi_n), p_n \subseteq p_{n+1}$. Let $q = \bigcup\{p_n; n \in \mathrm{FN}\}$. We have $(\forall\, m > n)(p_m \in \mathrm{Sp}(\pi_n))$ and, by (c) for π_n, we get $q = \bigcup\{p_m; m > n\} \in \mathrm{Sp}(\pi_n)$ and $q \in \bigcap\{\mathrm{Sp}(\pi_n); n \in \mathrm{FN}\} = \mathrm{Sp}(\pi)$. □

In the next definition, we want to define levels in trees of depth Ω in such a way that each level is countable (we need this in Lemma 9.8). Thus it is impossible to define levels in such a simple way as in the previous section (because even the ω-th level would have 2^ω elements). First, we define levels in the tree P, we transfer them to the other trees by a natural isomorphism between $\mathrm{Sp}(\pi)$ and P. By this definition we also achieve that the trees with the same "beginning" have equal first levels.

Definition 9.4. Let $\leq$ be a well-ordering of P by the type Ω such that $p \subseteq q \Rightarrow p \leq q$. Let $\pi \in \mathcal{P}$. Let $p \in \mathrm{Sp}(\pi)$; we define $\alpha(p)$ as the (unique) element of Ω such that $\langle \{q \in \mathrm{Sp}(\pi); q \subset p\}, \subseteq \rangle$ is isomorphic with $\alpha(p) \cap \Omega$. Let us define for $p \in \mathrm{Sp}(\pi)$

$$F(p) = \{\langle p(\beta), \alpha(p \restriction \beta)\rangle; \beta \in \mathrm{dom}\,(p) \,\&\, p \restriction \beta \in \mathrm{Sp}(\pi)\}.$$

F is an isomorphism between $\mathrm{Sp}(\pi)$ and P. Let us define

(i) $p \in \pi^{\alpha} \Leftrightarrow_{\mathrm{df}} p \in \mathrm{Sp}(\pi) \,\&\, \langle \{q \in P; q \leq F(p)\}, \leq \rangle \overset{\sim}{\preccurlyeq} \alpha \cap \Omega$.

(ii) $\pi^{(\alpha)} =_{\mathrm{df}} \{p^\frown 0, p^\frown 1; p \in \pi^{\alpha}\}$.

Observation. **(i)** $\pi^{\alpha} \preccurlyeq \mathrm{FN}$.

(ii) $\bigcup\{\pi^{(\alpha)}; \alpha \in \Omega\}$ is cofinal in π.

(iii) Let $p \neq q$, $p, q \in \pi^{(\alpha)}$. Then $\neg(p \subseteq q \vee q \subseteq p)$.

(iv) Let $\alpha \in \Omega$. Then $\pi = \bigcup\{\pi_p; p \in \pi^{(\alpha)}\}$.

We are interested in the system of conditions $\mathcal{P}_{\mathcal{A}}$, where $\mathcal{A} = \{a; a \subseteq N\}$. We will write (as in the previous section)

$$\langle \pi_{a_1}, \dots, \pi_{a_m} \rangle \in \mathcal{P}_a$$

for $a = \{a_1, \dots, a_m\}$. Variables a, b range over $\mathcal{A}$.

Definition 9.5. Let $\pi = \langle \pi_{a_1}, \dots, \pi_{a_m} \rangle \in \mathcal{P}_a, \alpha \in \Omega$. Then

(i) $p \varepsilon \pi \Leftrightarrow_{\mathrm{df}} p = \langle p_1, \dots, p_m \rangle \,\&\, p_1 \in \pi_{a_1}, \dots, p_m \in \pi_{a_m}$.

(ii) If $p \varepsilon \pi$, then $\pi_p =_{\mathrm{df}} \langle (\pi_{a_1})_{p_1}, \dots, (\pi_{a_m})_{p_m} \rangle$.

(iii) $\pi^{(\alpha)} =_{\mathrm{df}} \{\langle p_1, \dots, p_m \rangle; p_1 \in \pi^{(\alpha)}_{a_1}, \dots, p_m \in \pi^{(\alpha)}_{a_m}\}$.

Observation. **(i)** Let $\pi \in \mathcal{P}_a, p \in \pi$. Then $\pi_p \in \mathcal{P}_a$.

(ii) The systems $\mathcal{P}_a$ and $\mathcal{P}_{\mathcal{A}}$ are the systems of conditions.

The following theorem summarizes the results from Sections 2 to 4 for our system of conditions.

Theorem 9.6. *Let $\mathfrak{G} = \mathfrak{G}_a$ or $\mathfrak{G} = \mathfrak{G}_{\mathcal{A}}$. Then $P[\mathfrak{G}] \Vdash \mathrm{AST}$.*

Now we will work with the forcing $[\mathcal{P}_a\, \mathfrak{R}_a]$. We are going to prove that if $\pi \Vdash \mathrm{rank}(\mathcal{D}) = a$ holds for some $\mathcal{D}, \mathcal{E} \in \mathfrak{R}_a$ and $\pi \in \mathcal{P}_a$, then $\varrho \Vdash \mathcal{E} \in \mathcal{L}(\langle \check{X}, \mathcal{D} \rangle)$ for some $\varrho \subseteq \pi$ and X. The proof is similar as in the previous section, only it is done over Ω rather than over ω. At the limit step we take the intersections of (countably many) conditions constructed in the previous steps. In the single step of construction, we must go through a countable class of pairs from given level. Thus we will construct a countable sequence of conditions (instead of a finite one) and then we will take its intersection.

Lemma 9.7. **(i)** *Let $\alpha \in \Omega$. Then $\varrho = \bigcup\{\varrho_p; p \in \varrho^{(\alpha)}\}$.*

(ii) *Let $\alpha \in \Omega$, let φ be a formula such that $(\forall\, p \in \pi^{(\alpha)})(\pi_p \Vdash \varphi)$. Then $\pi \Vdash \varphi$.*

Proof: See Lemma 8.5. $\square$

Let $\mathcal{D}, \mathcal{E} \in \mathfrak{R}_a$ be given.

Notation. Let $z_\alpha, \alpha \in \Omega$ be an enumeration of all sets. Let us take

$$\begin{aligned}\Phi(\varrho,\alpha,p,q,x,\sigma) \Leftrightarrow_{\text{ df}} [(\alpha \in \Omega \,\&\, p,q \in \varrho^{(\alpha)} \,\&\, p \neq q) \Rightarrow \\ \Rightarrow (\sigma \subseteq \varrho \,\&\, \sigma^{(\alpha)} = \varrho^{(\alpha)} \,\&\, (\sigma_p \Vdash z_\alpha \in \mathcal{E} \vee \sigma_p \Vdash z_\alpha \notin \mathcal{E}) \,\&\, \\ \& \, ((\sigma_p \Vdash x \in \mathcal{D} \,\&\, \sigma_q \Vdash x \notin \mathcal{D}) \vee (\sigma_q \Vdash x \in \mathcal{D} \,\&\, \sigma_p \Vdash x \notin \mathcal{D})))].\end{aligned}$$

Lemma 9.8. *Let $\varrho \Vdash \operatorname{rank}(\mathcal{D}) = a$. Then $(\forall\, \alpha,p,q)(\exists\, x,\sigma)\Phi(\varrho,\alpha,p,q,x,\sigma)$.*

PROOF: See Lemma 8.6. □

Lemma 9.9. *Let $\pi \Vdash \operatorname{rank}(\mathcal{D}) = a$. Then there exists X such that $\pi \Vdash \mathcal{E} \in \mathcal{L}(\langle \check{X}, \mathcal{D}\rangle)$.*

PROOF: As in Lemma 8.7, we take a formula $\Psi(\varrho,\alpha,p,q,x,\sigma)$ such that

$$\begin{aligned}&(\Psi(\varrho,\alpha,p,q,x,\sigma) \Rightarrow \Phi(\varrho,\alpha,p,q,x,\sigma)) \,\& \\ &\& \,(\forall\, \varrho \subseteq \pi)(\forall\, \alpha \in \Omega)(\exists!\langle x,\sigma\rangle)\Psi(\varrho,\alpha,p,q,x,\sigma).\end{aligned}$$

Let $\pi \Vdash \operatorname{rank}(\mathcal{D}) = a$. We will find $\varrho \subseteq \pi$ and X such that $\varrho \Vdash \mathcal{E} \in \mathcal{L}(\langle \check{X}, \mathcal{D}\rangle)$. We will construct by induction over Ω a sequence of the conditions π_α such that for $\beta > \alpha$ the following holds:

- **(a)** $\pi_\beta \subseteq \pi_\alpha$, $\pi_0 = \pi$,
- **(b)** $\pi_\beta^{(\alpha)} = \pi_\alpha^{(\alpha)}$,
- **(c)** for $p \in \pi_\alpha^{(\alpha)}$, there holds $(\pi_{\alpha+1})_p \Vdash z_\alpha \in \mathcal{E} \vee (\pi_{\alpha+1})_p \Vdash z_\alpha \notin \mathcal{E}$.

Simultaneously, we will construct a sequence of countable classes Y_p for $p \in \pi_\alpha^{(\alpha)}$ such that

- **(d)** $(\pi_{\alpha+1})_p \Vdash \check{Y}_p \subseteq \chi_{\mathcal{D}}$,
- **(e)** $(p_1,p_2 \in \pi_\alpha^{(\alpha)} \,\&\, p_1 \neq p_2) \Rightarrow \neg\operatorname{Fnc}(Y_{p_1} \cup Y_{p_2})$.

Construction. Let $\pi_0 = \pi$.

Let β be a limit ordinal. Then $\pi_\beta = \bigcap\{\pi_\gamma; \gamma \in \beta \cap \Omega\}$.

Let π_α be given. We will construct $\pi_{\alpha+1}$. Let $\{p_1,q_1\},\dots,\{p_j,q_j\},\dots, j \in \mathrm{FN}$ be a sequence of all pairs of different elements of $\pi_\alpha^{(\alpha)}$. The ordering can be selected unambiguously due to the axiom of constructibility. We will construct a decreasing sequence of conditions $\sigma_0,\dots,\sigma_j,\dots$ in the following way:

$\sigma_0 = \pi_\alpha$,
σ_j is the unique condition such that $(\exists\, x)\Psi(\sigma_{j-1},\alpha,p_j,q_j,x,\sigma_j)$.

We denote this unique x by x_j. We take

$$\pi_{\alpha+1} = \bigcap\{\sigma_j; j \in \mathrm{FN}\}.$$

Now we will construct Y_p for $p \in \pi_\alpha^{(\alpha)}$. For each j such that $p \in \{p_j, q_j\}$, we add one element into Y_p, namely

$$\begin{aligned} &\langle 0, x\rangle, \quad \text{if } (\sigma_j)_p \Vdash x_j \notin \mathcal{D}, \\ &\langle 1, x\rangle, \quad \text{if } (\sigma_j)_p \Vdash x_j \in \mathcal{D}. \end{aligned}$$

There are no other elements in Y_p.

Correctness of the construction is proved as in the previous section (for the limit step it is trivial).

Let $\varrho = \bigcap\{\pi_a; \alpha \in \Omega\}$. We will prove that ϱ is a condition. Trivially ϱ is a tree. From (b) it follows that $\pi_\alpha^{(\alpha)} = \pi_k^{(\alpha)}$ for $k > \alpha$ and $\varrho^{(\alpha)} = \pi_\alpha^{(\alpha)}$. It follows that $\mathrm{Sp}(\varrho''\{n\})$ is cofinal in $\varrho''\{n\}$ and closed under countable monotonous unions.

The rest of proof is the same as in the previous section (classes Y_p are suitably coded). □

Metatheorem 9.10. $\mathrm{Con}(\overline{\mathrm{AST}}) \Rightarrow \mathrm{Con}(\mathrm{AST} + \neg(\mathrm{SC}))$.

DEMONSTRATION: As in Metatheorem 8.11, we can prove (in $\overline{\mathrm{AST}} + \overline{(Q)}$) that $P[\mathcal{P}_\mathcal{A}, \mathfrak{G}_\mathcal{A}] \Vdash \mathrm{AST} + \neg(\mathrm{SC})$. □

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(*Received* August 16, 1990)