On ω^2-saturated families

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Abstract. If there is no inner model with measurable cardinals, then for each cardinal λ there is an almost disjoint family $\mathcal{A}_\lambda$ of countable subsets of λ such that every subset of λ with order type $\geq \omega^2$ contains an element of $\mathcal{A}_\lambda$.

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1. Introduction.

In this paper we use the standard set-theoretical notation throughout, cf. [7]. The usual ordering of ordinals will be denoted by $<_{\mathrm{On}}$. For $A \subset \mathrm{On}$, write $\mathrm{tp}(A)$ for the order type of $\langle A, <_{\mathrm{On}}\rangle$.

Given a set $X \subset \mathrm{On}$ and an ordinal α, take $\big[X\big]^\alpha = \{a \subset X : |a| = |\alpha|\}$ and $\big(X\big)^\alpha = \{a \subset X : \mathrm{tp}(a) = \alpha\}$. For $\mathcal{A} \subset \big[X\big]^\omega$ and $Y \subset X$, let

$$\mathrm{I}_{\mathcal{A}} = \{a \subset X : |a \setminus \cup\mathcal{C}| < \omega \text{ for some finite } \mathcal{C} \subset \mathcal{A}\}$$

and $\mathrm{I}^+_{Y,\mathcal{A}} = \big[Y\big]^\omega \setminus \mathrm{I}_{\mathcal{A}}$.

An almost disjoint family $\mathcal{A} \subset \big[X\big]^\omega$ is called *ω^2-saturated* (*saturated*) *for* $Y \subset X$, iff for each $b \in \big(Y\big)^{\omega^2}$ ($b \in \mathrm{I}^+_{Y,\mathcal{A}}$) there is an $a \in \mathcal{A}$ with $a \subset b$.

Let $S_2(\alpha)$ ($S(\alpha)$) mean that *"there exists an almost disjoint, ω^2-saturated* (*saturated*) *family on α"*. For an ordinal β, take

$$\mathrm{cov}(\big[\beta\big]^\omega) = \min\{|\mathcal{B}| : \mathcal{B} \subset \big[\beta\big]^\omega \text{ and } \forall a \in \big[\beta\big]^\omega\ \exists b \in \mathcal{B}\ a \subset b\}.$$

In [5], the following problem was raised: for what cardinals λ is there an almost disjoint family of countable subsets of λ which refines $\big[\lambda\big]^{\omega_1}$? B. Balcar, J. Dočkálková and P. Simon [1] showed $S_2(\kappa)$ for $\kappa < (2^\omega)^{+\omega}$. P. Komjath [8] proved that if V=L, then for each $\lambda < \aleph_{\omega_1}$ there is an almost disjoint family $\mathcal{A} \subset \big[\lambda\big]^\omega$ that refines $\big[\lambda\big]^{\omega_1}$. A. Hajnal, I. Juhász and L. Soukup [6] showed that if one adds ω_1 dominating reals to the ground model iteratedly, then in the generic extension $\mathrm{S}(\kappa)$ holds for each κ. M. Goldstern, H. Judah and S. Shelah proved that if $S(\omega)$, $\lambda^\omega = \lambda^+$ and $\square_\lambda$ for each singular cardinal λ with cofinality ω, then $S(\alpha)$ for each α. The author of the present paper noticed that $\lambda^\omega = \lambda^+$ can be replaced by the assumption $\mathrm{cov}(\big[\lambda\big]^\omega) = \lambda^+$ in their proof, see [4]. Using their technique, we prove the following result.

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Theorem 1.1. *Assume that $\Box_\lambda$ holds and $\operatorname{cov}([\lambda]^\omega) = \lambda^+$ for each singular cardinal λ with cofinality ω. Then $S_2(\kappa)$ holds for each κ.*

Although it is still unknown whether one can prove $\mathrm{S}(\kappa)$ or $S_2(\kappa)$ for each κ in ZFC, this theorem shows that the failure of $S_2(\kappa)$ for some κ is a large cardinal assumption: it demands the failure of the covering lemma for K.

2. Proof of the theorem.

Given a set X of cardinality λ and a sequence $\mathcal{X} = \{x_\alpha : \alpha < \lambda^+\} \subset [X]^\omega$, a family $\left\langle A^k_\alpha : k < \omega, \alpha < \lambda^+ \right\rangle$ is called $\langle X, \mathcal{X}\rangle$-*nice* iff conditions (A)–(E) below hold:

(A) $A^k_\alpha \subset X$, $|A^k_\alpha| < \lambda$,
(B) $A^k_\alpha \subset A^{k+1}_\alpha$, $\bigcup\limits_{k\in\omega} A^k_\alpha = X$,
(C) $\forall \alpha < \beta\ \exists k_{\alpha,\beta}\ \forall k \ge k_{\alpha,\beta}\ A^k_\alpha \subset A^k_\beta$,
(D) $x_\alpha \subset A^0_{\alpha+1}$,
(E) if $\operatorname{cf}(\alpha) > \omega$, then

$$\bigcup_k [A^k_\alpha]^\omega \subset \bigcup_{\gamma<\alpha} \bigcup_l [A^l_\gamma]^\omega.$$

Lemma 2.1. *Given a set X of cardinality $\lambda > \operatorname{cf}(\lambda) = \omega$ and a sequence $\mathcal{X} = \{x_\alpha : \alpha < \lambda^+\} \subset [X]^\omega$, if $\Box_\lambda$ holds, then there is an $\langle X, \mathcal{X}\rangle$-nice family.*

Proof: It was proved in [4]. Since the property (E) was not explicitly claimed and to make this note self-contained, we give a proof.

Let $\langle C_\alpha : \alpha < \lambda^+\rangle$ be a $\Box_\lambda$-sequence, fix an increasing sequence of cardinals, $\langle \lambda_k : k < \omega\rangle$, which is cofinal in λ and write $X = \{\xi_\alpha : \alpha < \lambda\}$.

We will construct the family $\left\langle A^k_\alpha : k < \omega, \alpha < \lambda^+ \right\rangle$ by induction on α.

Take $A^k_0 = \{\xi_\alpha : \alpha < \lambda_k\}$. Assume $\left\langle A^k_\gamma : k < \omega, \gamma < \alpha \right\rangle$ is constructed.

If $\alpha = \beta + 1$, then put $A^k_\alpha = A^k_\beta \cup x_\beta$.

If α is limit, then take $C^*_\alpha = C'_\alpha \cup (C_\alpha \setminus \sup C'_\alpha)$, where C'_α is the set of limit points of C_α, pick $l_\alpha \in \omega$ with $|C'_\alpha| \le \lambda_{l_\alpha}$ and put

$$A^k_\alpha = \begin{cases} \emptyset & \text{if } k < l_\alpha, \\ \bigcup\limits_{\gamma \in C^*_\alpha} A^k_\gamma & \text{if } k \ge l_\alpha. \end{cases}$$

By induction on α, it is straightforward that $|A^k_\alpha| \le \lambda_k$ and the family $\left\langle A^k_\alpha : k < \omega, \alpha < \lambda^+ \right\rangle$ satisfies (A)–(E). $\Box$

Lemma 2.2. *Assume that $X \subset \mathrm{On}$, $X \subset \bigcup_n A_n$, $\mathcal{A} \subset \bigcup_n \left[A_n\right]^\omega$ is an almost disjoint family which is ω^2-saturated for all A_n. If $S_2(\mathrm{tp}(X))$, then there is an almost disjoint family $\mathcal{B} \supset \mathcal{A}$ with $\mathcal{B} \setminus \mathcal{A} \subset \left[X\right]^\omega$ such that $\mathcal{B}$ is ω^2-saturated for X.*

Proof: Since ω^2 cannot be the sum of finitely many smaller ordinals, the family $\mathcal{A}$ is ω^2-saturated for $\bigcup_{m \le n} A_m$ and so we can assume that

$$A_0 \subset A_1 \subset \dots A_n \subset \dots$$

Fix an almost disjoint family $\mathcal{C} \subset \left(X\right)^\omega$ witnessing $S_2(\mathrm{tp}(X))$ and take $\mathcal{D} = \{c \in \mathcal{C} : |n : c \cap (A_{n+1} \setminus A_n) \ne \emptyset| = \omega\}$. For $d \in \mathcal{D}$, pick a set $d^* \in \left[d\right]^\omega$ with $|d^* \cap A_n| < \omega$ for each $n < \omega$. Put $\mathcal{B} = \mathcal{A} \cup \{d^* : d \in \mathcal{D}\}$.

First let us observe that $\mathcal{B}$ is almost disjoint. Indeed, if $a \in \mathcal{A}$, then there is an n with $a \subset A_n$, so for each $d \in \mathcal{D}$, we have $|a \cap d^*| \le |A_n \cap d^*| < \omega$.

To show that $\mathcal{B}$ is ω^2-saturated for X, consider a $Y \in \left(X\right)^{\omega^2}$ and we will find a $b \in \mathcal{B}$ with $b \subset Y$.

Write $Y = \bigcup_m Y_m$, where $Y_0 <_{\mathrm{rm\ On}} Y_1 <_{\mathrm{On}} \dots <_{\mathrm{On}} Y_m <_{\mathrm{On}} \dots$ and $\mathrm{tp}(Y_m) = \omega$. Put $Z_m = \bigcup_{l \le m} Y_l$.

Let $n \in \omega$. If $\mathrm{tp}(Y \cap A_n) = \omega^2$, then there is an $a \in \mathcal{A}$ with $a \subset Y$. So we can assume that $\mathrm{tp}(Y \cap A_n) < \omega^2$. Thus we can choose a natural number $f(n) \ge n$ such that $Y_m \cap A_n$ is finite for each $m \ge f(n)$.

Put

$$Y^* = Y \setminus \bigcup \{Y_m \cap A_n : m, n \in \omega, m \ge f(n)\}.$$

Then $Y_m \setminus Y^* = \bigcup_{m \ge f(n)} (Y_m \cap A_n)$ is finite. So $\mathrm{tp}(Y^* \cap Y_m) = \omega$ and $\mathrm{tp}(Y^*) = \omega^2$. On the other hand, $Y^* \cap A_n \subset \bigcup_{m < f(n)} Y_m \subset Z_{f(n)}$.

We will choose $c_k \in \mathcal{C}$ and $m_k \in \omega$ by induction on k such that $c_k \subset (Y^* \setminus Z_{m_{k-1}}) \cap Z_{m_k}$. To simplify our notation, put $m_{-1} = -1$ and $Z_{-1} = \emptyset$. If m_{k-1} is chosen, pick a $c_k \in \mathcal{C} \cap (Y^* \setminus Z_{m_{k-1}})^\omega$. If $c_k \in \mathcal{D}$, then $c_k^* \in \mathcal{B} \cap \left(Y\right)^\omega$, and so we are done. Thus we can assume that $c_k \notin \mathcal{D}$. So there is an n with $c_k \subset A_n$. Taking $m_k = f(n)$, it follows that $c_k \subset Y^* \cap A_n \subset Z_{f(n)} = Z_{m_k}$. So the inductive step can be carried out.

After constructing the sequence $\{c_k : k < \omega\}$, fix for each $k \in \omega$ a partition (c_k^0, c_k^1) of c_k into infinite pieces and take $W = \bigcup_k c_k^0$. Since $W \in \left(X\right)^{\omega^2}$, there is a $c \in \mathcal{C}$ with $c \subset W$. If $c \notin \mathcal{D}$, then there were an n with $c \subset A_n$. Thus $c \subset Y^* \cap A_n \subset Z_{f(n)}$. Hence $c \subset \bigcup_{m_k \le f(n)} c_k^0$ and so $c \cap c_k$ is infinite for some k. But $c \ne c_k$ because $c \cap c_k^1 = \emptyset$. But it is a contradiction, because $\mathcal{C}$ is almost disjoint. Thus $c \in \mathcal{D}$ and $c^* \in \mathcal{B} \cap \left(Y\right)^\omega$, which proves that $\mathcal{B}$ is really ω^2-saturated for X.

□

Lemma 2.3. *Assume that λ is a singular cardinal with cofinality ω, $\square_\lambda$ holds and $\mathrm{cov}([\lambda]^\omega) = \lambda^+$. If $S_2(\alpha)$ for each $\alpha < \lambda$, then $S_2(\beta)$ for each $\beta < \lambda^+$.*

Proof: Let $\lambda \leq \beta < \lambda^+$ and fix a sequence $\mathcal{X} = \{x_\nu : \nu < \lambda^+\} \subset [\beta]^\omega$ witnessing $\mathrm{cov}([\beta]^\omega) = \lambda^+$.

By Lemma 2.1, there is a $\langle \beta, \mathcal{X} \rangle$-nice family $\left\langle A^k_\alpha : k < \omega, \alpha < \lambda^+ \right\rangle \subset [\beta]^{<\lambda}$. By induction on $\nu < \lambda^+$, we will define almost disjoint families $\mathcal{A}_\nu \subset (\beta)^\omega$ such that

(i) $\mathcal{A}_\nu \subset \bigcup_k (A^k_\nu)^\omega$,

(ii) $\mathcal{A}_\mu \subset \mathcal{A}_\nu$ for $\mu < \nu$,

(iii) $\mathcal{A}_\nu$ is ω^2-saturated for A^k_ν for each $k \in \omega$.

To simplify our notation, write $A^{-1}_\nu = \emptyset$ and $X^k_\alpha = A^k_\alpha \setminus A^{k-1}_\alpha$.

Case 1. $\nu = 0$.

Choose almost disjoint, ω^2-saturated families $\mathcal{A}_{0,k} \subset (X^k_0)^\omega$ for each $k \in \omega$ and take $\mathcal{A}_0 = \bigcup_k \mathcal{A}_{0,k}$.

Case 2. $\nu = \mu + 1$.

For each $k \in \omega$, apply Lemma 2.2 taking X^k_ν as X, A^n_μ as A_n for each $n \in \omega$ and $\mathcal{A}_\mu$ as $\mathcal{A}$ to get the family $\mathcal{A}_{\nu,k}$ which is ω^2-saturated for X^k_ν. Put $\mathcal{A}_\nu = \bigcup_k \mathcal{A}_{\nu,k}$. By (C), $\mathcal{A}_\nu \subset \bigcup_k (A^k_\nu)^\omega$.

Case 3. *ν is a limit ordinal with cofinality ω.*

Fix an increasing, cofinal sequence of ordinals $\{\nu_i : i < \omega\}$ in ν. Take $\mathcal{A}' = \bigcup_{\mu<\nu} \mathcal{A}_\mu$. Let $\{A'_n : n \in \omega\}$ be an enumeration of $\{A^k_{\nu_i} : i, k \in \omega\}$. Then $\mathcal{A} \subset \bigcup_n (A'_n)^\omega$ by (C). For each $k \in \omega$, apply Lemma 2.2 taking X^k_ν as X, A'_n as A_n for each $n \in \omega$ and $\mathcal{A}'$ as $\mathcal{A}$ to get the family $\mathcal{A}_{\nu,k}$ which is ω^2-saturated for X^k_ν. Take $\mathcal{A}_\nu = \bigcup_k \mathcal{A}_{\nu,k}$.

Case 4. *ν is limit with uncountable cofinality.*

Simply take $\mathcal{A}_\nu = \bigcup_{\mu<\nu} \mathcal{A}_\mu$. It works by (E).

The inductive construction is done. Put $\mathcal{A} = \bigcup_{\nu<\lambda^+} \mathcal{A}_\nu$. It is obviously almost disjoint and ω^2-saturated for β by (D). So $S_2(\beta)$ is proved. $\square$

Proof of Theorem 1.1: We will prove $S_2(\beta)$ by induction on β. If $\beta < (2^\omega)^{+\omega}$, then $S_2(\beta)$ holds by [1].

Assume now that we know $S_2(\alpha)$ for each $\alpha < \beta$. Let $\kappa = |\beta|$ and write $\beta = \{x_\mu : \mu < \kappa\}$. Let $X_\nu = \{x_\mu : \mu < \nu\}$ for $\nu \leq \kappa$. We will define almost disjoint, ω^2-saturated families $\mathcal{A}_\nu \subset (X_\nu)^\omega$ for $\nu \leq \kappa$ such that $\mathcal{A}_\mu \subset \mathcal{A}_\nu$ whenever $\mu < \nu$. Let $\mathcal{A}_0 = \emptyset$. If $\nu = \mu + 1$, put $\mathcal{A}_\nu = \mathcal{A}_\mu$. If ν is limit, then take $\mathcal{A}^*_\nu = \bigcup_{\mu<\nu} \mathcal{A}_\mu$. If $\mathrm{cf}(\nu) > \omega$, then $\mathcal{A}_\nu = \mathcal{A}^*_\nu$ is ω^2-saturated. If $\mathrm{cf}(\nu) = \omega$ then we can apply Lemma 2.2

to get an ω^2-saturated extension $\mathcal{A}_\nu$ of $\mathcal{A}_\nu^*$ provided $S_2(\mathrm{tp}(X_\nu))$ holds. But this holds by the induction hypothesis for $\nu < \kappa$ and by Lemma 2.3 for $\nu = \kappa$. So $\mathcal{A}_\kappa$ is an ω^2-saturated family on β. $\square$

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