Measurable cardinals and category bases

Andrzej Szymański

Abstract. We show that the existence of a non-trivial category base on a set of regular cardinality with each subset being Baire is equiconsistent to the existence of a measurable cardinal.

Keywords: category base, precipitous ideal, measurable cardinal

Classification: 54E52, 03E35, 03E55

The existence of a non-trivial measure m on a set (of regular cardinality) such that each its subset is m-measurable, is equiconsistent to the existence of a measurable cardinal [S].

The existence of a non-trivial topology on a set of regular cardinality such that each its subset has the Baire property, is equiconsistent to the existence of a measurable cardinal [KT].

Category bases constitute a common generalization of measure and topological structures. In this case the family of Baire sets plays the role of the σ-field of measurable sets or the σ-field of sets with the Baire property. Therefore it is natural to ask about the existence of a non-trivial category base on a set of regular cardinality such that each its subset would be a Baire set. We show that the answer is the same as for the measure or topology, namely, it is equiconsistent to the existence of a measurable cardinal.

Following J.C. Morgan [M1], we say that a pair (X,C) is a category base, if the elements in C, called regions, satisfy the following axioms:

(1) Every point of X belongs to some region, i.e., $X = \bigcup C$.

(2) Let A be a region and let D be any non-empty family of disjoint regions which has the power less than the power of C.

(a) If $A \cap (\bigcup D)$ contains a region, then there is a region $B \in D$ such that $A \cap B$ contains a region.

(b) If $A \cap (\bigcup D)$ contains no region, then there is a subregion of A which is disjoint from every region in D.

As is readily verified, every topology or all subsets of positive measure with respect to a complete σ-finite measure is a category base. With respect to a given category base, J.C. Morgan has defined abstract versions of the concepts of nowhere dense sets or null sets, sets with the Baire property, and so on.

A set is singular, if every region contains a subregion which is disjoint from it. A countable union of singular sets is called a meager set. A set which is not meager is called an abundant set. A set is abundant everywhere in a region, if every subregion of the region intersects the set on an abundant set. A set is a Baire set, if every region contains a subregion, whose intersection with either the set or its complement is meager.

For a given category base (X, C) let $M(C)$ and $B(C)$ denote the family of all meager subsets of X and Baire subsets of X, respectively. In the case that C is a topology on X, the singular sets coincide with nowhere dense sets, $M(C)$ coincides with the family of first category sets and $B(C)$ coincides with the family of sets with the Baire property. In the case that C is the family of all subsets of positive measure with respect to a σ-finite complete measure, the singular and meager sets coincide with the null-sets and $B(C)$ coincides with the σ-field of all measurable sets. We are going to utilize also the following basic facts from the theory of category bases.

Fact 1. Let (X, C) be a category base. If A is a region and B is a Baire set such that $A \cap B$ is abundant, then there exists an abundant everywhere in itself subregion D of A such that $D - B$ is meager.

Fact 2. The intersection of two regions is either a singular set or contains a region.

For proofs of these two particular facts as well as for a more extensive treatment of category bases, we refer to [M1]. The lemma below is new and it constitutes the basic ingredient in the proof of our main theorem.

Let (X, C) be a category base. We say that R is a category decomposition of X, if R is a disjoint family of regions such that each region intersects a member of R on a non-singular set. Notice few elementary facts about category decompositions.

Fact 3. Let R be a category decomposition of X. Then:

(i) $X - (\bigcup R)$ is singular;
(ii) if $E \subset X$ and $E \cap A$ is singular for every $A \in R$, then E is singular;
(iii) if $F \subset X$ and $F \cap A$ is meager for every $A \in R$, then F is meager.

Let S be a Baire set. An $M(C)$-partition of S is a maximal collection W of abundant Baire subsets of S such that $E \cap F \in M(C)$ for any distinct $E, F \in W$.

Lemma. *Let S be an abundant Baire set. Then for any category decomposition R of X and for any $M(C)$-partition W of S, there exists a category decomposition Q of X satisfying the following:*

(a) *every region in Q is a subregion of a region in R;*
(b) *if $A \in Q$ and $A \cap S$ is abundant, then A is abundant everywhere in itself and there exists $B \in W$ such that $A - B$ is meager.*

Proof: Let $C = \{C_\alpha : \alpha < \kappa\}$, where $\kappa = |C|$. For every $\beta < \kappa$ we will define a region A_β so that the following are satisfied:

(1) the family $\{A_\alpha : \alpha \le \beta\}$ is a disjoint family for every $\beta < \kappa$;
(2) C_β intersects some A_α with $\alpha \le \beta$ on a non-singular set;
(3) A_β is a subregion of some region from R;
(4) if $A_\beta \cap S$ is abundant, then A_β is abundant everywhere in itself and there exists $B \in W$ such that $A_\beta - B$ is meager.

We proceed to define A_0. Since R is a category decomposition of X, there exists $D \in R$ such that $C_0 \cap D$ is not singular. By virtue of Fact 2, there exists a region $A \subset C_0 \cap D$. If $A \cap S$ is meager, then we set $A_0 = A$. If $A \cap S$ is abundant then, since W is an $M(C)$-partition of S, there exists $B \in W$ such that $A \cap B$ is abundant. According to Fact 1, there exists a subregion $A^{'}$ of A which is abundant everywhere in itself and yet $A^{'} - B$ is meager. We then set A_0 to be $A^{'}$.

Let $\beta < \kappa$ and suppose A_α has been well defined for every $\alpha < \beta$. If $C_\beta \cap A_\alpha$ is non-singular for some $\alpha < \beta$, then we set $A_\beta = A_0$. If $C_\beta \cap A_\alpha$ is singular for each $\alpha < \beta$, then there exists a subregion A of C_β disjoint with every $A_\alpha, \alpha < \beta$. We may assume that A is also a subregion of some region from R. If $A \cap S$ is meager, then we set $A_\beta = A$. If $A \cap S$ is abundant, then, arguing as in the first part, there exist $B \in W$ and a subregion $A^{'}$of A such that $A^{'}$ is abundant everywhere in itself and $A^{'} - B$ is meager. Then we set $A_\beta = A^{'}$.

It follows immediately from the conditions (1)–(4) that the family $Q = \{A_\alpha : \alpha < \kappa\}$ is a category decomposition of X satisfying the conditions (a) and (b). □

Now we will need some facts about ideals. Let κ be an infinite cardinal. A set I of subsets of κ is an ideal over κ, if

(i) $\{\alpha\} \in I$ for each $\alpha \in \kappa$ and $\kappa \notin I$.
(ii) If $X \in I$ and $Y \subset X$, then $Y \in I$.
(iii) If $X \in I$ and $Y \in I$, then $X \cup Y \in I$.

A σ-complete ideal is an ideal which is closed under countable unions. If (X, C) is a non-trivial category base in the sense that one-point sets are singular and the set X is abundant, then $M(C)$ is a σ-ideal over $|X|$. Customarily it is convenient to adopt the following terminology about ideals. Let I be an ideal over κ. Elements of I are called sets of I-measure zero; elements of $I^+ = P(\kappa) - I$ are called sets of positive I-measure; elements of $I^* = \{\kappa - x : x \in I\}$ are called sets of I-measure one; when the context is clear, we drop the prefix "I-" in all above phrases.

Let S be a set of positive measure. An I-partition of S is a maximal collection W of subsets of S of positive measure such that $X \cap Y \in I$ for any distinct $X, Y \in W$. An I-partition W_1 of S is a refinement of an I-partition W_2 of S, $W_1 \leq W_2$, if every element of W_1 is a subset of an element of W_2.

The ideal I is weakly precipitous, if I is σ-complete and whenever S is a set of positive measure and $\{W_n : n \in \omega\}$ are I-partitions of S such that $W_0 \geq W_1 \geq \cdots \geq W_n \geq \ldots$, then there exists a sequence $X_0 \supseteq X_1 \supseteq \cdots \supseteq X_n \supseteq \ldots$ such that $X_n \in W_n$ for each n, and $\bigcap\{X_n : n \in \omega\} \neq \emptyset$. The following is of a basic importance for us.

Fact 4. If a regular uncountable cardinal κ carries a weakly precipitous ideal, then κ is measurable in some transitive model of ZFC.

This strong theorem is due to T. Jech, M. Magidor, W. Mitchell and K. Prikry [JP]. Originally, the theorem was formulated for precipitous ideals but the proof works for weakly precipitous ideals as well. For more extensive treatment about ideals, we refer to Jech's book [J].

Theorem. CON (ZFC + *"there exists a non-trivial category base* (X, C) *such that* $|X|$ *is a regular cardinal and* $B(C) = P(X)$*"), if and only if* CON (ZFC + *"there exists a measurable cardinal"*).

Proof: Assume first the consistency of the existence of a measurable cardinal. According to a theorem of Solovay [S], it implies the consistency of the existence of a set X of regular cardinality not exceeding the power of the continuum and a probabilistic non-trivial measure m on $P(X)$. Hence the family C of all sets of positive m-measure is a non-trivial category base on X such that $B(C) = P(X)$.

To prove the converse implication, we rely on Fact 4 and we will try to discover a weakly precipitous ideal. In fact, if (X, C) is a non-trivial category base such that $|X|$ is a regular cardinal and $B(C) = P(X)$, then $M(C)$ is weakly precipitous.

Clearly, $M(C)$ is σ-complete. Let $S \subset X$ be a set of positive measure and let $\{W_n : n \in \omega\}$ be an appropriate sequence of partitions of S. We define inductively a sequence $\{Q_n : n \in \omega\}$ of category decompositions of X. We set Q_0 as a category decomposition of X obtained by an application of Lemma for $R = \{X\}$ and $W = W_0$. We set Q_{n+1} as a category decomposition of X obtained by an application of Lemma for $R = Q_n$ and $W = W_{n+1}$. Let $Q_{n^*} = \{A \in Q_n : A \cap S \text{ is abundant }\}$ and let $B(A)$ denote a unique element in W_n such that $A - B(A)$ is meager whenever $A \in Q_{n^*}$.

Observe that both sets $E_n = S - (\bigcup Q_{n^*})$ and $F_n = \bigcup\{A - B(A) : A \in Q_{n^*}\}$ are meager for each $n \in \omega$. To see this, notice that both E_n and F_n intersect every member of the family Q_n on a meager set. Since Q_n is a category decomposition of X, they are meager, according to Fact 3 (iii).

Consider now the set $S^* = S - (\bigcup\{E_n \cup F_n : n \in \omega\})$. Since S is abundant, S^* is not empty and let p be its arbitrary point. Since $S^* \subset S - (\bigcup\{E_n : n \in \omega\}) = S \cap \bigcap\{\bigcup Q_{n^*} : n \in \omega\}$, for each $n \in \omega$ there exists a unique element A_n of Q_{n^*} containing the point p. Let $B_n = B(A_n)$ be a unique element of the partition W_n such that $A_n - B_n$ is meager. Since $A_n - B_n \subset F_n$, $p \in B_n$ for every $n \in \omega$. At the end let us prove that the selected sets B_n form a decreasing sequence. Notice at first that this fact is true about the sequence A_n, since each Q_n is a disjoint family being a refinement of the preceding one. Let $B \in W_{n-1}$ be such that $B_n \subset B$. If B were different from B_{n-1}, then $B \cap B_{n-1}$ would be meager. Since $A_{n-1} - B_{n-1}$ is meager, $B \cap A_{n-1}$ is meager and so is $B \cap A_n$. From the other side, every A_n is abundant and $A_n - B_n$ is meager. Since $B_n \subset B, B \cap A_n$ would be abundant; a contradiction. □

References

[J] Jech T., *Set Theory*, Academic Press, 1978.

[JP] Jech T., Magidor M., Mitchell W., Prikry K., *Precipitous ideals*, Journal of Symbolic Logic **45**, 1–8.

[KT] Kunen K., Szymański A., Tall F., *Baire irresolvable spaces and ideal theory*, Acta Math. Silesiana, No. 14 (1986), 98–107.

[M1] Morgan J.C.II, *Point Set Theory*, Marcel Dekker, Inc., 1990.
[S] Solovay R., *Real valued measurable cardinals*, in Axiomatic Set Theory, Proc. Symp. Pure Math. 13.1 (D. Scott ed.), 397–428, Amer. Math. Soc., 1971.

Slippery Rock University, Slippery Rock, PA 16057, USA

(*Received* June 7, 1990)