The trace theorem $W_p^{2,1}(\Omega_T) \ni \ f \ \mapsto \nabla_x f \in W_p^{1-1/p,1/2-1/2p}(\partial\Omega_T)$ revisited

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Abstract. Filling a possible gap in the literature, we give a complete and readable proof of this trace theorem, which also shows that the imbedding constant is uniformly bounded for $T \downarrow 0$. The proof is based on a version of Hardy's inequality (cp. Appendix).

Keywords: trace theory, anisotropic Sobolev spaces

Classification: 46E35

Introduction.

The imbedding theorem described in the title can be found in LADYSHENSKAYA et al. [L/S/U, Chapter II, Lemma 3.4]. However, none of the references cited there seems to contain a complete proof. The theorem is also stated in IL'IN [I, Theorem 8.4]; but there too, no proof is given. Things look even worse, if we ask for the dependence of the imbedding constant $c(T)$ on the height T of the space-time cylinder (for small T). In some applications of this trace theorem to nonlinear problems, one needs $c(T) \leq c_0$ for all T small (cf. WEIDEMAIER [W], particularly the Appendix). However, the formulation in IL'IN [I, Theorem 8.4], exhibits an explosion of $c(T)$ for $T \downarrow 0$. To settle things, we shall give in this note a detailed proof for the imbedding, which also shows the uniformity of $c(T)$ for $T \downarrow 0$.

The paper is organized as follows: in Chapter 1 we deduce an integral representation for $\nabla_x f$ in terms of $\partial_t f, \partial_x^2 f$, which is the basis for the estimates in Chapter 2.

Let us fix the notation: $\Omega_T := \Omega \times (0,T)$ with the typical point $(x,t) \in \Omega_T$; here $\Omega \subset \mathbb{R}^n$. The prime characterizes $(n-1)$-dimensional quantities : thus we write $x \in \mathbb{R}^n$ as $x = (x', x_n), x' \in \mathbb{R}^{n-1}; Q^{n-1}(\underline{a}', \underline{b}')$ is the open parallelepiped $\prod_{j=1}^{n-1}(a_i, b_i)$, when $\underline{a}' = (a_1, \dots, a_{n-1}), \underline{b}' = (b_1, \dots, b_{n-1}); Q^{n-1}(\lambda) := Q^{n-1}(-\lambda\underline{1}', \lambda\underline{1}')$ for $\lambda \in \mathbb{R}$; here
$\underline{1}' := (1, \cdots, 1) \in \mathbb{N}^{n-1}; Q_+^n(\lambda) := Q^{n-1}(\lambda) \times (0, \lambda)$; the superscript $\check{}$ always indicates the deletion of a coordinate (the n-th. one, if not further specified) , e.g. $\overset{i}{\check{y}} = (y_1, \cdots, y_{i-1}, y_{i+1}, \cdots, y_n) \quad (1 \leq i \leq n)$ and $\check{Q}^{n+1}(\underline{a}, \underline{b}) := \prod_{\substack{i=1 \\ i \neq n}}^{n+1} (a_i, b_i)$.
$W_p^{2,1}(\Omega_T) := \{u \,|\, \partial_x^\alpha u, \partial_t u \text{ (distr. sense)} \in L_p(\Omega_T) \ \forall\ |\alpha| \leq 2\}$ with the obvious norm.

I thank Prof. V.A. Solonnikov, Leningrad, for valuable hints

For a bounded domain $\Omega \subset \mathbb{R}^n, \partial\Omega \in C^2$ means that $\partial\Omega$ is a C^2-hypersurface. The spaces $W_p^{\alpha,\beta}(\partial\Omega_T)$ $\quad(\alpha,\beta \in (0,1))$ are defined as usual, via a partition of unity on $\partial\Omega$, and using local charts. We use the notation c^* to emphasize the non-dependence of the constant c on the quantity T (for T small).

1. Integral representation.

Our starting point is an integral representation for $\partial^{\underline{\nu}} f$ in terms of f: if f is smooth and defined on $\overline{Q^{n-1}(-\lambda\underline{1}', 2\lambda\underline{1}')\times[0,2\lambda]\times[0,3T]}$, then we have (cf. IL'IN/ SOLONNIKOV [I/S, p. 70, (6)] with $m_i = 0, k_i = l_i$)

$$\partial^{\underline{\nu}} f(x,t) = \frac{A}{T^r} \underset{Q^{n+1}(0,T^{\underline{\kappa}})}{\int\cdots\int} f((x,t)+y)\Pi(y,T)\,dy + $$
$$+\sum_{i=1}^{n+1} B_i \int_0^T v^{-(1+r)} \underset{Q^{n+1}(0,v^{\underline{\kappa}})}{\int\cdots\int} f((x,t)+y)\Pi_i(\overset{i}{\check{y}},v)\partial_i^{l_i}\psi_i(y_i,v)\,dy\,dv$$

for $(x,t)\in\overline{Q_+^n}(\lambda)\times[0,T], T\le T_0(\lambda)$ and $\nu_j \le l_j - 1$, where (cp. [I/S, pp. 69–70])

$$\Pi(y,T) := \prod_{j=1}^{n+1} \partial_j^{l_j}\chi_j(y_j,T)$$
$$\chi_j(y_j,T) := y_j^{l_j-\nu_j-1}\int_{y_j}^{T^{\kappa_j}} (T^{\kappa_j}-s)^{\mu_j}s^{\lambda_j}\,ds\ ,$$
$$\Pi_i(\overset{i}{\check{y}},v) := \prod_{\substack{j=1\\ j\neq i}}^{n+1} \partial_j^{l_j}\chi_j(y_j,v),$$
$$\psi_i(y_i,v) := y_i^{l_i+\lambda_i-\nu_i}\cdot(v^{\kappa_i}-y_i)^{\mu_i}$$

with certain parameters $\mu_j,\lambda_j\in\mathbb{N}$ and certain $A, B_i\in\mathbb{R}$; here $T^{\underline{\kappa}} := (T^{\kappa_1},\cdots,T^{\kappa_{n+1}}),\ r := \underline{\kappa}\cdot(\underline{1}+\underline{\lambda}+\underline{\mu}),\ \underline{1} := (1,\cdots,1)\in\mathbb{N}^{n+1}$.
In the sequel we fix $\underline{l} := (2,\cdots,2,1)\in\mathbb{N}^{n+1}, \underline{\kappa} = (\underline{\kappa}',\kappa_n,\kappa_{n+1}) := \frac{1}{\underline{l}} = (\frac{1}{2},\cdots,\frac{1}{2},1)$ and choose the parameters μ_j,λ_j so large that $\partial_j^k\psi_j(y_j,v)$ vanishes for $y_j = 0,\ y_j = T^{\kappa_j},\ 1\le k\le l_j$. Hence, integrating by parts and introducing $K_i(y,v) := \Pi_i(\overset{i}{\check{y}},v)\,\psi_i(y_i,v)$ $(0\le y_i\le v^{\kappa_i})$, we have shown that

$$\partial^{\underline{\nu}} f(x,t) = \frac{A}{T^r} \underset{Q^{n+1}(0,T^{\underline{\kappa}})}{\int\cdots\int} f((x,t)+y)\Pi(y,T)\,dy + \tag{1.1}$$
$$+\sum_{i=1}^{n+1} \tilde{B}_i \int_0^T v^{-(1+r)} \underset{Q^{n+1}(0,v^{\underline{\kappa}})}{\int\cdots\int} \partial_i^{l_i} f((x,t)+y)K_i(y,v)\,dy\,dv.$$

The kernels Π, K_i in this representation satisfy (uniformly w.r.t. $y \in Q^{n+1}(0, v^{\underline{\kappa}})$)

$$(1.2) \qquad |\partial_y^{\underline{\alpha}}\Pi(y,v)| \le c \cdot v^{r-\underline{\kappa}\cdot(\underline{1}+\underline{\nu}+\underline{\alpha})} \qquad \forall\ |\underline{\alpha}| \le 2$$

$$(1.3) \qquad |\partial_{n+1}^s K_i(y,v)| \le c \cdot y_n^\varepsilon \cdot v^{r+1-\underline{\kappa}\cdot(\underline{1}+\underline{\nu})-\varepsilon\kappa_n - s}$$

$$(\partial_{n+1} := \partial_{y_{n+1}}, 0 \le s \le 1,\ 1 \le i \le n+1,\ \varepsilon \in [0,1)).$$

For the proof of these two inequalities, we first note that $\partial_j^{l_j+\alpha_j}\chi_j(y_j,v)$ is a linear combination of terms of the form $(v^{\kappa_j}-y_j)^{\rho_1}y_j^{\rho_2}$ with $\rho_1+\rho_2 = \mu_j+\lambda_j-\nu_j-\alpha_j,\ \rho_2 > 0$ (for λ_j large) and consequently

$$|\partial_j^{l_j+\alpha_j}\chi_j(y_j,v)| \le c \cdot y_j^\varepsilon \cdot v^{-\kappa_j(\varepsilon+\alpha_j)} \cdot v^{\kappa_j(\mu_j+\lambda_j-\nu_j)} \qquad (0 \le y_j \le v^{\kappa_j})$$

for arbitrary $\varepsilon \in [0,1[$; this implies (for $1 \le k \le n-1$)

$$|\partial_{n+1}^s \Pi_k(\overset{k}{\check{y}},v)| \le c \cdot y_n^\varepsilon \cdot v^{-\kappa_n\varepsilon-\kappa_{n+1}\cdot s} \cdot v^{\underline{\kappa}(\underline{\mu}+\underline{\lambda}-\underline{\nu})-\kappa_k\delta_k}$$
$$|\partial_{n+1}^s \Pi_n(\overset{n}{\check{y}},v)| \le c \cdot v^{-\kappa_{n+1}\cdot s} \cdot v^{\underline{\kappa}\cdot(\underline{\mu}+\underline{\lambda}-\underline{\nu})-\kappa_n\delta_n}$$
$$|\Pi_{n+1}(\overset{n+1}{\check{y}},v)| \le c \cdot y_n^\varepsilon \cdot v^{-\kappa_n\cdot\varepsilon} \cdot v^{\underline{\kappa}\cdot(\underline{\mu}+\underline{\lambda}-\underline{\nu})-\kappa_{n+1}\delta_{n+1}},$$

where $\delta_j := \mu_j + \lambda_j - \nu_j$. The definition of ψ_i easily implies

$$|\psi_k(y_k,v)| \le c \cdot v^{\kappa_k\cdot(l_k+\delta_k)}$$
$$|\psi_n(y_n,v)| \le c \cdot y_n^\varepsilon \cdot v^{-\kappa_n\cdot\varepsilon} \cdot v^{\kappa_n\cdot(l_n+\delta_n)}$$
$$|\partial_{n+1}^s\psi_{n+1}(y_{n+1},v)| \le c \cdot v^{-s\cdot\kappa_{n+1}} \cdot v^{\kappa_{n+1}\cdot(l_{n+1}+\delta_{n+1})}\ ;$$

since $K_i(y,v) = \Pi_i(\overset{i}{\check{y}},v)\psi_i(y_i,v),\ \kappa_i l_i = 1\ (1 \le i \le n+1),\ \kappa_{n+1} = 1,\ r = \underline{\kappa}\cdot(\underline{1}+\underline{\lambda}+\underline{\mu})$, these formulas yield (1.3). For (1.2) compare IL'IN/ SOLONNIKOV [I/S, p. 72].

2. Estimates.

Our aim in this chapter is to prove the imbedding $W_p^{2,1}(\Omega_T) \ni f \mapsto \nabla_x f \in W_p^{1-\frac{1}{p},\frac{1}{2}(1-\frac{1}{p})}(\partial\Omega_T)$ with the imbedding constant c^* independent of T (for T small); here we let Ω be a bounded domain in $\mathbb{R}^n$ with boundary of the class C^2. Flattening the boundary locally, it is no restriction to assume that Ω is a cube i.e. $\Omega = Q_+^n(\lambda)$. Since $C^2(\overline{Q_+^n}(\lambda)\times[0,T])$ is dense in $W_p^{2,1}(Q_+^n(\lambda)\times(0,T))$ (cf. RÁKOSNÍK [R, Theorem 3]) and since the Hestenes-Whitney extension method (cf. ADAMS [A, p. 83]) yields a linear continuous extension operator
$E_T : W_p^{2,1}(Q_+^n(\lambda) \times (0,T)) \to W_p^{2,1}(Q_+^n(2\lambda) \times (0,2T))$ with
$E_T(C^2(\overline{Q_+^n}(\lambda) \times [0,T])) \subset C^2(\overline{Q_+^n}(2\lambda) \times [0,2T])$ and

$\|E_T\|_{W_p^{2,1}(Q_+^n(\lambda)\times(0,T))\to W_p^{2,1}(Q_+^n(2\lambda)\times(0,2T))} \leq B^*$ uniformly for all small T, it is sufficient to prove

$$\|\nabla_x f\|_{W_p^{1-\frac{1}{p},\frac{1}{2}(1-\frac{1}{p})}(Q^{n-1}(\lambda)\times(0,T))} \leq c^* \cdot \|f\|_{W_p^{2,1}(Q_+^n(2\lambda)\times(0,2T))}$$

for all $f \in C^2(\overline{Q_+^n}(2\lambda) \times [0,2T])$. The most difficult part in this inequality is the estimate for the time-regularity of the trace, i.e.

$$\text{(2.1)} \qquad |\nabla_x f|_{\mathcal{L}_p^{0,\frac{1}{2}(1-\frac{1}{p})}(Q^{n-1}(\lambda)\times(0,T))} \leq c^* \cdot \|f\|_{W_p^{2,1}(Q_+^n(2\lambda)\times(0,2T))},$$

where $|g|^p_{\mathcal{L}_p^{0,\beta}(Q^{n-1}(\lambda)\times(0,T))} := \int\limits_0^T h^{-(1+p\beta)} \, \|\Delta_{t,h} g\|^p_{p,Q^{n-1}(\lambda)\times(0,T-h)} \, dh$ for $\beta \in (0,1)$, when $(\Delta_{t,h}g)(x',t) := g(x',t+h) - g(x',t)$ and $\|\cdot\|_{p,X} := \|\cdot\|_{L^p(X)}$. The estimate for the spatial regularity follows from the more elementary trace theorem $W_p^1(\Omega) \to W_p^{1-\frac{1}{p}}(\partial\Omega)$ (cp. KUFNER et al. [K/J/F, 6.8.13 Theorem, p. 337]) by an easy scaling argument (in t). In the sequel, we shall prove (2.1). For this purpose, we start from the representation (1.1) for $\partial_j f$ $(1 \leq j \leq n)$: splitting $\int_0^T (\cdots)\, dv = \int_0^h (\cdots)\, dv + \int_h^T (\cdots)\, dv$ in the sum in the second line in (1.1) we get

$$\partial_j f(\cdot) = H_1(\cdot) + \sum_{i=1}^{n+1} \tilde{B}_i\{H_2^{(i)}(\cdot) + H_3^{(i)}(\cdot)\},$$

where

$$H_1(\cdot) := \frac{A}{T^r} \int\limits_{Q^{n+1}(0,\, T^{\underline{\kappa}})} \!\!\!\cdots\!\! \int f(\cdot + y)\Pi(y,T)\, dy,$$

$$\text{(2.2.)} \qquad H_2^{(i)}(\cdot) := \int_0^h v^{-(1+r)} \int\limits_{Q^{n+1}(0,\, v^{\underline{\kappa}})} \!\!\!\cdots\!\! \int \partial_i^{l_i} f(\cdot + y) \cdot K_i(y,v)\, dy\, dv,$$

$$H_3^{(i)}(\cdot) := \int_h^T v^{-(1+r)} \int\limits_{Q^{n+1}(0,\, v^{\underline{\kappa}})} \!\!\!\cdots\!\! \int \partial_i^{l_i} f(\cdot + y) \cdot K_i(y,v)\, dy\, dv.$$

In the sequel, we set $(\gamma H_1)(x',t) := H_1(x',0,t)$; we find

$$\text{(2.3)} \qquad \|\Delta_{t,h}(\gamma H_1)\|_{p,Q^{n-1}(\lambda)\times(0,T-h)} \;\leq\; h \cdot \|\partial_t\,(\gamma H_1)\|_{p,Q^{n-1}(\lambda)\times(0,T)}$$

(use $|\Delta_{t,h} f(\tau)| \leq \int_0^h |f'(\tau+s)| ds$ and Minkowski's integral inequality (cp. WHEEDEN/ ZYGMUND [W/Z, p. 143])); now

$$|\partial_t\,(\gamma H_1)(x',t)| \leq \frac{A}{T^r} \cdot \|\Pi(\cdot,T)\|_{\infty,Q^{n+1}(0,T^{\underline{\kappa}})} \cdot |Q^{n+1}(0,\,T^{\underline{\kappa}})|^{1/p'}$$
$$\cdot \, \|\partial_t\, f((x',0,t)+\cdot)\|_{p,Q^{n+1}(0,T^{\underline{\kappa}})}$$

by (2.2) and Hölder's inequality; hence

$$(2.4) \qquad \le c^* \cdot T^{-|\underline{\kappa}|\cdot(1-1/p')-\kappa_j} \cdot \|\partial_t f((x',0,t)+\cdot)\|_{p,Q^{n+1}(0,T^{\underline{\kappa}})}$$

by the kernel-estimate (1.2). Now observe that

$$\|\partial_t f((x',0,t)+\cdot)\|^p_{p,Q^{n+1}(0,T^{\underline{\kappa}})} = \\ = \int_0^{T^{\kappa_n}} \|\partial_t f(x'+\cdot,y_n,t+\cdot)\|^p_{p,\check{Q}^{n+1}(0,T^{\underline{\kappa}})}\, dy_n,$$

which easily implies via Fubini's theorem

$$(2.5) \quad \left(\int \!\!\cdots\!\! \int_{Q^{n-1}(\lambda)\times(0,T)} \|\partial_t f((x',0,t)+\cdot)\|^p_{p,Q^{n+1}(0,T^{\underline{\kappa}})}\, dx'\, dt \right)^{1/p} \le \\ \le |\check{Q}^{n+1}(0,T^{\underline{\kappa}})|^{1/p} \|\partial_t f\|_{p,Q^n((-\lambda\,\underline{1}',0),(\lambda\,\underline{1}'+T^{\underline{\kappa}'},T^{\kappa_n}))\times(0,2T)}.$$

Hence, by the last inequality, (2.4) and since $|\check{Q}^{n+1}(0,T^{\underline{\kappa}})| = T^{|\underline{\kappa}|-\frac{1}{2}}$ and $\kappa_j = \frac{1}{2}$:

$$\text{r.h. side in (2.3)} \\ \le c^* \cdot h \cdot T^{-\frac{1}{2}(1+\frac{1}{p})} \cdot \|\partial_t f\|_{p,\,Q^n((-\lambda\,\underline{1}',0),\,(\lambda\,\underline{1}'+T^{\underline{\kappa}'},T^{1/2}))\times(0,2T)}$$

so that, abbreviating $\rho = \rho(p) := \frac{1}{2}(1-\frac{1}{p})$,

$$|\gamma H_1|_{\mathcal{L}_p^{0,\rho}(Q^{n-1}(\lambda)\times(0,T))} \le \\ \le c^* \cdot T^{-\frac{1}{2}(1+\frac{1}{p})} \left(\int_0^T h^{-1+p(1-\rho)}\, dh \right)^{1/p} \|\partial_t f\|_{p,\,Q^n(\underline{a},\,\underline{b})\times(0,2T)}$$

with $\underline{a} := (-\lambda\,\underline{1}',0)$ and $\underline{b} := (\lambda\,\underline{1}'+T^{\underline{\kappa}'},T^{1/2})$; now $1-\rho = \frac{1}{2}(1+\frac{1}{p})$ and the T factors in the last inequality cancelled, as desired.

Let us turn our attention to $H_2^{(i)}$: trivially, for $h \le T$,

$$(2.6) \qquad \|\Delta_{t,h}(\gamma H_2^{(i)})\|_{p,Q^{n-1}(\lambda)\times(0,T-h)} \le 2 \cdot \|\gamma H_2^{(i)}\|_{p,Q^{n-1}(\lambda)\times(0,T)};$$

furthermore, using the kernel estimate (1.3) (with $s=0$), we get

$$(2.7) \quad |\gamma H_2^{(i)}(x',t)| \le \\ \le c^* \cdot \int_0^h v^{-(1+|\underline{\kappa}|+\varepsilon\kappa_n)+\frac{1}{2}} \int \!\!\cdots\!\! \int_{Q^{n+1}(0,v^{\underline{\kappa}})} y_n^{\varepsilon} \cdot |\partial_i^{l_i} f((x',0,t)+y)|\, dy\, dv\,;$$

we now represent the integrand as

$$\{ v^{-\frac{1}{p'}(1+|\underline{\kappa}|)+\frac{1}{2}(\rho-\varepsilon\cdot\kappa_n)}\} \cdot \{ v^{-\frac{1}{p}(1+|\underline{\kappa}|-\frac{1}{2})+\frac{1}{2}(\rho-\varepsilon\kappa_n)} \cdot y_n^{\varepsilon} \cdot |\partial_i^{l_i} f((x'0,t)+y)| \}$$

(note that $1/2=\rho+1/2p$); we choose $\varepsilon\in(0,\rho/\kappa_n)$; Hölder's inequality (with p',p) in y-v space then yields

$$\text{l.h.s. in (2.7)} \le c^* \cdot \left(\int_0^h v^{-1+\frac{p'}{2}(\rho-\varepsilon\cdot\kappa_n)}\,dv \right)^{1/p'} \cdot I^{1/p} \tag{2.8}$$

with

$$I := \int_0^h \int\limits_{Q^{n+1}(0,v^{\underline{\kappa}})} \!\!\cdots\!\! \int v^{-(1+|\underline{\kappa}|-\frac{1}{2})+\frac{p}{2}(\rho-\varepsilon\cdot\kappa_n)} \cdot y_n^{\varepsilon p} \cdot |\partial_i^{l_i} f((x',0,t)+y)|^p\,dy\,dv\,,$$

where in the first integral we took into account that $|Q^{n+1}(0,v^{\underline{\kappa}})| = v^{|\underline{\kappa}|}$; the first integral is clearly proportional to $h^{\frac{1}{2}(\rho-\varepsilon\cdot\kappa_n)}$. Thus, after a computation as in (2.5), we get

$$\|\gamma H_2^{(i)}\|_{p,Q^{n-1}(\lambda)\times(0,T)} \le c^* \cdot h^{\frac{1}{2}(\rho-\varepsilon\cdot\kappa_n)} \cdot \tilde{I}^{1/p} \tag{2.9}$$

with

$$\tilde{I} := \int_0^h v^{-(1+|\underline{\kappa}|-\frac{1}{2})+\frac{p}{2}(\rho-\varepsilon\cdot\kappa_n)} |\check{Q}^{n+1}(0,v^{\underline{\kappa}})| \int\limits_{Q^{n+1}(\underline{a},\underline{b}(v))} \!\!\cdots\!\! \int z_n^{\varepsilon\cdot p} \cdot |\partial_i^{l_i} f(z)|^p\,dz\,dv\,,$$

where $\underline{a} := (-\lambda\,\underline{1}',0,0)$, $\underline{b}(v) := (\lambda\,\underline{1}'+v^{\underline{\kappa}'}, v^{\kappa_n}, T+v)$; since $b(v)\le b(h)$, we can continue

$$\begin{aligned} \tilde{I} &\le \int_0^h v^{-1+\frac{p}{2}(\rho-\varepsilon\cdot\kappa_n)}\,dv \int\limits_{Q^{n+1}(\underline{a},\underline{b}(h))} \!\!\cdots\!\! \int z_n^{\varepsilon\cdot p} \cdot |\partial_i^{l_i} f(z)|^p\,dz \\ &\le c^* \cdot h^{(\rho-\varepsilon\cdot\kappa_n)\cdot p/2} \int_0^{h^{\kappa_n}} z_n^{\varepsilon\cdot p} \cdot \varphi(z_n)\,dz_n \end{aligned}$$

with $\varphi(z_n) := \|\partial_i^{l_i} f(\cdot,z_n,\cdot)\|^p_{p,\,Q^{n-1}(-\lambda\,\underline{1}',\lambda\,\underline{1}'+T^{\underline{\kappa}'})\times(0,2T)}$ by Fubini's theorem and since $h\le T$; consequently, by (2.6), (2.9) and the last line

$$\begin{aligned} |\gamma H_2|^p_{\mathcal{L}_p^{0,\rho}(Q^{n-1}(\lambda)\times(0,T))} &\le \\ &\le c^* \cdot \int_0^T h^{-(1+p\cdot\varepsilon\cdot\kappa_n)} \int_0^{h^{\kappa_n}} z_n^{\varepsilon\cdot p} \cdot \varphi(z_n)\,dz_n\,dh \end{aligned} \tag{2.10}$$

and by the version of Hardy's inequality from Lemma, (i) in the Appendix

$$\le c^* \cdot (p\cdot\varepsilon\cdot\kappa_n)^{-1} \cdot \int_0^{T^{\kappa_n}} \varphi(z_n)\,dz_n$$
$$= c^* \cdot (p\cdot\varepsilon\cdot\kappa_n)^{-1} \cdot \|\partial_i^{l_i} f\|^p_{p,\,Q^{\,n}((-\lambda\,\underline{1}',0),\,(\lambda\,\underline{1}'+T^{\underline{\kappa}'},T^{1/2}))\times(0,2T)}\ ,$$

which is the desired result for $H_2^{(i)}$.

Finally, let us turn to $H_3^{(i)}$; we again use (2.3) and observe that the correct expression for $\partial_t\,(\gamma H_3^{(i)})$ is obtained just by replacing K_i (in the definition of $H_3^{(i)}$) by $\partial_{n+1}\,K_i$ (integrate by parts); after estimating $|\partial_{n+1}\,K_i|$ according to (1.3), we arrive at

(2.11) $\quad |\partial_t\,(\gamma H_3^{(i)})(x',t)| \le$

$$\le\ c^* \cdot \int_h^T v^{-(1+|\underline{\kappa}|+\frac{1}{2}+\varepsilon\cdot\kappa_n)} \underset{Q^{\,n+1}(0,\,v^{\underline{\kappa}})}{\int\cdots\int} y_n^{\varepsilon}\cdot|\partial_i^{l_i} f((x'0,t)+y)|\,dy\,dv$$

(cp. (2.7); here the v-exponent is smaller by one, since $\partial_{n+1}\,K_i$ entails (in (1.3)) the additional factor v^{-1}); in the last integral we write the integrand in the form

$$\{v^{-\frac{1}{p'}(1+|\underline{\kappa}|)-(1-\rho-\delta)}\}\cdot\{v^{-\frac{1}{p}(1+|\underline{\kappa}|-\frac{1}{2})-(\varepsilon\kappa_n+\delta)}\cdot y_n^{\varepsilon}\cdot|\partial_i^{l_i} f(\cdots)|\}$$

(note that $-\frac{1}{2} = \frac{1}{2p}+\rho-1$), where we introduced $\delta\in(0,1-\rho)$. Now apply Hölder's inequality (with p',p) in y-v space and get

$$\text{r.h.s. in (2.11)} \le c^*\cdot\left(\int_h^T v^{-1-p'\cdot(1-\rho-\delta)}\,dv\right)^{1/p'}\cdot J^{1/p}$$

with

$$J := \int_h^T v^{-(1+|\underline{\kappa}|-\frac{1}{2})-p(\varepsilon\cdot\kappa_n+\delta)} \underset{Q^{\,n+1}(0,\,v^{\underline{\kappa}})}{\int\cdots\int} y_n^{\varepsilon p}\cdot|\partial_i^{l_i} f((x',0,t)+y)|^p\,dy\,dv\,;$$

proceeding as in the argument leading from (2.8) to (2.9), the last estimate allows us to conclude

$$\|\partial_t\,(\gamma H_3^{(i)})\|_{p,\,Q^{\,n-1}(\lambda)\times(0,T)} \le$$
$$\le c^*\cdot h^{-(1-\rho-\delta)}\cdot\left(\int_h^T v^{-1-p(\varepsilon\cdot\kappa_n+\delta)}\int_0^{v^{\kappa_n}} z_n^{\varepsilon\cdot p}\cdot\varphi(z_n)\,dz_n\,dv\right)^{1/p}$$

with $\varphi(\cdot)$ as before (since $v\le T$); by (2.3)

$$|\gamma H_3^{(i)}|^p_{\mathcal{L}_p^{0,\rho}(Q^{\,n-1}(\lambda)\times(0,T))}$$
$$\le c^*\cdot\int_0^T h^{-1+p\delta}\int_h^T v^{-1-p\cdot(\varepsilon\cdot\kappa_n+\delta)}\int_0^{v^{\kappa_n}} z_n^{\varepsilon\cdot p}\cdot\varphi(z_n)\,dz_n\,dv\,dh$$
$$\le c^*\cdot(p\cdot\delta)^{-1}\cdot\int_0^T v^{-1-p\cdot\varepsilon\cdot\kappa_n}\int_0^{v^{\kappa_n}} z_n^{\varepsilon p}\cdot\varphi(z_n)\,dz_n\,dv$$

by Appendix, Lemma (ii); now we may continue as after (2.10) and the desired result for $H_3^{(i)}$ follows.

Thus (2.1) is proved for all $T\le T_0(\lambda)=\lambda^2$.

Appendix.

We note a version of Hardy's inequality.

Lemma. *Suppose that $f \in L_1(0, T^\gamma)$ is nonnegative, $0 < T \leq \infty; \varepsilon, \gamma > 0$. Then*

(i) $\int_0^T x^{-1-\varepsilon\cdot\gamma} \int_0^{x^\gamma} y^\varepsilon \cdot f(y)\, dy\, dx \leq (\gamma\cdot\varepsilon)^{-1} \int_0^{T^\gamma} f(y)\, dy,$

(ii) $\int_0^T x^{-1+\varepsilon\cdot\gamma} \int_{x^\gamma}^{T^\gamma} y^{-\varepsilon} \cdot f(y)\, dy\, dx \leq (\gamma\cdot\varepsilon)^{-1} \int_0^{T^\gamma} f(y)\, dy.$

PROOF: These inequalities are proved in BESOV/ IL'IN/ NIKOL'SKII [B/I/N, 2.15, p. 28] (even in a more general form) for $T = \infty$. For T finite they follow easily by applying the version for $T = \infty$ to the extension by zero of f to $\mathbb{R}^+$. □

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(*Received* August 6, 1990)