A characterization of Corson-compact spaces

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Abstract. We characterize Corson-compact spaces by means of countable elementary substructures.

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First, let us review some definitions and facts concerning elementary substructures.

Let $\mathcal{H}$ be an arbitrary non-empty set. A non-empty subset $\mathcal{M}$ of $\mathcal{H}$ is said to be an elementary substructure of $\mathcal{H}$ ($\mathcal{M} \prec \mathcal{H}$, for short), if for any formula $\varphi(x_1, \dots, x_n)$ of the language of set theory with the only free variables $x_1, \dots, x_n$ and for any $a_1, \dots, a_n \in \mathcal{M}$ $\varphi[a_1, \dots, a_n]$ is true iff it is true in $\mathcal{H}$.

A frequently used argument is the following fact which is known as Tarski Criterion for elementary substructures:

> A subset $\mathcal{M}$ of $\mathcal{H}$ forms an elementary substructure of $\mathcal{H}$ if and only if for every formula $\varphi(x_0, x_1, \dots, x_n)$ and every $a_1, \dots, a_1 \in \mathcal{M}$ such that there exists an $a \in \mathcal{H}$ such that $\varphi(a, a_1, \dots, a_n)$ is true in $\mathcal{H}$, there is a $b \in \mathcal{M}$ such that $\varphi(b, a_1, \dots, a_n)$ is true in $\mathcal{H}$ (and therefore in $\mathcal{M}$).

Remark that if there is a unique $a \in \mathcal{H}$ satisfying $\varphi(a, a_1, \dots, a_n)$ (in $\mathcal{H}$), then a belongs to $\mathcal{M}$ provided $\mathcal{M} \prec \mathcal{H}$ and $a_i \in \mathcal{M}$, $i = 1, \dots, n$. For a cardinal Θ, $\mathcal{H}(\Theta)$ denotes the set of all sets whose transitive closure has size less Θ (see Kunen [7]). For any sentence φ which is true (in V), there exist sufficiently large regular cardinals Θ such that φ is true in $\mathcal{H}(\Theta)$. This is the reason why we are interested in elementary substructures of $\mathcal{H}(\Theta)$, where Θ is regular and uncountable. When we investigate an object, say a topological space, we always assume Θ to be "large enough" without discussion how large it needs to be. Throughout the paper, we make the following assumption. If $\mathcal{M}$ is an elementary substructure, $\mathcal{M}$ contains all sets we need for the investigation of our object – for example, the set X, the set of all open subsets of X and the family $C(X)$ of all real-valued continuous functions defined on X. This will be expressed by saying that "$\mathcal{M}$ is a suitable elementary substructure".

The base of all our considerations is the following

Theorem 1 (Löwenheim–Skolem–Tarski)**.** *For each infinite set $\mathcal{H}$ and each subset $X \subseteq \mathcal{H}$, there exists an elementary substructure $\mathcal{M}$ of $\mathcal{H}$ such that $X \subseteq \mathcal{M}$ and $|\mathcal{M}| \leq \max\{|X|, \omega\}$.*

The following facts are well known.

Fact 2. If Θ is a regular uncountable cardinal, $\mathcal{M} \prec \mathcal{H}(\Theta)$ and A is a countable set, $A \in \mathcal{M}$, then $A \subseteq \mathcal{M}$.

For any uncountable set $\mathcal{H}$, $[\mathcal{H}]^\omega$ denotes the set of all countable subsets of $\mathcal{H}$. A family $C \subseteq [\mathcal{H}]^\omega$ is said to be unbounded if for every $X \in [\mathcal{H}]^\omega$ there is a $Y \in C$ with $X \subseteq Y$. We say C is closed if, whenever $X_n \in C$ and $X_n \subseteq X_{n+1}$ for each $n \in \omega$, then $\bigcup\{X_n : n \in \omega\} \in C$.

Fact 3. $\{\mathcal{M} \in [\mathcal{H}]^\omega : \mathcal{M} \prec \mathcal{H}\}$ is a closed unbounded subset of $[\mathcal{H}]^\omega$.

Fact 4. If C_1, C_2 are closed unbounded subsets of $[\mathcal{H}]^\omega$, then $C_1 \cap C_2$ is also a closed unbounded subset of $[\mathcal{H}]^\omega$.

The reader is referred to Kunen [7] or Dow [4] for more information on elementary substructures.

Now we are going to construct for each Hausdorff compact space X and each suitable elementary substructure $\mathcal{M}$ (of $\mathcal{H}(\Theta)$) a relatively small compact space $X(\mathcal{M})$ and a mapping $\varphi^X_{\mathcal{M}}$ from X onto $X(\mathcal{M})$.[1] Let $C(X)$ denote the set of all real-valued continuous functions defined on X. $\varphi^X_{\mathcal{M}}$ corresponds to the mapping which relates each point $x \in X$ to the point $(fx)_{C(X)\cap\mathcal{M}}$ from the product space $\mathbb{R}^{C(X)\cap\mathcal{M}}$. That is, $X(\mathcal{M})$ is the continuous image of X with the property that for any pair of distinct points $x_1, x_2 \in X$, we have $\varphi^X_{\mathcal{M}}(x_1) \neq \varphi^X_{\mathcal{M}}(x_2)$ iff there is a function $f \in C(X) \cap \mathcal{M}$ with $f(x_1) \neq f(x_2)$. Hence, $\varphi^X_{\mathcal{M}}(x_1) \neq \varphi^X_{\mathcal{M}}(x_2)$ iff there exist open subsets $U, V \in \mathcal{M}$ of X such that $x \in U, y \in V$ and $\mathrm{cl}(U) \cap \mathrm{cl}(V) = \emptyset$.

Lemma 5. *Let $i : X \to \mathbb{R}^T$ be a continuous embedding of the Hausdorff compact space X into $\mathbb{R}^T$. Then $\varphi^X_{\mathcal{M}}$ is isomorphic to the composition of i and $\pi_{\mathcal{M}}$, where $\pi_{\mathcal{M}}$ denotes the projection mapping $\mathbb{R}^T \to \mathbb{R}^{T\cap\mathcal{M}}$.*

Proof: It is enough to show that for every function $f \in C(X) \cap \mathcal{M}$ and any pair of distinct points $x_1, x_2 \in X$ with $f(x_1) \neq f(x_2)$, we have $\pi_{\mathcal{M}}(ix_1) \neq \pi_{\mathcal{M}}(ix_2)$. Since X is compact, we may find – by means of some elementary observations – a continuous function $g : \mathbb{R}^T \to \mathbb{R}$ such that $f = g\cdot i$. Since $i, f \in \mathcal{M}$, we may assume that $g \in \mathcal{M}$. It is well known (see Engelking [5, 3.4.H]) that g depends on countably many coordinates, i.e. there exists a countable set $A \subseteq T$ and a continuous function $h : \mathbb{R}^A \to \mathbb{R}$ such that $g = h\cdot\pi_A$. We may assume that $A \in \mathcal{M}$. Since A is countable, it follows from Fact 2 that $A \subset \mathcal{M}$. Now it is easy to derive the existence of an index $\alpha \in A$ with $\pi_\alpha(ix_1) \neq \pi_\alpha(ix_2)$. Consequently, $\pi_{\mathcal{M}}(ix_1) \neq \pi_{\mathcal{M}}(ix_2)$. □

The following definition plays the decisive role in this paper.

Definition 6. Let X be a Hausdorff compact space and $\mathcal{M}$ a suitable elementary substructure (of $\mathcal{H}(\Theta)$). $\varphi^X_{\mathcal{M}}$ is called an $\mathcal{M}$-retraction, if $\varphi^X_{\mathcal{M}}$ maps $\mathrm{cl}(X \cap \mathcal{M})$ homeomorphic on $X(\mathcal{M})$.

A compact space X is called Corson-compact, if X is homeomorphic to a subset of

$$\Sigma(\mathbb{R}^T) = \{x \in \mathbb{R}^T : \mathrm{supp}(x) \ \text{ is countable}\},$$

[1]This construction may be defined for arbitrary uniform spaces as will be shown in Bandlow [2].

where $\operatorname{supp}(x) = \{t \in T : x_t \neq 0\}$ for $x \in \mathbb{R}^T$, for some set T. Of course, $\Sigma(\mathbb{R}^T)$ is a subspace of $\mathbb{R}^T$ with the usual product topology. Our main result is the following

Theorem 7. *Let X be a Hausdorff compact space. The following assertions are equivalent:*

(a) *X is Corson-compact.*
(b) *There are a sufficiently large regular uncountable cardinal Θ and a closed unbounded family $C \subseteq [\mathcal{H}(\Theta)]^\omega$ of countable elementary substructures of $\mathcal{H}(\Theta)$ such that $\varphi^X_{\mathcal{M}}$ is an $\mathcal{M}$-retraction for every $\mathcal{M} \in C$.*
(c) *For every sufficiently large regular uncountable cardinal Θ there exists a closed unbounded family $C \subseteq [\mathcal{H}(\Theta)]^\omega$ of countable elementary substructures of $\mathcal{H}(\Theta)$ such that $\varphi^X_{\mathcal{M}}$ is an $\mathcal{M}$-retraction for every $\mathcal{M} \in C$.*

Remark. Other characterizations of Corson-compact spaces were given by Gul'ko [6] and Shapirovskii [7]. I believe that our concept is more convenient for applications. In a subsequent paper, we will use our characterization to investigate the space of all real-valued continuous functions defined on a Corson-compact space in the topology of pointwise convergence.

The proof of the theorem breaks in several lemmas.

Lemma 8. (a) $\to$ (b).

PROOF: Let $i : X \to \Sigma(\mathbb{R}^T)$ be an embedding of the Hausdorff compact space X into $\Sigma(\mathbb{R}^T)$. Suppose $\mathcal{M}$ is a suitable elementary substructure (of $\mathcal{H}(\Theta)$). It is enough to show that $\varphi^X_{\mathcal{M}}$ is an $\mathcal{M}$-retraction. For the sake of simplicity, we identify X with $i(X)$. If $x \in X \cap \mathcal{M}$, then it follows from Fact 2 that $\operatorname{supp}(x) \subseteq \mathcal{M}$. Hence, $\operatorname{supp}(X \cap \mathcal{M}) \subseteq \mathcal{M}$ and, consequently, $\operatorname{supp}(\operatorname{cl}(X \cap \mathcal{M})) \subseteq \mathcal{M}$. Now it follows from Lemma 5 that $\varphi^X_{\mathcal{M}}$ restricted to $\operatorname{cl}(X \cap \mathcal{M})$ is a one-to-one mapping. Since $\varphi^X_{\mathcal{M}}$ always maps $\operatorname{cl}(X \cap \mathcal{M})$ onto $X(\mathcal{M})$, this implies that $\varphi^X_{\mathcal{M}}$ is an $\mathcal{M}$-retraction. $\square$

Lemma 9. (b) $\to$ (c).

The idea of the proof of this implication is standard and is based on the following

Fact 10 (Devlin [3])**.** Let $\mathcal{A}$ and $\mathcal{B}$ be uncountable sets, $\mathcal{A} \subseteq \mathcal{B}$.

(a) If $C \subseteq [\mathcal{B}]^\omega$ is closed and unbounded, then $\{X \cap \mathcal{A} : X \in C\}$ contains a closed unbounded subfamily of $[\mathcal{A}]^\omega$.
(b) If $C \subseteq [\mathcal{A}]^\omega$ is closed and unbounded, then $\{X \in [\mathcal{B}]^\omega : X \cap \mathcal{A} \in C\}$ is a closed unbounded subfamily of $[\mathcal{B}]^\omega$.

PROOF OF LEMMA 9: Let X be a Hausdorff compact space, Θ a regular uncountable cardinal and C_0 a closed unbounded set of countable elementary substructures of $\mathcal{H}(\Theta)$, such that $\varphi^X_{\mathcal{M}}$ is an $\mathcal{M}$-retraction for every $\mathcal{M} \in C_0$.

Let μ be an arbitrary sufficiently large regular uncountable cardinal. "Sufficiently large" means, for instance, that X and $C(X)$ are elements of $\mathcal{H}(\mu)$ and, therefore, $C(X) \subseteq \mathcal{H}(\mu)$ and $X \subseteq \mathcal{H}(\mu)$. Suppose that $\mu < \Theta$. By Facts 10 (a) and 4, we can find a closed unbounded subset C of $[\mathcal{H}(\mu)]^\omega$, consisting of elementary substructures of $\mathcal{H}(\mu)$ and satisfying the property that for each $\mathcal{N} \in C$ there exists an elementary substructure $\mathcal{M} \in C_0$ with $\mathcal{N} = \mathcal{M} \cap \mathcal{H}(\mu)$. This implies $\mathcal{N} \cap X = \mathcal{M} \cap X$ and $\mathcal{N} \cap C(X) = \mathcal{M} \cap C(X)$. Hence, $\varphi^X_{\mathcal{M}}$ and $\varphi^X_{\mathcal{N}}$ are isomorphic and $\mathrm{cl}(X \cap \mathcal{M}) = \mathrm{cl}(X \cap \mathcal{N})$. Thus $\varphi^X_{\mathcal{N}}$ is an $\mathcal{N}$-retraction.

The proof for the case $\mu > \Theta$ is quite similar. $\square$

Lemma 11. *Let the Hausdorff compact space X be as in Theorem* 7 (b). *Then $t(X) = \omega$.*

Proof: Let x be a point of X and A a subset of X such that $x \in \mathrm{cl}(A) \setminus A$. Let $\mathcal{M} \prec \mathcal{H}(\Theta)$ be such that $x, A \in \mathcal{M}$ and $\varphi^X_{\mathcal{M}}$ is an $\mathcal{M}$-retraction. We claim that $x \in \mathrm{cl}(A \cap \mathcal{M})$. Otherwise, by the construction of $\varphi^X_{\mathcal{M}}$, for every point $y \in \mathrm{cl}(A \cap \mathcal{M})$, there exists a function $f_y \in C(X) \cap \mathcal{M}$ with $f_y(x) \neq f_y(y)$. Since $\mathrm{cl}(A \cap \mathcal{M})$ is compact and $\mathcal{M} \prec \mathcal{H}(\Theta)$, we can find a function $g \in C(X) \cap \mathcal{M}$ which separates x and $\mathrm{cl}(A \cap \mathcal{M})$. Hence, there exists an open subset $U \in \mathcal{M}$ of X with $x \in U$ and $U \cap A \cap \mathcal{M} = \emptyset$. Since U and A are elements of $\mathcal{M}$, this implies that $U \cap A = \emptyset$, i.e. $x \notin \mathrm{cl}(A)$. This contradiction proves the lemma. $\square$

Lemma 12. *Let X be a Hausdorff compact space, Θ a regular uncountable cardinal and $C_0 \subseteq [\varphi^X_{\mathcal{M}}]^\omega$ a closed unbounded family of countable elementary substructures of $\mathcal{H}(\Theta)$ such that $\varphi^X_{\mathcal{M}}$ is an $\mathcal{M}$-retraction for every $\mathcal{M} \in C_0$. Furthermore let $\vartheta > \Theta$ be a regular uncountable cardinal and $\mathcal{N}$ an elementary substructure of $\mathcal{H}(\vartheta)$ with $X, C_0 \in \mathcal{N}$. Then $\varphi^X_{\mathcal{N}}$ is an $\mathcal{N}$-retraction.*

Proof: The assertion "$(\forall\, x \in X)(\exists\, \mathcal{M} \in C_0)(x \in \mathcal{M})$" holds in $\mathcal{H}(\vartheta)$, hence in $\mathcal{N}$, since $X, C_0 \in \mathcal{N}$. Therefore, for every point $x \in X \cap \mathcal{N}$, there exists an $\mathcal{M} \in C_0 \cap \mathcal{N}$ with $x \in \mathcal{M}$. From Fact 2, it follows that $\mathcal{M} \subseteq \mathcal{N}$. Since C_0 is closed, there exists for every countable set $A \subseteq X \cap \mathcal{N}$ an $\mathcal{M} \in C_0$ with $A \subseteq \mathcal{M} \subseteq \mathcal{N}$.

Let x_1, x_2 be a pair of distinct points of $\mathrm{cl}(X \cap \mathcal{N})$. We have to show that $\varphi^X_{\mathcal{N}}(x_1) \neq \varphi^X_{\mathcal{N}}(x_2)$, i.e. there must exist a function $f \in C(X) \cap \mathcal{N}$ with $f(x_1) \neq f(x_2)$. Since $t(X) = \omega$, there exists a countable set $A \subseteq X \cap \mathcal{N}$ such that $x_1 \in \mathrm{cl}(A)$ and $x_2 \in \mathrm{cl}(A)$. Let $\mathcal{M} \in C_0$ be such that $A \subseteq \mathcal{M} \subseteq \mathcal{N}$. Then $x_1, x_2 \in \mathrm{cl}(X \cap \mathcal{M})$ and we find a function $f \in C(X) \cap \mathcal{M}$ with $f(x_1) \neq f(x_2)$. $\square$

Lemma 13. *Let $f : X \to Y$ be a continuous mapping from the Hausdorff compact space X onto the Hausdorff compact space Y. Suppose further that $\mathcal{M}$ is an elementary substructure* (*of $\mathcal{H}(\Theta)$*) *such that $f \in \mathcal{M}$ and $\varphi^X_{\mathcal{M}}$ is an $\mathcal{M}$-retraction. Then $\varphi^Y_{\mathcal{M}}$ is also an $\mathcal{M}$-retraction.*

Proof: One readily sees that $f(X \cap \mathcal{M}) = Y \cap \mathcal{M}$. Let x, y be a pair of distinct points of $\mathrm{cl}(Y \cap \mathcal{M})$ and choose open subsets U, V of Y such that $x \in U, y \in V$ and $U \cap V = \emptyset$. Since $\varphi^X_{\mathcal{M}} \mid_{\mathrm{cl}(X \cap \mathcal{M})}$ is a homeomorphism onto $X(\mathcal{M})$ and $\mathrm{cl}(X \cap \mathcal{M}) \setminus f^{-1}(U)$ is compact, there exists a function $f \in C(X) \cap \mathcal{M}$ which separates $f^{-1}\{x\} \cap \mathrm{cl}(X \cap \mathcal{M})$ and $\mathrm{cl}(X \cap \mathcal{M}) \setminus f^{-1}(U)$. Thus there exists a closed subset $F \in \mathcal{M}$ of X such that $f^{-1}\{x\} \cap \mathrm{cl}(X \cap \mathcal{M}) \subseteq F \cap \mathrm{cl}(X \cap \mathcal{M}) \subseteq f^{-1}(U)$. Analogously we can find a closed subset $H \in \mathcal{M}$ of X satisfying

$$f^{-1}\{y\} \cap \mathrm{cl}(X \cap \mathcal{M}) \subseteq H \cap \mathrm{cl}(X \cap \mathcal{M}) \subseteq f^{-1}(V).$$

We claim that $f(F)\cap f(H)=\emptyset$. Assume, on the contrary, $f(F)\cap f(H)\neq\emptyset$. Since $F,H\in\mathcal{M}$, there exist points $x'\in F\cap\mathcal{M}$ and $y'\in H\cap\mathcal{M}$ with $f(x')=f(y')$. This contradicts $F\cap\operatorname{cl}(X\cap\mathcal{M})\subseteq f^{-1}(U)$ and $H\cap\operatorname{cl}(X\cap\mathcal{M})\subseteq f^{-1}(V)$. Of course, $f(F)\in\mathcal{M}$ and $f(H)\in\mathcal{M}$. Therefore we can find a function $h\in C(Y)\cap\mathcal{M}$ which separates $f(F)$ and $f(H)$. This implies $\varphi^Y_{\mathcal{M}}(x)\neq\varphi^Y_{\mathcal{M}}(y)$. □

We have arrived at the final assertion.

Lemma 14. *Let X be as in Theorem* 7 (b). *Then there exists a set T and a homeomorphic embedding from X into $\Sigma(\mathbb{R}^T)$.*

PROOF: By induction on $\tau=w(X)$. For $\tau=\omega$, this is trivial. Suppose the assertion holds for the Hausdorff compact spaces of weight $<\tau$. Using Lemma 12, one can find a regular uncountable cardinal ϑ and an increasing sequence $\langle\mathcal{N}_\alpha:\omega\le\alpha<w(X)\rangle$ of elementary substructures of $\mathcal{H}(\vartheta)$, such that

(1) $|\mathcal{N}_\alpha|<\tau$ for all $\alpha,\omega\le\alpha<\tau$,
(2) $\varphi^X_{\mathcal{N}_\alpha}$ is an $\mathcal{N}_\alpha$-retraction for all $\alpha,\omega\le\alpha<\tau$,
(3) $\mathcal{N}_\alpha=\bigcup\{\mathcal{N}_\beta:\omega\le\beta<\alpha\}$ for all limit ordinals $\alpha,\omega\le\alpha<\tau$,
(4) $C(X)\cap(\bigcup\{\mathcal{N}_\alpha:\omega\le\alpha<\tau\})$ separates an arbitrary pair of distinct points of X.

Now we make use of the inductive assumption. By Lemma 13, there exist a set T_α and a homeomorphic embedding

$$q_\alpha:X(\mathcal{N}_\alpha)\to\Sigma(\mathbb{R}^T\alpha)$$

for every $\alpha,\omega\le\alpha<\tau$. Of course, one may assume that the T_α are pairwise disjoint. We set $Z_\alpha=\operatorname{cl}(X\cap\mathcal{N}_\alpha)$ and identify Z_α with $X(\mathcal{N}_\alpha)$. Instead of $\varphi^X_{\mathcal{N}_\alpha}$, we consider a mapping $\varphi_\alpha:X\to X$, where $\varphi_\alpha(X)=Z_\alpha$, $\omega\le\alpha<\tau$.

Now we define the mapping $q:X\to\Sigma(\mathbb{R}^T)$, where $T=\bigcup\{T_{\alpha+1}:\omega\le\alpha<\tau\}$ by setting

$$(q(x))_t=(q_{\alpha+1}(\varphi_{\alpha+1}(x)))_t-(q_{\alpha+1}(\varphi_\alpha(x)))_t$$

for all $x\in X$ and $t\in T_{\alpha+1}$, $\omega\le\alpha<\tau$, and

$$(q(x))_t=(q_\alpha(\varphi_{\omega(x)}))_t$$

for all $x\in X$ and $t\in T_\omega$.
(Remark that the idea of this definition is due to Amir and Lindenstrauss [1].)

q is obviously a continuous mapping from X into $\mathbb{R}^T$. First, let us check that q is injective. Suppose we are given two points $x,y\in X$, $x\neq y$. Then there exists an ordinal $\beta,\omega\le\beta<\tau$,, such that $\varphi_\beta(x)\neq\varphi_\beta(y)$ and $\varphi_\gamma(x)=\varphi_\gamma(y)$ for all γ with $\omega\le\gamma<\beta$. If $\beta=\omega$, then $(q(x))_t\neq(q(y))_t$ for any $t\in T_\omega$ and hence $q(x)\neq q(y)$. If $\beta>\omega$, then, by the condition (3), β is a successor ordinal, i.e. $\beta=\alpha+1$ for an $\alpha,\omega\le\alpha<\tau$. From $\varphi_\alpha(x)=\varphi_\alpha(y)$ and $\varphi_{\alpha+1}(x)\neq\varphi_{\alpha+1}(y)$, it follows that $(q(x))_t\neq(q(y))_t$ for any $t\in T_{\alpha+1}$.

To complete the proof, we have to show that for every point $x\in X$ the set $B_x=\{t\in T:(q(x))_t\neq 0\}$ is at most countable. Assume, on the contrary, that

B_x is uncountable. Then we can choose a subset $B \subseteq B_x$ such that $\mu = \sup B$ satisfies $\mathrm{cf}(\mu) > \omega$. Since $t(X) = \omega$ (see Lemma 11), we have $Z_\mu = \bigcup\{Z_\alpha : \omega \leq \alpha < \tau\}$. Therefore, one can find an ordinal $\alpha_0 < \mu$ such that $\varphi_\mu(x) \in Z_{\alpha_0}$ and, consequently, $\varphi_\alpha(x) = \varphi_\mu(x)$ for every α with $\alpha_0 < \alpha < \mu$. Hence $(q(x))_t = 0$ for every $t \in T_{\alpha+1}, \alpha_0 < \alpha < \mu$, which contradicts $B \subseteq B_x$; B is countable.
This concludes the proof of the lemma and of the theorem. □

Remark. The proof of Lemma 13 is a new proof of the fact that a Hausdorff continuous image of a Corson-compact space is Corson-compact (Gul'ko [6]).

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