Existence of solutions of perturbed O.D.E.'s in Banach spaces

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Abstract. We consider a perturbed Cauchy problem like the following

$$\text{(PCP)} \begin{cases} x' = A(t,x) + B(t,x) \\ x(0) = x_0 \end{cases}$$

and we present two results showing that (PCP) has a solution. In some cases, our theorems are more general than the previous ones obtained by other authors (see [4], [8], [9], [11], [13], [17], [18]).

Keywords: perturbed Cauchy problem, semi-inner product, measure of noncompactness

Classification: 34G05, 34G20

1. Introduction.

Let $I = [0,1]$ and X be a closed subset of a Banach space E. If $x_0 \in X$ and A, B are two functions defined on $I \times X$ with values into E, we are interested in solving the following perturbed Cauchy problem

$$\text{(PCP)} \quad \begin{cases} x' = A(t,x) + B(t,x) \\ x(0) = x_0 \end{cases}$$

under several assumptions on A and B; essentially, A will satisfy dissipative conditions and B compactness type ones, as it has been done by a lot of authors (see [4], [11], [13], [17], [18]). We always assume that there is a subinterval $J = [0,a]$ of I and a sequence of equicontinuous and a.e. derivable functions $x_n : J \to X$ such that there is $K > 0$ such that $\|x_n(t') - x_n(t'')\| \leq K|t' - t''|$ on J, $n \in N$, and

$$\lim_n \|x_n'(t) - [A(t, x_n(t)) + B(t, x_n(t))]\| = 0 \quad \text{a.e. on } J$$

and we look for conditions about A and B forcing a suitable subsequence of (x_n) to converge (to a solution x of (PCP)).

In this paper, we use the following notions of semi-inner product and Kuratowski measure of noncompactness (see [3]).

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Definition 1. Let $x, y \in E$. We define $Fx = \{x^* \in E^* : x^*(x) = \|x\|^2 = \|x^*\|^2\}$ and $(y,x)_+ = \max\{x^*(y) : x^* \in Fx\}, (y,x)_- = \min\{x^*(y) : x^* \in Fx\}$.

We have the following properties of semi-inner products:

(i) $(x+y,z)_\pm \le (x,z)_\pm + (y,z)_\pm$ and $|(x,y)_\pm| \le \|x\|\,\|y\|$,

(ii) if $x : (a,b) \to X$ is differentiable at t and $\phi(t) = \|x(t)\|$, then $\phi(t)D^-\phi(t) \le (x'(t), x(t))_-$.

Definition 2. Given a bounded subset X of E, we define the Kuratowski measure of non compactness $\alpha(X)$ as follows:
$\alpha(X) = \inf\{\varepsilon > 0$: there exist bounded subsets A_i of X with $X = \bigcup_{i=1}^n A_i$ and $\operatorname{diam} A_i < \varepsilon\}$.

The measure α has the following properties:

(j) $\alpha(A+B) \le \alpha(A) + \alpha(B), \alpha(kA) = |k|\alpha(A) \quad \forall\, k \in \mathbb{R}$,

(jj) $\alpha(A) = 0 \Leftrightarrow A$ is relatively compact,

(jjj) $\alpha(A) \le \alpha(B)$ if $A \subseteq B, \alpha(A \cup B) = \max\{\alpha(A), \alpha(B)\}$,

(jv) $\alpha(\overline{\mathrm{co}}(A)) = \alpha(A)$, where $\overline{\mathrm{co}}(A)$ is the closed, convex hull of A,

(v) $\alpha(A) \le \operatorname{diam} A$.

2. Existence results.

First of all, we consider the following groups of hypotheses used in [14] (see also [3]) and in the recent paper [9] in order to get a sequence of approximate solutions defined on J as described in the Introduction.

(H1) (see [14]). *Let the function $f = A + B$ be continuous and bounded. Further, if $X_r = X \cap \{x : \|x - x_0\| \le r\}$, $r > 0$, assume that*

$$\lim_{h\to 0^+} h^{-1} d(x + hf(t,x), X_r) = 0 \qquad \text{for all} \quad t \in I,\, x \in X. \tag{0}$$

(H2) (see [9]). *Let X be separable and convex. Let the function $f = A + B$ be bounded, satisfying* (0) *and the following Carathéodory assumptions:*

(C1) *the functions $t \to f(t,x)$ are strongly measurable, for all $x \in X$;*

(C2) *the functions $x \to f(t,x)$ are continuous, for almost all $t \in I$.*

(H3) (see [9]). *Let X be convex. Let the function $f = A + B$ be bounded satisfying* (0), (C1), (C2). *Further assume that there are two functions $L : I \to E$ and $H : E \to \mathbb{R}^+$ such that*

$$\begin{cases} L \in L^1(I,E), H \text{ is bounded on bounded sets} \\ \|f(t',x) - f(t'',x)\| \le \|L(t',x) - L(t'',x)\| H(x)(1 + \|f(t',x)\|), \\ \hfill t',\, t'' \in I,\ x \in X. \end{cases} \tag{1}$$

Remark 1. Note that we do not assume $\mathring{X} \ne \emptyset$, as some authors did (see [18]).

Remark 2. (H3) requires the existence of L and H verifying (1); this is quite a restrictive hypothesis, that, however, has been used successfully by a lot of authors studying nonlinear evolution equations (see [2], [10], [12], [15]).

Now, we present our results about the existence of solutions for (PCP); in the sequel, we shall consider the subset Z of X defined by $Z = \{x_n(t) : t \in I, n \in N\}$; note that Z is bounded.

Theorem 1. *Assume that one hypothesis among* (H1), (H2) *and* (H3) *is verified. Moreover, suppose that there exist two functions* $\varphi_A, \varphi_B \in L^1(I, \mathbb{R})$ *such that* $\|A(t,x)\| \le \varphi_A(t), \|B(t,x)\| \le \varphi_B(t)$ *for almost all* $t \in I, x \in Z$ *and that the following other facts are true:*

(2) *there is a function* $\ell_A \in L^1(J, \mathbb{R}^+)$ *such that*

$$(A(t,x) - A(t,y), x-y)_- \le \ell_A(t)\|x-y\|^2 \ t \ \text{ a.e. in } \ J, x, y \in Z;$$

(3) *there is a function* $\ell_B \in L^1(J, \mathbb{R}^+)$ *such that*

$$\alpha(B(t,Y)) \le \ell_B(t)\alpha(Y) \ t \ \text{ a.e. in } \ J, Y \subseteq Z;$$

(4) *for each* $\varepsilon > 0$, *there is a (closed) subset* J_ε *of* $J, m(J \setminus J_\varepsilon) < \varepsilon$ *such that* $B_{J_\varepsilon \times Z}$ *is uniformly continuous.*

Then (PCP) *has a solution on* J.

Proof: For each $\varepsilon > 0$, there is $J_\varepsilon \subset J$, closed, $m(J \setminus J_\varepsilon) < \varepsilon$ such that the following facts are true:

(5) $B_{J_\varepsilon \times Z}$ is uniformly continuous,
(6) $\ell_{A|J_\varepsilon}, \ell_{B|J_\varepsilon}$ are continuous,
(7) $\int_{J\setminus J_\varepsilon} \varphi_A(s)\,ds + \int_{J \setminus J_\varepsilon} \varphi_B(s)\,ds < \varepsilon$.

Repeating the proof of the first part of Theorem 4 in [11], we can get a partition $\{B_{K_1,\dots,K_m}\}$ of $\mathbb{N}$ in such a way that, for $r, s \in B_{K_1,\dots,K_m}$ and with $\mu(t) = \alpha(\{x_n(t)\})$, we have

$$\|B(t, x_r(t)) - B(t, x_s(t))\| \le 5\varepsilon + \ell_B(t)\mu(t) \ \text{ on } \ J_\varepsilon\,. \tag{8}$$

Using (i) and (ii) of Definition 1 and observing that $p_{rs}(t) = \|x_r(t) - x_s(t)\|$ is a.e. differentiable, because absolutely continuous, we get from (8) with $r, s \in B_{K_1,\dots,K_m}$

$$\begin{aligned} p_{rs}(t)p'_{rs}(t) \le \ell_A(t)p^2_{rs}(t) + \ell_B(t)p_{rs}(t)\mu(t) + 5\varepsilon p_{rs}(t) + \\ + (\|h_r(t)\| + \|h_s(t)\|)p_{rs}(t) \end{aligned}$$

for almost all $t \in J_\varepsilon$, where h_r, h_s are suitable functions with $\int_J \|h_r(s)\| + \|h_s(s)\|\,ds \to 0$ as $r, s \to \infty$.

On the other hand, it is very easy to see that

$$p'_{rs}(t) \le 2[\varphi_A(t) + \varphi_B(t)] + \|h_r(t)\| + \|h_s(t)\|.$$

Hence we have for a.a. $t \in J$, since $p_{rs}(0) = 0$ and $p'_{rs}(t_0) = 0$ whenever

$p_{rs}(t_0) = 0$ and $p'_{rs}(t_0)$ exists, $r, s \in B_{K_1,\dots,K_m}$,

$$
\begin{aligned}
p_{rs}(t) &= \int_0^t p'_{rs}(s)\,ds = \int_{[0,t]\cap J_\varepsilon} p'_{rs}(s)\,ds + \int_{[0,t]\setminus J_\varepsilon} p'_{rs}(s)\,ds \leq \\
&\leq \int_{[0,t]\cap J_\varepsilon} [\ell_A(s)p_{rs}(s) + \ell_B(s)\mu(s) + 5\varepsilon]\,ds + \int_{[0,t]\setminus J_\varepsilon} 2[\varphi_A(s) + \varphi_B(s)]\,ds + \\
&+ \int_J 2[\|h_r(s)\| + \|h_s(s)\|]\,ds \leq 8\varepsilon + \int_0^t \ell_B(s)\mu(s)\,ds + \\
&+ \int_0^t \ell_A(s)p_{rs}(s)\,ds
\end{aligned}
\tag{9}
$$

for r, s sufficiently large.

It is very easy to see that (9) implies the following inequality, $r, s \in B_{K_1,\dots,K_m}$,

$$
p_{rs}(t) \leq \left[8\varepsilon + \int_0^t \ell_B(s)\mu(s)\,ds\right] \exp\left(\int_0^t \ell_A(s)\,ds\right) \tag{10}
$$

for r, s sufficiently large. By using (jjj) and (v) of Definition 2, we can easily prove that (10) gives the following inequality

$$
\mu(t) \leq \left[8\varepsilon + \int_0^t \ell_B(s)\mu(s)\,ds\right] M^*, \tag{11}
$$

M^* being a positive number greater than $\exp(\int_0^t \ell_A(s)\,ds)$ for all $t \in J$. Hence, by (11), $\mu(t) \equiv 0$ on J, taking into account that ε is arbitrary. The proof is complete. $\square$

Remark 3. The proof of Theorem 1 is very similar to that one of Theorem 4 of [11], that is, however, generalized by virtue of the hypothesis (4); indeed, in [11], B is assumed to be uniformly continuous.

We shall see in a subsequent remark that our improvement is not only a technicality.

The next result makes use of similar assumptions concerning A and B; this time we shall assume the validity of (4) with respect to A; in this way, A and B are allowed to satisfy more general assumptions than (2) and (3).

Theorem 2. *Assume that one hypothesis among* (H1), (H2) *and* (H3) *is verified. Moreover, suppose there exist two functions* $\varphi_A, \varphi_B \in L^1(I, \mathbb{R})$ *such that* $\|A(t,x)\| \leq \varphi_A(t), \|B(t,x)\| \leq \varphi_B(t)$ *for almost all* $t \in I, x \in Z$.

Let the following other facts be true:

(12) *there exists a function* $\omega_A : J \times \mathbb{R}^+ \to \mathbb{R}^+$ *verifying Carathéodory hypotheses like* (C1) *and* (C2) *such that*

$$
(A(t,x) - A(t,y), x - y)_- \leq \omega_A(t, \|x - y\|)\|x - y\|\ t \ \text{ a.e. in } \ J, x, y \in Z;
$$

(13) *there exists a function* $\omega_B : J \times \mathbb{R}^+ \to \mathbb{R}^+$ *verifying Carathéodory hypotheses like* (C1) *and* (C2) *such that for each subset* Y *of* Z *and almost all* $t \in J$ *we have*

$$\lim_{h \to 0^+} \alpha(B([t-h,t],Y)) \leq \omega_B(t, \alpha(Y)),$$

where $h > 0$ *is such that* $t - h > 0$*;*

(14) $\omega_A + \omega_B$ *is such that the only absolutely continuous function* $u : J \to \mathbb{R}^+$ *for which* $u(0) = 0, u'(t) \leq \omega_A(t, u(t)) + \omega_B(t, u(t))$ *is the identically null function;*

(15) *for each* $\varepsilon > 0$ *there is a closed subset* J_ε *of* $J, m(J \setminus J_\varepsilon) < \varepsilon$*, such that* $A_{|I_\varepsilon \times Z}$ *is uniformly continuous.*

Then (PCP) *has a solution on* J*.*

PROOF: It was proved in the paper [11] that (12) implies that

$$\alpha(Y) - \alpha(\{x + hA(t,x) : x \in Y\}) \leq h\omega_A(t, \alpha(Y)) \tag{16}$$

for each $h > 0, t \in J$ and $Y \subset Z$. Put $\mu(t) = \alpha(\{x_n(t)\}), t \in J$. It is well known that μ is an absolutely continuous real function defined on J. Consider the following inequalities, with t a.e. in J, $h > 0$ and $t - h > 0$:

$$\begin{aligned}
&\mu(t) - \mu(t-h) = \alpha(\{x_n(t)\}) - \alpha(\{x_n(t-h)\}) = \\
&= \alpha(\{x_n(t)\}) - \alpha(\{x_n(t) - hA(t, x_n(t))\}) + \\
&+ \alpha(\{x_n(t) - hA(t, x_n(t))\}) - \alpha(\{x_n(t-h)\}) \leq \\
&\leq h\omega_A(t, \alpha(\{x_n(t)\})) + \alpha(\{[x_n(t) - x_n(t-h)] - hA(t, x_n(t))\}) \leq \\
&\leq h\omega_A(t, \alpha(\{x_n(t)\})) + h\alpha\left(\left\{h^{-1}\int_{t-h}^{t} [A(s, x_n(s)) - A(t, x_n(t))]\, ds\right\}\right) + \\
&+ h\alpha\left(\left\{h^{-1}\int_{t-h}^{t} B(s, x_n(s))\, ds\right\}\right) \leq \\
&\leq h\omega_A(t, \alpha(x_n(t))) + h\alpha\left(\left\{h^{-1}\int_{t-h}^{t} [A(s, x_n(s)) - A(t, x_n(t))]\, ds\right\}\right) + \\
&+ h\alpha(B([t-h,t], \{x_n[t-h,t]\})),
\end{aligned} \tag{17}$$

where we used Corollary 8 on page 48 of [5]. Dividing by $h > 0$, we get

$$\begin{aligned}
&\frac{\mu(t) - \mu(t-h)}{h} \leq \\
&\quad \leq \omega_A(t, \mu(t)) + \alpha\left(\left\{h^{-1}\int_{t-h}^{t} [A(s, x_n(s)) - A(t, x_n(t))]\, ds\right\}\right) + \\
&\qquad\qquad + \alpha(B([t-h,t], \{x_n[t-h,t]\})).
\end{aligned} \tag{18}$$

Now, we need two remarks. Consider the function

$$\mathcal{A}(t) = t \to \{A(t, x_n(t))\}$$

from J to $\ell^\infty(E)$ (= the Banach space of all bounded sequences of E endowed with the sup norm). By virtue of [15] and the equicontinuity of (x_n), $\mathcal{A}$ verifies Lusin Theorem (see [6]); hence $\mathcal{A}$ is strongly measurable; since $\|\mathcal{A}(t)\|_{\ell^\infty(E)} \leq \varphi_A(t)$ almost everywhere, $\mathcal{A}$ is also Bochner integrable. Hence we have ([17])

$$\lim_{h\to 0^+} h^{-1} \int_{t-h}^{t} \|\mathcal{A}(t) - \mathcal{A}(s)\| \, ds = 0$$

almost everywhere on J. This implies that the diameter of the set

$$\left\{ h^{-1} \int_{t-h}^{t} [A(t, x_n(t)) - A(s, x_n(s))] \, ds : n \in N] \right\}$$

tends to zero as $h \to 0^+$. Hence we can say that

$$\lim_{h\to 0^+} \alpha \left(\left\{ h^{-1} \int_{t-h}^{t} [A(t, x_n(t)) - A(s, x_n(s))] \, ds \right\} \right) = 0.$$

The other remark we shall use, is the following one: by a result due to Ambrosetti ([1]), we know that there is $t^* \in [t, t+h]$ such that $\alpha(\{x_n[t, t+h]\}) = \alpha(\{x_n(t^*)\})$. Since $\alpha(\{x_n(\centerdot)\})$ is continuous (in particular at t), for each $\sigma > 0$ there is $\delta_0 > 0$ such that $|\tilde{t} - t| < \delta_0$ implies $|\alpha(\{x_n(t)\})| < \sigma$. On the other hand, $u \to \omega_B(t, u)$ is continuous; hence, given $\sigma > 0$, it is possible to determine $h^* > 0$ such that, for $h \in]0, h^*]$, we have

$$\omega_B(t, \alpha(\{x_n(t^*)\})) \leq \omega_B(t, \alpha(\{x_n(t)\})) + \sigma.$$

Taking $h \to 0^+$ in (18), our hypotheses and the above couple of remarks show that

$$\mu'(t) \leq \omega_B(t, \alpha(\{x_n(t)\})) + \sigma + \omega_A(t, \alpha(\{x_n(t)\}));$$

the arbitrarity of σ gives that

$$\mu'(t) \leq \omega_B(t, \mu(t)) + \omega_A(t, \mu(t)) \tag{19}$$

for t a.e. in J.

Since $\mu(0) = 0$, (19) gives $\mu(t) = 0$ on J. We are done. □

Remark 4. As observed by Martin ([13]), a typical situation in which (PCP) can be applied, is the following integro-differential equation

$$\frac{\partial u(t,s)}{\partial t} = f(t, s, u(t,s)) + \int_0^1 g(t, s, \tau, u(t, \tau)) \, d\tau \quad (t, s) \in [0,1]^2,$$

where one can put, for instance, $E = C([0,1]), X \subset E$,

$$\begin{aligned} A(t,x)(s) &= f(t, s, x(s)) && (t, s, x) \in [0,1]^2 \times X, \\ B(t,x)(s) &= \int_0^1 g(t, s, \tau, x(\tau)) \, d\tau && (t, s, x) \in [0,1]^2 \times X. \end{aligned}$$

Observe, in particular, that if

$$t \to f(t,s,u) \;\text{ is measurable, for all }\; (s,u) \in [0,1] \times \mathbb{R},$$
$$(s,u) \to f(t,s,u) \;\text{ is continuous, for almost all }\; t \in [0,1],$$

then A verifies (C1) and (C2). Since Z is bounded, there is $M > 0$ such that $|x_n(t)(s)| \leq M$ for all $n \in N$, $t, s \in [0,1]$. Hence if one considers the restriction of f to $[0,1]^2 \times [-M,M]$, by using again the result from [16], given $\varepsilon > 0$, there is a (closed) subset I_ε of I, $m(I \setminus I_\varepsilon) < \varepsilon$, for which $f_{|I_\varepsilon \times [0,1] \times [-M,M]}$ is (uniformly) continuous. It is very easy to show that this implies that $A_{|I_\varepsilon \times Z}$ is uniformly continuous. In the same way, we can show that (4) of Theorem 1 is true, even if B is not uniformly continuous on the whole of $I \times X$. Hence Theorem 1 actually extends Theorem 4 of [11].

This example also shows that assuming (2), (3), (4) (or (12), (13), (15), in the present case), is some time useful; in the present setting A and B are just continuous with respect to $x \in X$, but however verify (4) and (15) when we restrict our interest to $I \times Z$; note that (4) and (15) imply that for almost all $t \in J$, the functions $x \to A(t,x)$ and $x \to B(t,x)$ are uniformly continuous; but, thanks to (4) and (15), we are not requiring this on whole of X, just on Z.

We observe that both Theorem 1 and Theorem 2 improve (at least partially) the previous results due to Deimling ([4]), Emmanuele ([8], [9]), Martin ([13]), Hu Shou Chuan ([11]), Schechter ([17]), Volkmann ([18]).

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