When is every order ideal a ring ideal?

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Abstract. A lattice-ordered ring $\mathbb{R}$ is called an *OIRI-ring* if each of its order ideals is a ring ideal. Generalizing earlier work of Basly and Triki, OIRI-rings are characterized as those f-rings $\mathbb{R}$ such that $\mathbb{R}/\mathbb{I}$ is contained in an f-ring with an identity element that is a strong order unit for some nil l-ideal $\mathbb{I}$ of $\mathbb{R}$. In particular, if $P(\mathbb{R})$ denotes the set of nilpotent elements of the f-ring $\mathbb{R}$, then $\mathbb{R}$ is an OIRI-ring if and only if $\mathbb{R}/P(\mathbb{R})$ is contained in an f-ring with an identity element that is a strong order unit.

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1. Introduction.

Throughout, $\mathbb{R}$ will denote a lattice ordered ring or l-ring. That is, $\mathbb{R}$ is a lattice and a ring in which the sum and product of nonnegative elements is nonnegative. The set of nonnegative elements of a subset $\mathbb{S}$ of $\mathbb{R}$ is denoted by $\mathbb{S}^+$. If $a \in \mathbb{R}$, let $a^+ = a \vee 0, a^- = (-a) \vee 0$, and $|a| = a^+ + a^-$. Let $\mathbb{R}^+ = \{a^+ : a \in \mathbb{R}\}$. For unfamiliar terminology, see [BKW] or [LZ].

By an *order ideal* $\mathbb{I}$ of $\mathbb{R}$ is meant a subgroup of $\mathbb{R}(+)$ such that if $x \in \mathbb{I}$ and $|y| \leq |x|$, then $y \in \mathbb{I}$.

Definition 1.1. If every order ideal of an l-ring $\mathbb{R}$ is a ring ideal, then $\mathbb{R}$ will be called an *OIRI-ring*. Equivalently, $\mathbb{R}$ is an OIRI-ring if and only if for every x, z in $\mathbb{R}$, there is a positive integer n such that $|xz| \vee |zx| \leq n|x|$.

In [BT], M. Basly and A. Triki characterized (without using the name) those OIRI-rings that are archimedean semiprime algebras over the reals. These algebras admit natural norms which have particularly nice properties. We do not discuss such norms. Instead, in what follows, we characterize OIRI-rings within the class of l-rings with a theorem that includes the Basly–Triki characterization as a special case. We pause to recall some definitions.

If $\mathbb{R}$ and $\mathbb{S}$ are l-rings and $\phi : \mathbb{R} \to \mathbb{S}$ is a homomorphism that preserves the lattice as well as the ring operations, then ϕ is called an *l-homomorphism*. The kernel $\ker \phi$ of an l-homomorphism ϕ is called an l-ideal. Equivalently, $\mathbb{I}$ is an l-ideal if and only if it is both a ring ideal and an order ideal. The intersection of all the prime l-ideals of $\mathbb{R}$ will be denoted by $P(\mathbb{R})$ and if $P(\mathbb{R}) = \{0\}$, then $\mathbb{R}$ is said to be *semiprime* (or *reduced* in [BKW]).

$\mathbb{R}$ is said to be an *f-ring* if whenever a, b, c are in $\mathbb{R}^+$

(i) $a \wedge b = 0$ implies $a \wedge bc = a \wedge cb = 0$.

On the other hand, in [BKW], an f-ring is defined to be:

(ii) an l-ring that is a subdirect product of totally ordered rings.

It is clear that the (ii) implies (i), and it is shown in [FH] that (i) and (ii) are equivalent if and only if the prime ideal theorem for Boolean algebras holds. This latter is implied by the Axiom of Choice, but not conversely; see [J].

It is clear from (i) that the class F of f-rings is a *variety*, i.e. every sub-l-ring and every l-homomorphic image of a member of F is in F, and every subdirect product of members of F is in F.

The proof of the next result makes only minor modifications in an argument in [BT].

Throughout, $\mathbb{N}$ will denote the set of positive integers.

Proposition 1.2. *Every OIRI-ring is f-ring.*

Proof: Suppose $a, b, c \in \mathbb{R}^+$ and $a \wedge b = 0$. By assumption there is an $n \in \mathbb{N}$ such that $bc \leq nb$. So, $0 \leq a \wedge bc \leq a \wedge nb \leq n(a \wedge b) = 0$. Thus $a \wedge bc = 0$, and similarly, $a \wedge cb = 0$. Hence $\mathbb{R}$ is an f-ring. $\square$

This proposition and the characterization in [BT] use only the weaker definition (i) of f-ring. In the sequel, we will need to use (ii), so we assume henceforth that the prime ideal theorem for Boolean Algebras holds.

Our main result is that an f-ring $\mathbb{R}$ is an OIRI-ring if and only if $\mathbb{R}/\mathbb{A}(\mathbb{R})$ is contained in an f-ring with an identity element that is a strong order unit if and only if $\mathbb{R}/P(\mathbb{R})$ is contained in an f-ring with an identity element that is a strong order unit, where $\mathbb{A}(\mathbb{R})$ (resp. $P(\mathbb{R})$) denotes the smallest l-ideal containing all left and all right annihilators (resp. all nilpotent elements) of $\mathbb{R}$.

2. Characterizing OIRI-rings.

We will make use below of some known facts about f-rings established in 9.2.6 and 9.7.8 of [BKW].

2.1. If $\mathbb{R}$ is an f-ring, then the intersection $P(\mathbb{R})$ of all the prime l-ideals of $\mathbb{R}$ is the set of nilpotent elements of $\mathbb{R}$, and $\mathbb{R}/P(\mathbb{R})$ is a subdirect sum of totally ordered rings without proper divisors of 0.

2.2. Every semiprime f-ring is a sub-f-ring of an f-ring with identity element 1.

2.3. If $\mathbb{R}$ is an f-ring, then there is a family $\{\phi_\alpha : \alpha \in \Gamma\}$ of l-homomorphisms of $\mathbb{R}$ onto totally ordered rings such that $\bigcap\{\ker \phi_\alpha : \alpha \in \Gamma\} = \{0\}$, in which case $\mathbb{R}$ is a subdirect sum of the totally ordered rings $\mathbb{R}/\ker \phi_\alpha$ as α ranges over Γ. This notation will be used whenever we need to describe an f-ring $\mathbb{R}$ as a subdirect product of totally ordered rings in the sequel. Observe that if $a, b \in \mathbb{R}$, then either $a \leq b$ or there is a $\gamma \in \Gamma$ such that $\phi_\gamma(a) > \phi_\gamma(b)$.

The following lemma makes it easier to verify that an f-ring is an OIRI-ring.

Lemma 2.4. *If $\mathbb{R}$ is an f-ring and x and y are in $\mathbb{R}^+$, the following are equivalent:*

(a) $x^2 \leq nx$ *for some* $n \in \mathbb{N}$.

(b) $xy \vee yx \leq my$ *for some* $m \in \mathbb{N}$.

Proof: Assume (a) and that there is a $\gamma \in \Gamma$ such that $\phi_\gamma(xy) > (n+1)\phi_\gamma(y)$. Then, since $x \in \mathbb{R}^+, \phi_\gamma(x^2y) \geq (n+1)\phi_\gamma(xy) > n\phi_\gamma(xy)$. By (a), since $y \in \mathbb{R}^+$, $x^2y \leq nxy$, whence $\phi_\gamma(x^2y) \leq n\phi_\gamma(xy)$. This contradiction shows that $xy \leq (n+1)y$. Similarly, $yx \leq (n+1)y$, so (b) holds with $m = n+1$.

Obviously, (b) implies (a), and the proof of the lemma is complete. □

Our first theorem follows immediately from the lemma and Proposition 1.2.

Theorem 2.5. *If $\mathbb{R}$ is an l-ring, the following are equivalent:*

(a) $\mathbb{R}$ *is an OIRI-ring.*
(b) $\mathbb{R}$ *is an f-ring and for each $x \in \mathbb{R}^+$ there is an $n \in \mathbb{N}$ such that $x^2 \leq nx$.*

Remark 2.6. It is easy to verify that any l-homomorphic image, sub-l-ring, or direct sum of OIRI-rings is an OIRI-ring. Clearly, the real field R is an OIRI-ring, but no infinite direct product of copies of R is an OIRI-ring, so the class of OIRI-rings fail to form a variety.

A subset of a ring $\mathbb{R}$ is called *nil* if each of its elements is nilpotent. By 2.1, the l-ideal $P(\mathbb{R})$ of an f-ring is nil.

Theorem 2.7. *If $\mathbb{I}$ is a nil l-ideal of an f-ring $\mathbb{R}$, then $\mathbb{R}$ is an OIRI -ring if and only if $\mathbb{R}/\mathbb{I}$ is an OIRI-ring.*

Proof: By the remark, if $\mathbb{R}$ is an OIRI-ring, then so is its l-homomorphic image $\mathbb{R}/\mathbb{I}$. So, assume that $\mathbb{R}/\mathbb{I}$ is an OIRI-ring, and let σ denote an 1-homomorphism of $\mathbb{R}$ onto $\mathbb{R}/\mathbb{I}$. If $x \in \mathbb{R}^+$, then by assumption, there is an $n \in \mathbb{N}$ such that $\sigma(x^2) \leq n\sigma(x)$. Hence there is a $p \in \mathbb{I}$ such that

$$x^2 \leq nx + p. \tag{\dagger}$$

Representing $\mathbb{R}$ as a subdirect product of totally ordered rings as in 2.3, suppose there is a $\gamma \in \Gamma$ such that

$$\phi_\gamma(x^2) > (n+1)\phi_\gamma(x). \tag{$*$}$$

If $\phi_\gamma(p) \leq \phi_\gamma(x)$, then (†) implies that $\phi_\gamma(x^2) \leq (n+1)\phi_\gamma(x)$, contrary to ($*$). Hence $\phi_\gamma(x) < \phi_\gamma(p)$. Since p is nilpotent, so is $\phi_\gamma(p)$, and it follows that $z = \phi_\gamma(x)$ is nilpotent.

If m is the least element of $\mathbb{N}$ such that $z^m = 0$ and $m > 1$, then ($*$) implies $(n+1)z^{m-1} \leq z^m = 0$. So $z^{m-1} = 0$ and hence $z = 0$, contrary to ($*$). Hence $\mathbb{R}$ is an OIRI-ring. □

Definitions 2.8. Suppose $\mathbb{R}$ is an f-ring.

(a) If $\mathbb{R}$ can be embedded in an f-ring with identity element 1, then $\mathbb{R}$ is said to be *unitable*.
(b) If $\mathbb{R}$ has an identity element 1, let

$$Z(\mathbb{R}) = \{a \in \mathbb{R} : |a| \leq n1 \text{ for some } n \in \mathbb{N}\}.$$

(c) If $Z(\mathbb{R}) = \mathbb{R}$, it is customary to call 1 a *strong order unit*.

Remarks 2.9.

(i) As noted in 2.2, every semiprime f-ring is unitable.

(ii) By 9.7.14 of [BKW], the class of unitable f-rings is a variety (= primitive class in [BKW]). By Theorem 1.7 of [HI], it contains all f-rings $\mathbb{R}$ for which 0 is the only left or right annihilator of $\mathbb{R}$.

(iii) By 9.4.16 of [BKW], 0 and 1 are the only idempotents of a totally ordered ring $\mathbb{R}$ with identity element. So 0 is the only idempotent in a unitable totally ordered ring without an identity element.

It is easy to characterize OIRI-rings with an identity element.

Proposition 2.10. *An l-ring $\mathbb{R}$ with identity element 1 is an OIRI-ring $\mathbb{R}$ if and only if 1 is a strong order unit for $\mathbb{R}$.*

Proof: If 1 is a strong order unit for $\mathbb{R}$ and x, y are in $\mathbb{R}$, then there is an $n \in \mathbb{N}$ such that $|x| \leq n1$. So $|xy| \leq |x|\,|y| \leq n|y|$, and similarly, $|yx| \leq n|y|$. Hence $\mathbb{R}$ is an OIRI-ring.

Conversely, if $\mathbb{R}$ is an OIRI-ring, then the smallest order ideal containing 1, namely $Z(\mathbb{R})$, must be a ring ideal, so $Z(\mathbb{R}) = \mathbb{R}$, whence 1 is a strong order unit. $\square$

Combined with Proposition 1.2, this gives an alternate proof of the well-known fact that if $\mathbb{R}$ is an l-ring with identity, then $Z(\mathbb{R})$ is an f-ring.

The following theorem generalizes 2.10 above.

Theorem 2.11. *Suppose $\mathbb{R}$ is a unitable f-ring.*

(a) *If $\mathbb{R}$ is an OIRI-ring and $\mathbb{R}^*$ is an f-ring with identity element containing $\mathbb{R}$, then $\mathbb{R} \subset Z(\mathbb{R}^*)$.*

(b) *Conversely, if $\mathbb{R} \subset Z(\mathbb{R}^*)$ for some f-ring $\mathbb{R}^*$ with identity element containing $\mathbb{R}$, then $\mathbb{R}$ is an OIRI-ring.*

Proof of (a): Suppose there is an $x \in \mathbb{R}^+ \setminus Z(\mathbb{R}^*)$. Now, $x^2 \leq nx$ for some $n \in \mathbb{N}$ since $\mathbb{R}$ is an OIRI-ring. Adopting the notation of 2.3 above to $\mathbb{R}^*$, suppose there is a $\gamma \in \Gamma$ such that $\phi_\gamma(x) > (n+1)\phi_\gamma(1)$. Then $\phi_\gamma(x^2) \geq (n+1)\phi_\gamma(x) > n\phi_\gamma(x)$, but by the above, $\phi_\gamma(x^2) \leq n\phi_\gamma(x)$. This contradiction shows that $\mathbb{R} \subset Z(\mathbb{R}^*)$ and (a) holds. $\square$

Proof of (b): If $\mathbb{R} \subset Z(\mathbb{R}^*)$ and $x \in \mathbb{R}^+$, then there is an $n \in \mathbb{N}$ such that $x \leq n1$. So $x^2 \leq nx$, and it follows from 2.5 that $\mathbb{R}$ is an OIRI-ring. $\square$

Examples 2.12. There are OIRI-rings that fail to be unitable.

(a) Suppose $\mathbb{S}$ denotes the set of 2×2 matrices with entries from the real field R (with its usual total order) whose second row has zero entries. We abbreviate a typical member of $\mathbb{S}$ by $[a\ b]$ for $a, b \in R$. Note that $[a\ b] + [c\ d] = [(a+c)\,(b+d)]$ and $[a\ b]\,[c\ d] = [ac\ ad]$. If $\mathbb{P}^+ = \{[a\ b] : a > 0$ or $a = 0$ and $b \geq 0\}$, then $\mathbb{P}^+$ is the positive cone for a total order on $\mathbb{S}$. Clearly, $P(\mathbb{S}) = \{[0\ b] : b \in R\}$, so $\mathbb{S}/P(\mathbb{S})$ and the OIRI-ring R are isomorphic. Thus, $\mathbb{S}$ is an OIRI-ring by 2.7. Since $[1\ 0]$ is a nonzero idempotent of $\mathbb{S}$, it follows from 2.9 (iii) that $\mathbb{S}$ fails to be unitable.

$\mathbb{S}$ fails to be commutative, so we also supply:

(b) Let $\mathbb{T}$ denote the direct sum of R and R_0, where R_0 has the same addition as in R, while having trivial multiplication. If $(a, b) \in \mathbb{T}$, let $\mathbb{P}^+(\mathbb{T}) = \{(a, b) : a > 0$ or $a = 0$ and $b \geq 0\}$. With $\mathbb{P}^+(\mathbb{T})$ as positive cone, $\mathbb{T}$ becomes a totally ordered ring. Clearly, $P(\mathbb{T}) = \{(0, b) : b \in R_0\}$ and $\mathbb{T}/P(\mathbb{T})$ is isomorphic to R. As in the last example, $\mathbb{T}$ is an OIRI-ring. Since $(1, 0)$ is a nonzero idempotent, the commutative f-ring $\mathbb{T}$ fails to be unitable by 2.9 (iii).

The main results of this paper follow.

Theorem 2.13. *An f-ring $\mathbb{R}$ is an OIRI-ring if and only if there is a nil l-ideal $\mathbb{I}$ such that $\mathbb{R}/\mathbb{I}$ is contained in an f-ring with an identity element that is a strong order unit.*

PROOF: $P(\mathbb{R})$ is a nil l-ideal by 2.1, $\mathbb{R}/P(\mathbb{R})$ is unitable by 2.2, and is an OIRI-ring since it is an l-homomorphic image of an OIRI-ring. So there is an f-ring $\mathbb{R}^*$ with identity element containing $\mathbb{R}/P(\mathbb{R})$ and by 2.11, $\mathbb{R}/P(\mathbb{R}) \subset Z(\mathbb{R}^*)$. Hence the necessity holds since 1 is a strong order unit for $Z(\mathbb{R}^*)$.

Conversely, if $\mathbb{I}$ is a nil l-ideal such that $\mathbb{R}/\mathbb{I}$ is contained in an f-ring $\mathbb{S}$ with identity element for which $\mathbb{S} = Z(\mathbb{S})$, then $\mathbb{R}$ is an OIRI-ring by 2.11 and 2.7. ☐

Let $\mathbb{A}(\mathbb{R})$ denote the sum of the left annihilator $\mathbb{A}_l(\mathbb{R})$ of $\mathbb{R}$ and the right annihilator $\mathbb{A}_r(\mathbb{R})$ of the f-ring $\mathbb{R}$. By 2.1, each of these two latter ideals is contained in $P(\mathbb{R})$, as is their sum $\mathbb{A}(\mathbb{R})$. So $\mathbb{A}(\mathbb{R})$ is a nil l-ideal. Moreover, by 2.9 (ii), $\mathbb{R}/\mathbb{A}(\mathbb{R})$ is unitable. Thus we have:

Corollary 2.14. *If $\mathbb{R}$ is an f-ring, then the following are equivalent:*

(a) $\mathbb{R}$ *is an OIRI-ring.*
(b) $\mathbb{R}/\mathbb{A}(\mathbb{R})$ *is contained in an f-ring with an identity element that is a strong order unit.*
(c) $\mathbb{R}/P(\mathbb{R})$ *is contained in an f-ring with an identity element that is a strong order unit.*

The hypothesis that $\mathbb{R}$ is an f-ring in 2.13 and 2.14, cannot be weakened to the assumption that $\mathbb{R}$ is an l-ring as is shown by the next example.

Example 2.15. Let $\mathbb{U}$ denote the ring of upper triangular 2×2 matrices with real entries and ordered coordinatewise. Clearly, $\mathbb{U}$ is an l-ring and the set $\mathbb{T}$ of elements of $\mathbb{U}$ whose diagonal entries are 0 is a nil l-ideal such that $\mathbb{U}/\mathbb{T}$ is a direct sum of two copies of R. Thus $\mathbb{U}/\mathbb{T}$ is an OIRI-ring with strong order unit diag (1,1). But $\mathbb{U}$ fails to be an f-ring since the matrix whose first row is $[1\ -1]$ and whose second row is $[0\ 0]$ is its own square and fails to be nonnegative.

We close with the remark that Example 2.12 (a) shows that $\mathbb{A}(\mathbb{R})$ may not be replaced by $\mathbb{A}_r(\mathbb{R})$ in the statement of Corollary 2.14, since in that example, $\mathbb{A}_r(\mathbb{S}) = \{[0\ 0]\}$ and $\mathbb{S}$ is not unitable.

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