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On some translation invariant balayage spaces

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Abstract. It is well known that strong Feller semigroups generate balayage spaces provided the set of their excessive functions contains sufficiently many elements. In this note, we give explicit examples of strong Feller semigroups which do generate balayage spaces. Further we want to point out the role of the generator of the semigroup in the related potential theory.

Keywords: balayage spaces, strong Feller semigroups, non-local potential theory

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Introduction.

Since the pioneering work of Marcel Riesz [9] it is clear that a lot of results which hold for the classical Newton potential do also hold for more general potentials which are related to non-local operators. M. Riesz did not work with such operators but the kernels of these potentials are in modern terminology nothing but fundamental solutions of certain pseudo differential operators. In his monograph [8] N.S. Landkof considered in detail the non-local part of potential theory, in particular he handled these aspects of the theory which are related to the balayage theory.

In order to have an axiomatic approach to the theory of balayage and to incorporate probabilistic ideas into the theory, J. Bliedtner and W. Hansen introduced the concept of a balayage space. Their results are presented in [2]. They had been able to prove that a lot of known objects in potential theory can be treated within their considerations.

The purpose of this note is to give some examples of translation invariant balayage spaces. It turns out that as in the theory of translation invariant Dirichlet spaces and in the theory of translation invariant harmonic spaces, see [4] and [5], continuous negative definite functions play a fundamental part in our considerations.

The plan of the paper is following: In the first section we just recall some basic definitions and results from [2], while Section 2 is devoted to continuous negative definite functions and Feller semigroups. In this section we also present some ideas how one could look at non-local potential theory from another point of view. These ideas will be handled more precisely in a forthcoming paper. Finally, in Section 3 we prove our main result, i.e. we do the calculations in order to get new examples of balayage spaces.

M. Brzezina pointed out that it is possible to construct further examples of balayage spaces by using our examples. His results are given in [3].

1. The notion of a balayage space.

We will recall some basic definitions and results from the theory of balayage spaces as they are given in [2].

Let X be a locally compact topological space with countable base and denote by $B(X)$ the numerical Borel functions, i.e. the Borel functions $f : X \to \overline{\mathbb{R}}$. The space of continuous functions on X is denoted by $C(X)$. By definition a convex cone $S \subset C(X)$ is called a **function cone** if S contains a strictly positive function, the set of non-negative functions in S is linearly separating, i.e. for any $x, y \in X$, $x \neq y$, and any $\lambda \geq 0$ there exists $f \in S, f \geq 0$, such that $f(x) \neq \lambda f(y)$, and for any $f \in S$ there exists a non-negative function $g \in S$ such that for any $\varepsilon > 0$ there exists a compact set $C \subset X$ such that $|f(x)| \leq \varepsilon g(x)$ for $x \in X - C$. Let $F \subset B(X)$ and define

$$S(F) = \{\sup f_n, (f_n)_{n\in\mathbb{N}} \text{ is an increasing sequence in } F\}.$$

We say that F is **σ-stable** if $S(F) = F$. Further if $f : X \to \mathbb{R} \cup \{\infty\}$ is a numerical function we denote by f^* the lower semi-continuous regularization of f, i.e. $f(x) = \liminf_{y\to x} f(y), x \in X$. (In [2] the function f^* is denoted by $\hat{f}$, but $\hat{}$ is reserved for the Fourier transform!) With any convex cone W of non-negative lower semi-continuous numerical functions on X one can associate the **W-fine topology**. This is the coarsest topology on X which is finer than the initial topology and for which all functions of W are continuous.

Now we can state the fundamental

Definition 1.1 ([2, p. 57]).

Let X be a locally compact topological space with countable base and W a convex cone of non-negative lower semi-continuous functions. The pair (X, W) is called a **balayage space** if the following conditions hold:

1. The cone W is σ-stable.
2. For every subset $V \subset W$ we have $(\inf V)^{*f} \in W$, where $*f$ denotes the lower semi-continuous regularization with respect to the W-fine topology.
3. For $u, v_1, v_2 \in W$ such that $u \leq v_1 + v_2$ there exist $u_1, u_2 \in W$ such that $u = u_1 + u_2, u_1 \leq v_1$ and $u_2 \leq v_2$.
4. There exists a function cone P of non-negative continuous functions such that $W = S(P)$.

In [2] the theory of balayage spaces is developed and some known structures of potential theory are identified as balayage spaces. Further it is shown that one can obtain further examples of balayage spaces by using certain Feller semigroups. These are the results we will recall next.

A **kernel** K on X is a mapping $K : X \times \mathbf{B}(X) \to \mathbb{R}^+ \cup \{\infty\}$, $\mathbf{B}(X)$ denotes the Borel sets of X, such that $x \mapsto K(x, B)$ is a Borel measurable function for every $B \in \mathbf{B}(X)$ and $B \mapsto K(x, B)$ is a measure for every $x \in X$. A kernel K is called **sub-Markovian** if $K(x, X) \leq 1$. We call K **proper** if for every compact $C \subset X$ the function $x \mapsto K(x, C)$ is bounded.

Definition 1.2. A family $(P_t)_{t>0}$ of sub-Markovian kernels on X is called a **Feller-semigroup** if

$$P_{s+t}(x,B) = \int_X P_s(y,B)P_t(x,dy) \tag{1.1}$$

holds for all $x \in X$ and $B \in \mathbf{B}(X)$, $s,t > 0$, if the operators P_t defined on the bounded Borel functions by

$$(P_t f)(x) = \int_X f(y)P_t(x,dy) \tag{1.2}$$

map $C_\infty(X)$, the space of continuous functions on X vanishing at infinity, into itself, and if it is strongly continuous, i.e.

$$\lim_{t\to 0} P_t f = f \quad \text{uniformly on } X \text{ for all } f \in C_\infty(X). \tag{1.3}$$

A Feller semigroup $(P_t)_{t>0}$ is called a **strong Feller semigroup** if each of the operators P_t maps the bounded Borel functions into the set of bounded continuous functions. Given a Feller semigroup, we define the set of **excessive functions** of $(P_t)_{t>0}$ as

$$E_P = \{u \in B(X), u \geq 0 \text{ and } \sup_{t>0} P_t u = u\}.$$

The following result holds:

Theorem 1.1 ([2, p. 177])**.** *Let $(P_t)_{t>0}$ be a strong Feller semigroup on X. Suppose further that there exist strictly positive functions $u, v \in E_P \cap C(X)$ such that $\frac{u}{v} \in C_\infty(X)$ and that the kernel V_0 defined on $X \times \mathbf{B}(X)$ by $V_0(x,B) = \int_0^\infty P_t(x,B)\,dt$ is proper. Then (X, E_P) is a balayage space.*

Later we have to use the resolvent of a Feller semigroup. If $(P_t)_{t>0}$ is a Feller semigroup, we define its **resolvent** by

$$V_\lambda(x,B) = \int_0^\infty e^{-\lambda t} P_t(x,B)\,dt,\ \lambda > 0, \tag{1.4}$$

and the corresponding operator

$$V_\lambda f(x) = \int_0^\infty e^{-\lambda t}(P_t f)(x)\,dt \tag{1.5}$$

for $f \in B(X)$, $f \geq 0$. Note that whenever we will use the resolvent later on we have

$$V_0 f = \sup_{\lambda>0} V_\lambda f = \lim_{\lambda\to 0} V_\lambda f \tag{1.6}$$

for all Borel functions $f \geq 0$, where V_0 also denotes the operator associated with the kernel V_0. This kernel is called the **potential kernel** of $(P_t)_{t>0}$. Let us denote by

$$E_V = \{u \in B(X), u \geq 0 \text{ and } \sup_{\lambda>0} \lambda V_\lambda u = u\} \tag{1.7}$$

the set of **excessive functions of** $(V_\lambda)_{\lambda>0}$. From Corollary II.3.13 in [2] we find in the case of a strong Feller semigroup

$$E_P = E_V,$$

and therefore we can speak of excessive functions without further distinction.

2. Translation invariant Feller semigroups, related pseudo differential operators and some formal considerations.

We need

Definition 2.1. A function $a : \mathbb{R}^n \to \mathbb{C}$ is said to be **negative definite** if for all $m \in \mathbb{N}$ and $(x^1, \dots, x^m), x^j \in \mathbb{R}^n$ and $1 \leq j \leq m$, the matrix $(a(x^i) + \overline{a(x^j)} - a(x^i - x^j))_{i,j=1,\dots,m}$ is positive Hermitian, i.e. for all m-tuple $(c_1, \dots, c_m) \in \mathbb{C}^m$ it follows that

$$\sum_{i,j=1}^{m} (a(x^i) + \overline{a(x^j)} - a(x^i - x^j)) c_i \overline{c_j} \geq 0. \tag{2.1}$$

A good reference for negative definite functions is the book [1]. In this paper we are mainly concerned with continuous negative definite functions which are real-valued. The following results are taken from [1].

Lemma 2.1 ([1, §7]). *Let $a : \mathbb{R}^n \to \mathbb{R}$ be a continuous negative definite function. Then we have $a(\xi) = a(-\xi)$ and*

$$0 \leq a(0) \leq a(\xi) \quad \text{for all} \quad \xi \in \mathbb{R}^n, \tag{2.2}$$

and

$$a(\xi) \leq c_a (1 + |\xi|^2) \quad \text{for all} \quad \xi \in \mathbb{R}^n. \tag{2.3}$$

Furthermore, we have

Lemma 2.2 ([1, p. 89]). *Every translation invariant Feller semigroup on $\mathbb{R}^n$ is a convolution semigroup, i.e.*

$$P_t f = \mu_t * f, \; f \in C_\infty(\mathbb{R}^n) \quad \text{and} \quad t > 0, \tag{2.4}$$

where $(\mu_t)_{t>0}$ is a family of bounded measures on $\mathbb{R}^n$ such that $\mu_t(\mathbb{R}^n) \leq 1$, $\mu_t * \mu_s = \mu_{s+t}, s, t > 0$, and $\lim_{t\to 0} \mu_t = \delta_0$ vaguely, where δ_0 denotes Dirac's delta-function. Conversely, given a convolution semigroup $(\mu_t)_{t>0}$ with the properties stated above then (2.4) defines a translation invariant Feller semigroup on $\mathbb{R}^n$.

The connection between the notion of negative definite functions and Lemma 2.2 is given by

Theorem 2.1 ([1, p. 49]). *There is a one-to-one correspondence between convolution semigroups on $\mathbb{R}^n$ and continuous negative definite functions on $\mathbb{R}^n$. If $(\mu_t)_{t>0}$ is a convolution semigroup on $\mathbb{R}^n$, then there exists a unique continuous negative definite function $a : \mathbb{R}^n \to \mathbb{C}$ such that*

$$\hat{\mu}_t(\xi) = e^{-ta(\xi)}, \; t > 0 \quad \text{and} \quad \xi \in \mathbb{R}^n. \tag{2.5}$$

Conversely, given a continuous negative definite function $a : \mathbb{R}^n \to \mathbb{C}$, then (2.5) *determines a convolution semigroup on $\mathbb{R}^n$.*

Combining (2.4) with (2.5) and assuming that the formal calculation could be justified we get

$$\begin{aligned}(P_t f)(x) &= (\mu_t * f)(x) \\ &= (2\pi)^{-n} \int_{\mathbb{R}^n} e^{ix\cdot\xi} (\mu_t * f)\hat{}\,(\xi)\, d\xi \\ &= (2\pi)^{-n} \int_{\mathbb{R}^n} e^{ix\cdot\xi} e^{-ta(\xi)} \hat{f}(\xi)\, d\xi.\end{aligned} \tag{2.6}$$

Moreover, for the resolvent of $(P_t)_{t>0}$ we find again formally

$$\begin{aligned}V_\lambda f(x) &= \int_0^\infty e^{-\lambda t}(P_t f)(x)\, dt \\ &= (2\pi)^{-n} \int_0^\infty \int_{\mathbb{R}^n} e^{ix\cdot\xi} e^{-(\lambda + a(\xi))t} \hat{f}(\xi)\, d\xi\, dt \\ &= (2\pi)^{-n} \int_{\mathbb{R}^n} e^{ix\cdot\xi} \frac{1}{\lambda + a(\xi)} \hat{f}(\xi)\, d\xi.\end{aligned} \tag{2.7}$$

The goal of the next section is to make these formal calculations correct for a certain class of continuous negative definite functions and then to use (2.6), (2.7) and the following theorem due to J. Hawkes in order to apply Theorem 1.1.

Theorem 2.2 ([6, Lemma 2.1])**.** *Let $(P_t)_{t>0}$ be a translation invariant Feller semigroup. This semigroup is a strong Feller semigroup if and only if each of the measures $P_t(x, .)$, $t > 0$, has a density with respect to the Lebesgue measure.*

Let us have a further look at (2.7). Formally the operator V_λ is inverse to the operator

$$(a(D) + \lambda)u(x) = (2\pi)^{-n} \int_{\mathbb{R}^n} e^{ix\cdot\xi}(a(\xi) + \lambda)\hat{u}(\xi)\, d\xi, \tag{2.8}$$

which is a pseudo differential operator. Moreover, (2.8) with $\lambda = 0$ should be regarded as an operator densely defined on $C_\infty(\mathbb{R}^n)$ which is closable and a certain closed extension of it should be the generator of the semigroup under consideration, i.e. $P_t = e^{-a(D)t}$.

Finally suppose that

$$E(x) = (2\pi)^{-n} \int_{\mathbb{R}^n} e^{ix\cdot\xi} \frac{1}{a(\xi)}\, d\xi \tag{2.9}$$

exists. Then we find again by a formal calculation

$$a(D)E(x) = (2\pi)^{-n} \int_{\mathbb{R}^n} e^{ix\cdot\xi} a(\xi)\hat{E}(\xi)\, d\xi = \delta_0,$$

i.e. E is a fundamental solution of $a(D)$. For $a(0) > 0$ it follows that E exists at least in the sense of tempered distributions. For any Borel measure μ on $\mathbb{R}^n$ we can define the potential

$$E^\mu(x) = (E * \mu)(x) = \int_{\mathbb{R}^n} E(x-y)\,d\mu(y) \tag{2.10}$$

and we find

$$a(D)E^\mu(x) = \mu.$$

In particular, we have

$$a(D)E^\mu(x) = 0 \ \text{ in } \ \mathbb{R}^n - \operatorname{supp}\mu.$$

Defining a function (or distribution) $u : \mathbb{R}^n \to \mathbb{R}$ to be "harmonic" with respect to $a(D)$ in an open set Ω if and only if $a(D)u = 0$ in Ω, we find that the potential E^μ is harmonic with respect to $a(D)$ in the open set $\mathbb{R}^n - \operatorname{supp}\mu$. As pointed out in [7], Theorem 1, for $a(\xi) = |\xi|^{2r}$, $0 < r < 1$, this is a natural way to look at $2r$-harmonic functions in the sense of N.S. Landkof, see [8]. In a forthcoming paper we will develop this idea more systematically and rigorously. The purpose of the next section is to establish the fact that for a large class of continuous negative definite functions the corresponding Feller semigroup gives a balayage space.

3. Some balayage spaces generated by pseudo differential operators.

In this section $a : \mathbb{R}^n \to \mathbb{R}$ denotes a continuous negative definite function and further it is always assumed that

$$a(0) > 0 \tag{3.1}$$

and that for some $s \in (0, 2]$

$$a(\xi) \geq c_s|\xi|^s, \ \ |\xi| \geq \varrho, \tag{3.2}$$

holds with some $\varrho \geq 0$. Let $(P_t)_{t>0}$ be the Feller semigroup associated with a by Theorem 2.1 and Lemma 2.2. Now (3.2) and the fact that $e^{-ta(\xi)}$ is a positive definite function, see [1, Theorem 7.8], imply that

$$g_t(x) = (2\pi)^{-n} \int_{\mathbb{R}^n} e^{ix\cdot\xi} e^{-ta(\xi)}\,d\xi \tag{3.3}$$

exists, is non-negative and further $g_t(.-x)$ is a density of the measure $P_t(x, .)$ with respect to the Lebesgue measure. The last statement follows from the uniqueness of the Fourier transform. By Theorem 2.2 the semigroup $(P_t)_{t>0}$ is a strong Feller semigroup. By Corollary 1.26 in [10] we find that $g_t \in L^1(\mathbb{R}^n)$ and $\int_{\mathbb{R}^n} g_t(x)\,dx = e^{-ta(0)}$ for $t > 0$. Thus we have

$$P_t(x, B) = \int_B g_t(x-y)\,dy \tag{3.4}$$

and for $f \in L^1(\mathbb{R}^n)$ we get using (3.1) and Young's inequality

$$\int_0^\infty (P_t f)(x)\,dt = \int_0^\infty \int_{\mathbb{R}^n} g_t(x-y) f(y)\,dy\,dt \le \frac{1}{a(0)} \|f\|_{L^1}\,.$$

From this it follows that

$$(V_0 f)(x) = \int_0^\infty (P_t f)(x)\,dt \tag{3.5}$$

is proper, where we call an operator generated by a kernel proper if and only if the kernel is proper. Further it follows that G_λ defined by

$$G_\lambda(x) = \int_0^\infty e^{-\lambda t} g_t(x)\,dt, \quad \lambda > 0, \tag{3.6}$$

belongs to $L^1(\mathbb{R}^n)$ and

$$\begin{aligned} \|G_\lambda\|_{L^1} &= \int_{\mathbb{R}^n} \int_0^\infty e^{-\lambda t} g_t(x)\,dt\,dx = \int_0^\infty e^{-\lambda t} \int_{\mathbb{R}^n} g_t(x)\,dx\,dt \\ &= \int_0^\infty e^{-(\lambda + a(0))t}\,dt = \frac{1}{\lambda + a(0)}\,. \end{aligned}$$

We find that

$$\begin{aligned} V_\lambda f(x) &= \int_0^\infty e^{-\lambda t} (P_t f)(x)\,dt \\ &= \int_0^\infty \int_{\mathbb{R}^n} e^{-\lambda t} g_t(x-y) f(y)\,dy\,dt = (G_\lambda * f)(x) \end{aligned}$$

and again by Young's inequality V_λ is defined for $f \in L^p(\mathbb{R}^n), 1 \le p \le \infty$. Now we prove that the function $x \mapsto 1$ belongs to $E_P = E_V$. Indeed we have

$$\sup_{\lambda>0} \lambda \int_0^\infty \int_{\mathbb{R}^n} e^{-\lambda t} g_t(x-y)\,dy\,dt = \sup_{\lambda>0} \lambda \int_0^\infty e^{-(\lambda+a(0))t}\,dt = \sup_{\lambda>0} \frac{\lambda}{\lambda + a(0)} = 1.$$

Further it is known by Proposition II.3.8 in [2] that

$$V_0(\{f \in B(X), f \ge 0\}) \subset E_V\,.$$

But for $f \in S(\mathbb{R}^n)$ we find

$$\begin{aligned} V_0 f(x) &= \int_0^\infty \int_{\mathbb{R}^n} g_t(x-y) f(y)\,dy\,dt \\ &= (2\pi)^{-n} \int_0^\infty \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{i(x-y)\cdot\xi} e^{-ta(\xi)} f(y)\,d\xi\,dy\,dt \\ &= (2\pi)^{-n} \int_0^\infty \int_{\mathbb{R}^n} e^{ix\cdot\xi} e^{-ta(\xi)} \hat{f}(\xi)\,d\xi\,dt \\ &= (2\pi)^{-n} \int_{\mathbb{R}^n} e^{ix\cdot\xi} \frac{1}{a(\xi)} \hat{f}(\xi)\,d\xi, \end{aligned}$$

where the calculation is now rigorous since $a(\xi) \ge a(0) > 0$ and $S(\mathbb{R}^n)$ denotes the Schwartz space of tempered functions, see [10, p. 19]. Thus for any $f \in S(\mathbb{R}^n)$, $f \ge 0$, it follows that $V_0 f$ belongs to $E_V \cap C_\infty(\mathbb{R}^n)$. By Theorem 1.1 we finally have taking $v = 1$ and $u = V_0 f$ for some $f > 0, f \in S(\mathbb{R}^n)$,

Theorem 3.1. *Let $a : \mathbb{R}^n \to \mathbb{R}$ be a continuous negative definite function satisfying (3.1) and (3.2) and denote by $(P_t)_{t>0}$ the semigroup corresponding to a. Then $(\mathbb{R}^n, E_P)$ is a balayage space.*

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