Envelopes of holomorphy for solutions of the Laplace and Dirac equations

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Abstract. Analytic continuation and domains of holomorphy for solution to the complex Laplace and Dirac equations in $\mathbf{C}^n$ are studied. First, geometric description of envelopes of holomorphy over domains in $\mathbf{E}^n$ is given. In more general case, solutions can be continued by integral formulas using values on a real $n-1$ dimensional cycle in $\mathbf{C}^n$. Sufficient conditions for this being possible are formulated.

Keywords: envelope of holomorphy, integral formula, index, null-convexity, complex null cone, Lipschitz boundary

Classification: 32D10, 30G35

Introduction.

The aim of this paper is to give a geometric description of natural domains of holomorphy for solutions of the complex Laplace and Dirac equations. From one point of view, it is analogous to the study of domains of holomorphy for functions of several complex variables. Instead of holomorphic functions, we consider holomorphic solutions to complex partial differential equations.

The study of partial differential equations on domains in $\mathbf{C}^n$ was inspired by quantum field theory. One of the fundamental questions that arose in the physical context was that of analytic continuation of solutions and of domains of holomorphy. There is a substantial difference between even and odd dimensions. We will confine ourselves to the case of even dimension, $n = 2k$.

In the part 2, we consider the basic case of the continuation of solutions from $\mathbf{E}^n$ to $\mathbf{C}^n$. Given a domain in $\mathbf{E}^n$ we describe a corresponding domain in $\mathbf{C}^n$, the envelope of holomorphy, with the property that every solution on the original domain extends to a holomorphic solution on the envelope of holomorphy. These domains were previously described in [4], [5]. We give a new, constructive description of envelopes of holomorphy.

J. Ryan further generalized Euclidean domains to a certain class of real n-dimensional manifolds with boundary in $\mathbf{C}^n$ (see [4]). He used the generalized Cauchy integral formula to give other examples of envelopes of holomorphy. In the part 3, we apply more efficient integral formulas, introduced in the part 1, to show that we need only an $(n-1)$-dimensional real closed manifold (which may but need not be given as the boundary of an n-dimensional surface) to be able to holomorphically continue the solutions.

1. Integral formulas for the Dirac and Laplace equations.

For $z \in \mathbf{C}^n$, let $z = (z_1, ..., z_n)$, $z_i = x_i + iy_i$. Let $\mathbf{E}^n$ denote the Euclidean subspace of $\mathbf{C}^n$:

$$\mathbf{E}^n = \{z \in \mathbf{C}^n : y_i = 0 \text{ for } i = 1, ..., n\}.$$

Further, we use

$$\|z\| = \left(\sum_{i=1}^{n} x_i^2 + \sum_{i=1}^{n} y_i^2\right)^{\frac{1}{2}},$$

$$|z|^2 = \sum_{i=1}^{n} z_i^2.$$

The set

$$CN(p) = \left\{q \in \mathbf{C}^n : |p - q|^2 = 0\right\}$$

is called the complex null cone of a point $p \in \mathbf{C}^n$. The complex Laplace operator is defined by

$$\Delta_C = \sum_{i=1}^{n} \frac{\partial^2}{\partial z_i^2},$$

and its real version

$$\Delta_R = \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2}.$$

Let $\mathbf{C}_n^C$ be the complex Clifford algebra over $\mathbf{C}^n$ with the quadratic form $-(z_1^2 + ... + z_n^2)$. We denote by $e_i, i = 1, ..., n$ the canonical generators of $\mathbf{C}_n^C$ and define the complex Dirac operator

$$\mathbf{D}_C = \sum_{i=1}^{n} e_i \frac{\partial}{\partial z_i}$$

and its real version

$$\mathbf{D}_R = \sum_{i=1}^{n} e_i \frac{\partial}{\partial x_i}$$

which act on S-valued functions, where S is any left ideal in $\mathbf{C}_n^C$.

The integral formulas we are going to use are typical for even dimensions. They were first proved in $\mathbf{E}^n$ in [1] and they are analogous to the Cauchy formula for holomorphic functions of one complex variable. For Laplace equation, they coincide with well known formulas for harmonic functions. In the complexification, they are proved in homological form in [2]. This form is especially useful in higher dimensions.

It is a typical feature of these formulas that the closed differential form under the integral is not defined on the characteristic surface of the equation. In the euclidean case, the singularity is just the point p and the value of the integral over a closed $(n-1)$-dimensional cycle depends only on its class in the homology group $H_{n-1}(\mathbf{E}^n \setminus \{p\}, \mathbf{Z})$. In the complexification, the singularity is more complicated: it is the complex null cone $CN(p)$. This leads to the notion of index of a point $p \in \mathbf{C}^n$ with respect to an $(n-1)$-dimensional cycle, where we suppose that the cycle lies outside the singularity. In other words, we are interested in the homology group $H_{n-1}(\mathbf{C}^n \setminus CN(p), \mathbf{Z})$. The situation is much more complicated than in the euclidean case, where the group $H_{n-1}(\mathbf{E}^n \setminus \{p\}, \mathbf{Z})$ is clearly a free abelian group with one generator. To see that the same holds in the complexification, requires much more effort.

Theorem 1. *Let n be any positive integer. Then*

$$H_{n-1}(\mathbf{C}^n \setminus CN(p), \mathbf{Z}) \simeq \mathbf{Z}$$

and for $\epsilon > 0$ the sphere

$$S^{\epsilon}_{n-1}(p) = \{q \in \mathbf{C}^n : p - q \in \mathbf{E}^n, \|p - q\| = \epsilon\}$$

gives the generator of this group.

Definition 1. The number k for which γ is homological to $kS^{\epsilon}_{n-1}(p)$ is to be called and denoted by $\mathrm{ind}_{\gamma}(p)$.

Let Ω be a domain in $\mathbf{C}^n$. Let us consider solutions defined on Ω. In the standard euclidean formulation of the Cauchy integral formula in homological form, the domain Ω can be arbitrary, but the contour of integration must be homologically trivial in Ω. The reason is that in this case γ is also homological to a small sphere around p in $\Omega \setminus \{p\}$ and for the proof, it suffices to let the radius of the sphere go to zero. The situation is quite different in the complex case. The difficulty lies in the fact that it is no more true that a homologically trivial cycle must be homological to a small $(n-1)$-dimensional sphere around p in $\Omega \setminus CN(p)$. In other words, we cannot replace $\mathbf{C}^n$ by an arbitrary domain Ω in Theorem 1. We have to impose a restriction on Ω. The idea is that when γ is (during the deformation to a point) near the cone $CN(p)$ it must be possible to follow the rays on $CN(p)$ toward the point p. The following simple condition guarantees this.

Definition 2. We say that $\Omega \subseteq \mathbf{C}^n$ is null-convex with respect to a point $p \in \Omega$ if for all $q \in \Omega$ such that $|p - q|^2 = 0$, the whole segment $\overline{pq}$ lies in Ω.

Theorem 2 (Integral formula for the complex Dirac operator)**.** *Let f be a solution of the complex Dirac equation $\displaystyle\sum_{i=1}^{n} e_i \frac{\partial f}{\partial z_i} = 0$ on $\Omega \subseteq \mathbf{C}^n$, where Ω is a null-convex* *domain with respect to a point $p \in \Omega$. Let γ be an $(n-1)$-dimensional cycle in $\Omega \setminus CN(p)$ which is homologically trivial in Ω. Then*

$$f(p)\,\mathrm{ind}_\gamma\,(p) = \frac{1}{k_n}\int_\gamma \frac{-(p-z)}{|p-z|^{n-2}}\,dz f(z),$$

where k_n is the area of the unit sphere in $\mathbf{E}^n$.

Theorem 3 (Integral formula for the complex Laplace operator)**.** *Let f be a solution to the complex Laplace equation on a domain $\Omega \subseteq \mathbf{C}^n$ null-convex with respect to a point $p \in \Omega$. Let γ be an $(n-1)$-dimensional cycle in $\Omega \setminus CN(p)$ which is homologically trivial in Ω. Then*

$$f(p)\,\mathrm{ind}_\gamma\,(p) = \frac{1}{k_n}\int_\gamma \frac{\sum_{i=1}^n (-1)^i (p-z)_i d\hat{z}_i}{|p-z|^n} f(z) + \frac{\sum_{i=1}^n (-1)^i \dfrac{\partial f}{\partial z_i}\, d\hat{z}_i}{(n-2)|p-z|^{n-2}}\,,$$

where $d\hat{z}_i = dz_1 \wedge \ldots \wedge dz_{i-1} \wedge dz_{i+1} \wedge \cdots \wedge dz_n$.

The proofs of the theorems can be found in [2]. □

2. The holomorphic continuation of solutions from Euclidean domains.

All our further considerations are common to $\mathbf{D}_C$ and Δ_C. For simplicity, we denote the corresponding operator by $\mathbf{d}_C$, so $\mathbf{d}_C$ denotes either $\mathbf{D}_C$ or Δ_C. We use the symbol $\mathbf{d}_R$ in the same way. When we talk about a solution $f : \Omega \to \mathbf{C}_n^C$ we always mean that, for the Laplace operator, the solution has values in C, while for the Dirac operator in S.

Let $f(z_1, z_2, ..., z_n)$ be a holomorphic function of n complex variables satisfying the complex equation $\mathbf{d}_C f = 0$ in Ω. It follows immediately from Cauchy-Riemann equations that $\dfrac{\partial f}{\partial z_i} = \dfrac{\partial f}{\partial x_i}$. So the restriction of f to $\mathbf{E}^n$, a function of n real variables $f(x_1, x_2, \ldots, x_n)$, satisfies the real equation $\mathbf{d}_R f = 0$ in $\mathbf{E}^n \cap \Omega$. On the other hand, the real Laplace operator is elliptic and so is $\mathbf{D}_R$ (see [1] for the proof). All solutions to the equation $\mathbf{d}_R f = 0$ are therefore real analytic functions. Let $\Omega \subseteq \mathbf{E}^n$ be the domain of a solution. The corresponding power series at a point x has a nonzero radius of convergence and so defines a holomorphic function on a neighbourhood $U(x) \subseteq \mathbf{C}^n$. The open set $\tilde{\Omega} = \bigcup_{x\in\Omega} U(x)$ is then a neighbourhood (in $\mathbf{C}^n$) of the original domain, and the solution can be holomorphically continued to $\tilde{\Omega}$. Let a domain $M \subseteq \mathbf{E}^n$ be given. We want to describe the largest domain $\tilde{M} \subseteq \mathbf{C}^n$ to which all the solutions defined on M extend holomorphically.

Definition 3. Let M be a domain in $\mathbf{E}^n$. The component of the set $\mathbf{C}^n \setminus \bigcup_{x\in\partial M} CN(x)$ which contains the interior of M is called the envelope of holomorphy of M and is denoted by $\tilde{M}$.

Theorem 4. *Let M be a domain in $\mathbf{E}^n$.*

(i) *If f satisfies $\mathbf{d}_R f = 0$ on M, then there is a function $\tilde{f} : \tilde{M} \to \mathbf{C}_n^C$ such that $\mathbf{d}_C \tilde{f} = 0$ on $\tilde{M}$ and $\tilde{f}\mid_M = f$.*

(ii) *For each point $x \in \partial\tilde{M}$, there is a solution $\tilde{f}_x$ defined on $\tilde{M}$ which is unbounded in x.*

PROOF: (i) Put

$$\tilde{f}(p) = \frac{1}{\mathrm{ind}_\gamma\,(p)}\frac{1}{k_n}\int_\gamma \frac{-(p-z)}{|p-z|^{n-2}}dz f(z)$$

and

$$\tilde{f}(p) = \frac{1}{\mathrm{ind}_\gamma\,(p)}\frac{1}{k_n}\int_\gamma \frac{\sum_{i=1}^n(-1)^i(p-z)_i d\hat{z}_i}{|p-z|^n} f(z) + \frac{\sum_{i=1}^n(-1)^i\dfrac{\partial f}{\partial z_i}\,d\hat{z}_i}{(n-2)|p-z|^{n-2}}$$

for the Dirac and Laplace operators, respectively. The function $\tilde{f}$ is holomorphic and since the same formulas hold for $f(x)$, $x \in M$, we have $\tilde{f}\mid_M = f$.

(ii) Since $x \in \partial\tilde{M}$, there is $x_0 \in \partial M$ such that $|x - x_0|^2 = 0$. The map

$$\tilde{f}_x(z) = \frac{1}{|z-x_0|^{n-2}}$$

is an elementary solution of the Laplace equation. It is defined on $\tilde{M}$ and unbounded in x. For the Dirac equation, we take similarly

$$\tilde{f}_x(z) = \frac{z-x_0}{|z-x_0|^n}\,.$$

□

So $\tilde{M}$ has the required properties, but its definition is not constructive and gives almost no information about $\tilde{M}$. Our aim is to describe $\tilde{M}$ geometrically, as far as possible.

The first question is which parts of the null cones really form the boundary. The following theorem says in which directions the boundary of $\tilde{M}$ lies.

Theorem 5. *Let $p \in \partial M$ and let ∂M be smooth in p. Let n denote the unit inner normal vector to ∂M in p. Suppose that a point of the form $p + z$ lies on $\partial\tilde{M}$, where $z = x + iy$ is a null vector. Then there is a complex number c, $Re(c) > 0$ and a tangent vector $u \in T_p\partial M$ such that*

(1) $$z = c(n + iu).$$

PROOF: Let $\tilde{p} = p + z$ lie on $\partial\tilde{M}$. Then $CN(\tilde{p}) \cap \mathbf{E}^n$ lies in $\overline{M}$ and it has at least one point in common with ∂M, namely p. For the tangent space at the point p, the intersection with $CN(\tilde{p})$ is just p:

(2) $$CN(\tilde{p}) \cap (T_p\partial M)_p = \{p\}.$$

Take $u \in T_p\partial M$, $u \neq 0$. From (2), we get

$$|z-u|^2 \neq 0.$$

Let us examine the equation

$$|z-s|^2 = -2\sum_{i=1}^{n} z_i s_i + \|s\|^2 = 0.$$

It has a non-zero solution in $T_p\partial M$ if and only if there is a vector s such that

$$\sum_{i=1}^{n} s_i x_i \neq 0 \text{ and } \sum_{i=1}^{n} s_i y_i = 0.$$

Such a vector does not exist, hence there exist $a_1, a_2 \in R$ for which

$$x = a_1 n + a_2 y.$$

Moreover, $a_1 > 0$, because $CN(\tilde{p}) \cap \mathbf{E}^n$ lies inside $\overline{M}$. We put $c_1 = \dfrac{a_1}{a_2^2+1}, c_2 = \dfrac{-a_1 a_2}{a_2^2+1}$ and for $c = c_1 + ic_2$ and $u = -\dfrac{c_2 n + y}{c_1}$, we get the equality (1). □

The following lemma gives a correspondence between $\mathbf{C}^n \setminus \mathbf{E}^n$ and $(n-2)$-dimensional spheres in $\mathbf{E}^n$.

Lemma. *Let S be an $(n-2)$-dimensional sphere in $\mathbf{E}^n$. Then there is a point $z \in \mathbf{C}^n$ such that*

$$CN(z) \cap \mathbf{E}^n = S.$$

Proof: Let x be the center of S, a the radius and y the unit normal vector to the hyperplane spanned by S. Then $z = x + iay$ has the required property. □

Remark: The point z may be replaced by $\overline{z}$, but up to this change, the correspondence is one-to-one.

Two main theorems of this section follow. The first one describes the boundary of $\tilde{M}$ "almost everywhere" as a smooth $(2n-1)$-dimensional manifold.

Theorem 6. *Let $M \subseteq \mathbf{E}^n$ be a domain with a smooth boundary. Then the boundary of $\tilde{M}$ contains a smooth $(2n-1)$-dimensional manifold which is open and dense in $\partial\tilde{M}$.*

Proof: **1**. Consider first $h \in \partial\tilde{M}$ with the properties:

(i) $CN(h) \cap \partial M = \{p_0\}$.

(ii) We have (compare (1) in Theorem 5)

$$h = p_0 + z_0 = p_0 + c_0(n_0 + iu_0), \quad c_0 = c_1 + ic_2,\ c_1 > 0, \quad z_0 = x_0 + iy_0,$$

where $\|c_0\|$ is less than the maximal radius of spheres which lie in M in the plane perpendicular to y_0 and which touch ∂M in the point p_0.

The set of all such points h is open in $\partial\tilde{M}$. Our aim is to prove that $T_h\partial\tilde{M}$ exists. Let $t = (t_1, t_2, ..., t_{n-1})$ be cartesian coordinates on $T_{p_0}\partial M$ with respect to a basis $(\tau_1, \tau_2, ..., \tau_{n-1})$ such that $\tau_i \perp y_0$ for $i = 1, 2, ..., n-2$. We have $x_0 = c_1 n_0 - c_2 u_0$, so $x_0 \notin T_{p_0}\partial M$ and $p = (t_1, t_2, ..., t_{n-1}, x')$ are coordinates on $\mathbf{E}^n$, where the last coordinate is taken with respect to the vector x_0. The boundary of M is locally described by a function $f = f(t)$ in such a way that a point p lies in ∂M if and only if $p = p(t) = (t, f(t))$. Let us denote the unit sphere in $\mathbf{E}^n$ by S_{n-1} and the $(n-2)$-dimensional sphere in $\mathbf{E}^n$ perpendicular to n_0 by S_{n-2}. Let $v = (v_1, v_2, ..., v_{n-2})$ be coordinates on a neighbourhood of the point u_0 in S_{n-2} and let $u(v_1, ..., v_{n-2}, t_1, ..., t_{n-1})$ be a smooth map with values in S_{n-1} defined on a neighbourhood of the point (u_0, p_0) such that $u(u_0, p_0) = u_0$ and $u(v, t) \perp n(t)$, where $n(t)$ is the unit normal vector to ∂M in a point $p(t)$, and for fixed t, $u(v, t)$ is a diffeomorphism. By Theorem 5 and by our assumptions about the point h, the map

$$\Phi(t, c, v) = p(t) + c(n(t) + iu(v, t))$$

describes the boundary $\partial\tilde{M}$ in a neighbourhood of the point h, i.e. $b \in \partial\tilde{M}$ if and only if $b \in Im\Phi$. The map Φ is smooth. We have to prove that the rank of the tangent map is maximal, i.e. that the partial derivatives with respect to t_i, c_i, v_i are linearly independent. The vectors $\dfrac{\partial\Phi}{\partial c_1} = n_0 + iu_0, \dfrac{\partial\Phi}{\partial c_2} = in_0 - u_0$ span a two dimensional space N. The vectors $i\dfrac{\partial\Phi}{\partial v_i}$ are, by the assumption about Φ, independent, and they span an $(n-2)$-dimensional space Q in $\mathbf{E}^n$ which is perpendicular to n_0 and u_0, because

$$(\frac{\partial u}{\partial v_i}, u) = \frac{1}{2}\frac{\partial}{\partial v_i}(u, u) = 0.$$

Therefore Q is perpendicular to x_0 and y_0.

We introduce new coordinates $x = (c_1 n - c_2 u)/\|c\|$, $y = (c_2 n + c_1 u)/\|c\|$ and express vectors

$$\frac{\partial\Phi}{\partial t_i} = \frac{\partial p}{\partial t_i} + c(\frac{\partial n}{\partial t_i} + i\frac{\partial u}{\partial t_i})$$

with respect to x and y:

$$\frac{\partial\Phi}{\partial t_i} = \frac{\partial p}{\partial t_i} + \|c\|(\frac{\partial x}{\partial t_i} + i\frac{\partial y}{\partial t_i}).$$

Using identity $\dfrac{\partial}{\partial t_i}(y, x) = (\dfrac{\partial y}{\partial t_i}, x) + (y, \dfrac{\partial x}{\partial t_i}) = 0$, we get

$$i\frac{\partial y}{\partial t_i} = i\left[\frac{\partial y}{\partial t_i} - (\frac{\partial y}{\partial t_i}, x)x\right] - i\left[(y, \frac{\partial x}{\partial t_i})x + i(y, \frac{\partial x}{\partial t_i})y\right] - (y, \frac{\partial x}{\partial t_i})y.$$

The first term belongs to iQ, the second to N. So it is sufficient to prove that the vectors

$$\frac{\partial p}{\partial t_i} - \|c\| \left[-(y, \frac{\partial x}{\partial t_i}) y + \frac{\partial x}{\partial t_i} \right]$$

are independent, where the term in brackets is a projection of $\frac{\partial x}{\partial t_i}$ to the plane perpendicular to x and y. From the definition of f, we have

$$x = \frac{(\frac{\partial f}{\partial t_1}, ..., \frac{\partial f}{\partial t_{n-1}}, 1)}{\sqrt{1 + \sum_{i=1}^{n-1} \left(\frac{\partial f}{\partial t_i}\right)^2}},$$

$$\frac{\partial x}{\partial t_i} = \left(\frac{\partial^2 f}{\partial t_1 \partial t_i}, ..., \frac{\partial^2 f}{\partial t_{n-1} \partial t_i}, 0 \right).$$

Projecting this vector to the plane perpendicular to x and y, we get

$$\left(\frac{\partial^2 f}{\partial t_1 \partial t_i}, ..., \frac{\partial^2 f}{\partial t_{n-2} \partial t_i}, 0, 0 \right).$$

Further, we have $\frac{\partial p_j}{\partial t_i} = \delta_{ij}$. We will prove the linear independence of the vectors by proving that the $(n-1) \times n$ matrix

$$\begin{pmatrix} 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 1 & 0 \end{pmatrix} - \|c\| \begin{pmatrix} \frac{\partial^2 f}{\partial t_1^2} & \dots & \frac{\partial^2 f}{\partial t_{n-2} \partial t_1} & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots \\ \frac{\partial^2 f}{\partial t_1 \partial t_{n-1}} & \dots & \frac{\partial^2 f}{\partial t_{n-2} \partial t_{n-1}} & 0 & 0 \end{pmatrix}$$

has rank $n-1$. As the $(n-1)$-st column is independent, we will omit the two last columns and the last row and show that the symmetric $(n-2) \times (n-2)$ matrix B defined in this way is positively definite and so regular.

Let $w = (t_1, t_2, ..., t_{n-2})$ be a unit vector. We have

$$B(w, w) = 1 - \|c\| \frac{\partial^2 f}{\partial w^2} > 0,$$

because $\|c\|$ is less than the radius of the sphere which touches ∂M in the point p, so $\|c\| < \frac{1}{\frac{\partial^2 f}{\partial w^2}}$.

2. Now let $h \in \partial \tilde{M}$ be arbitrary. We shall show that in every neighbourhood of h, there is a point h' which satisfies the conditions (i) and (ii) from the first part of the proof. Let $\epsilon > 0$. Take $h_0 \in \tilde{M} \cap U(h, \epsilon)$ and let $S = CN(h_0) \cap \mathbf{E}^n$. So we have

$$\mathrm{dist}(S, \partial M) = \delta > 0.$$

Take $x \in \partial M$ such that

$$\operatorname{dist}(x, S) = \delta.$$

Then the sphere S shifted by the vector $\delta(\mathrm{Re} h_0 - x)$ meets ∂M just in the point x. By Lemma, there is a point h' for which $CN(h') \cap \mathbf{E}^n$ is just this shifted sphere. Such h' satisfies the conditions (i), (ii) and

$$\|h - h'\| \leq \|h - h_0\| + \|h_0 - h'\| < \epsilon + \delta \leq 2\epsilon .$$

□

Theorem 7. *Let $M \subseteq \mathbf{E}^n$, $p \in \partial M$ and ∂M be smooth in p. Let $\tilde{p} = p + z \in \partial \tilde{M}$, where $z = c(n + iu)$ be a vector such that $T_{\tilde{p}} \partial \tilde{M}$ exists. Then the vector*

$$z' = \overline{\frac{1}{c} z} = n - iu$$

is a normal vector to $\partial \tilde{M}$ at the point $\tilde{p}$.

Proof: It suffices to realize that all vectors $\dfrac{\partial \Phi}{\partial t_i}, \dfrac{\partial \Phi}{\partial c_i}, \dfrac{\partial \Phi}{\partial v_i}$ in the proof of Theorem 6 are perpendicular to $n - iu$. □

3. Generalized envelopes of holomorphy.

As we have already seen, every solution to the equation $\mathbf{d}_R f = 0$ on a domain $M \subseteq \mathbf{E}^n$ has a natural holomorphic extension to the envelope of holomorphy $\tilde{M}$. Now we adopt a more general point of view. We are interested in properties of solutions to the complex equation $\mathbf{d}_C f = 0$ and in this context, contours of integration which lie entirely in $\mathbf{E}^n$ in integral formulas from Theorem 2 and 3, have no special importance. If a solution to the equation $\mathbf{d}_C f = 0$ on a domain $\Omega \subseteq \mathbf{C}^n$ and a cycle $\gamma \subseteq \Omega$ are given in such a way that the integral formula holds, we can use it to extend the solution to a larger domain.

The integral formulas are of the form

$$f(p) \operatorname{ind}_\gamma (p) = \frac{1}{k_n} \int_\gamma \omega_d(f, p),$$

where ω_d is a closed differential form and the integral is not defined for p in $CN(\gamma) = \bigcup_{x \in \gamma} CN(x)$.

Let us consider the following case, where the conditions on Ω and γ are chosen in such a way that the integral formulas are valid. Let $\Omega \subseteq \mathbf{C}^n$ be a domain and γ be a cycle on its boundary such that $CN(\gamma) \cap \Omega = \emptyset$. Moreover let γ be homologically trivial in $\Omega \cup \gamma$ and $\operatorname{ind}_\gamma (p) \neq 0$ for some (and so for any) $p \in \Omega$.

Definition 4. Let us denote by $\tilde{\Omega}$ the corresponding component of the set $\mathbf{C}^n \setminus CN(\gamma)$ determined by Ω. For a continuous function f defined on $\Omega \cup \gamma$ and satisfying in Ω the equation $\mathbf{d}_C f = 0$, we define a function $\tilde{f}$ by

$$\tilde{f}(p) = \frac{1}{\operatorname{ind}_\gamma(p)} \frac{1}{k_n} \int_\gamma \omega_d(f, p).$$

The function $\tilde{f}$ is holomorphic. We need some further assumption to prove that it is the extension of f.

Theorem 8. *Suppose that there is a point $p \in \Omega$ such that for some $\epsilon < \operatorname{dist}(p, \partial\Omega)$, we have $\gamma \sim kS^\epsilon_{n-1}(p)$ in $\Omega \setminus CN(p)$ for some $k \in \mathbf{Z}$. Then $\tilde{f} \mid \Omega = f$.*

Proof: Let $B \subseteq \Omega$ be open ball with center p and radius ϵ. Then B is null-convex, so by Theorems 3 and 4

$$f(p) = \frac{1}{k} \frac{1}{k_n} \int_{kS_{n-1}} \omega_d(f, p).$$

It follows from the definition of index that $k = \operatorname{ind}_\gamma(p)$, so by the assumption $k \neq 0$. The differential form $\omega_d(f, p)$ is closed, so the value of the integral over homologically equivalent cycles is the same and $\tilde{f}(p) = f(p)$.

Let us take $y \in B$ such that $\|p - y\| < \frac{\epsilon}{2}$. Then $\gamma \sim kS^{\frac{\epsilon}{2}}_{n-1}(y)$ and by the same argument $\tilde{f}(y) = f(y)$. Since f and $\tilde{f}$ are holomorphic, we have $f = \tilde{f}$ on Ω. $\square$

Let us consider a real n-dimensional smooth manifold with boundary $M \subseteq \mathbf{C}^n$ and consider points $p \in \operatorname{int} M$ such that the following condition is satisfied:

$$\text{(A)} \qquad CN(p) \cap M = \{p\}.$$

The following theorem was proved by J.Ryan in [4].

Theorem 9. *If $M \subseteq \mathbf{C}^n$ is a real, n-dimensional smooth manifold with boundary which satisfies the condition* (A) *at every point $p \in M$ and if for every $p \in M$*

$$\text{(B)} \qquad CN(p) \cap (T_p M)_p = \{p\},$$

then each solution defined on a neighbourhood of M can be extended to $\tilde{M}$.

As an easy consequence of Theorem 8, we get the following substantial generalization of Theorem 9.

Theorem 10. *Let f be a solution to the equation $\mathbf{d}_C f = 0$ which is defined on a neighbourhood U of the manifold M and suppose that there is a point $p \in \operatorname{int} M$ with the property (A). Then there is a function $\tilde{f}$ defined on $\tilde{M}$ such that $\mathbf{d}_C \tilde{f} = 0$ and $\tilde{f} = f$ on M, where $\tilde{M}$ is the component of $\mathbf{C}^n \setminus CN(\partial M)$ determined by M.*

Proof: The smoothness of M implies that $\partial U(p, \epsilon)$ is transversal to M for sufficiently small ϵ. So $\partial U(p, \epsilon) \cap M$ is an $(n-1)$-dimensional cycle which, by the property (A), is homological to ∂M in $U \setminus CN(p)$. We have

$$H_{n-1}(U(p, \epsilon) \setminus CN(p), \mathbf{Z}) \simeq H_{n-1}(\mathbf{C}^n \setminus CN(p), \mathbf{Z}) \simeq \mathbf{Z}.$$

We can take $S^{\epsilon}_{n-1}(p)$ as a generator of this group, so for some k we have $\partial M \sim kS^{\epsilon}_{n-1}(p)$ in $U \setminus CN(p)$. The rest is a consequence of Theorem 8. □

The next theorem says that for null-convex domains we can always apply Theorem 8.

Theorem 11. *If Ω is a null-convex domain with respect to a point $p \in \Omega$, then $\gamma \sim kS^{\epsilon}_{n-1}(p)$ in $\Omega \setminus CN(p)$.*

Proof: Put $A = \mathbf{C}^n \setminus CN(p)$ and consider the standard Mayer-Vietoris sequence

$$H_n(A \cup \Omega, \mathbf{Z}) \to H_{n-1}(A \cap \Omega, \mathbf{Z}) \to H_{n-1}(A, \mathbf{Z}) \oplus H_{n-1}(\Omega, \mathbf{Z}) \to H_{n-1}(A \cup \Omega, \mathbf{Z})$$

which is exact and $H_k(A \cup \Omega, \mathbf{Z}) = 0$ for $k = n, n-1$, because $A \cup \Omega$ is star-convex with respect to p. Thus we have an isomorphism

$$H_{n-1}(A \cap \Omega) \xrightarrow{i} H_{n-1}(A, \mathbf{Z}) \oplus H_{n-1}(\Omega, \mathbf{Z}).$$

But $\gamma \sim 0$ in $\Omega \cup \gamma$ by the assumption and $\gamma \sim kS^{\epsilon}_{n-1}(p)$ in A for some k by Theorem 1. So $\gamma \sim kS^{\epsilon}_{n-1}(p)$ in $A \cap \Omega = \Omega \setminus CN(p)$. □

Theorem 10, though it has weaker assumptions than Theorem 9, is still not very natural. It contains an assumption on the interior of M which does not appear in the integral formulas at all. Our aim is to find natural conditions on Ω and γ which will guarantee that the function $\tilde{f}$ is an extension of f to $\tilde{\Omega}$.

In the following theorem, we still suppose that γ is the boundary of a smooth manifold M, but all other assumptions involve only γ and Ω.

Theorem 12. *Let $\gamma = \partial M$ and let $p_0 \in \gamma$ be such that $CN(p_0) \cap \gamma = \{p_0\}$ and $T_{p_0}M \cap CN(p_0) = \{p_0\}$. Then there is a point $\tilde{p} \in \operatorname{int} M$ such that $\gamma \sim kS^{\epsilon}_{n-1}(\tilde{p})$ in $\Omega \setminus CN(\tilde{p})$.*

Proof: We shall show that there is a point $\tilde{p} \in \operatorname{int} M$ such that $CN(\tilde{p}) \cap M = \{\tilde{p}\}$. Then we will have

$$\partial M \sim kS^{\epsilon}_{n-1}(\tilde{p}) \ \text{ in } \Omega \setminus CN(\tilde{p})$$

by Theorem 10.

Let us first prove the following statement.

(S) There are ϵ and $\delta > 0$ such that for every $y \in U(p_0, \delta) \cap M$

$$CN(y) \cap U(y, \epsilon) \cap M = \{y\}.$$

Let us suppose, to get a contradiction, that for each $n \in N$ there are $a_n, b_n \in M$ such that

$$\|p_o - a_n\| < \frac{1}{n}, \quad \|b_n - a_n\| < \frac{1}{n} \text{ and } (a_n - b_n) \in CN.$$

Let $t \in CN$ be an accumulating point of the sequence of unit vectors $(a_n - b_n)/\|a_n - b_n\|$. Let $\psi_1, \ldots, \psi_n$ be smooth functions defined on a neighbourhood U of p_0 such that $M \cap U$ is a subset of the set $N = \{z \in U : \psi_i(z) = 0, \quad i = 1, ..., n\}$ and the rank of the matrix

$$\left(\frac{\partial \psi_i}{\partial x_j}, \frac{\partial \psi_i}{\partial y_j} \right)_{i=1,\dots,n}^{j=1,\dots,n}$$

is maximal in the point p_0. In other words,

$$v \in T_{p_0} M \quad \text{iff} \quad \frac{\partial \psi_i}{\partial v} |_{p_0} = 0 \text{ for } i = 1, ..., n.$$

Since $t \notin T_{p_0} M$, there is an index i such that $\frac{\partial \psi_i}{\partial t} |_{p_0} \neq 0$. It follows from the smoothness of ψ_i that there is $\nu > 0$ such that for $y \in U(p_0, \nu)$ and $t' \in U(t, \nu)$ there is $\frac{\partial \psi_i}{\partial t'} |_y \neq 0$. So there is $n \in N$ such that for $w = (a_n - b_n)/\|a_n - b_n\|$ we have

$$\frac{\partial \psi_i}{\partial w} |_y \neq 0$$

in every point y of the segment $\overline{a_n b_n}$.

On the other hand, by Rolle theorem, there is $\xi \in \overline{a_n b_n}$ such that

$$\frac{\partial \psi_i}{\partial w} |_\xi = 0,$$

which is a contradiction. Thus we have proved the statement (S).

Put $\epsilon_1 = \operatorname{dist}(CN(p_0), (M \setminus U(p_0, \frac{\epsilon}{2}))$. If $\epsilon_2 < \min(\delta, \epsilon_1)$, then for $y \in U(p_0, \epsilon_2) \cap \operatorname{int} M$, we have

$$CN(y) \cap (M \setminus U(y, \frac{\epsilon}{2})) = \emptyset$$

by the choice of ϵ_2 and

$$CN(y) \cap M \cap U(y, \frac{\epsilon}{2}) = \{y\}$$

by (S). Thus we have found $y \in \operatorname{int} M$ for which $CN(y) \cap M = \{y\}$. □

Now we formulate the main theorem.

Definition 5. We say that a boundary of a domain $\Omega \subseteq \mathbf{C}^n$ is Lipschitz in a point $x_0 \in \partial\Omega$ if there are real numbers $\mu > 0$, $\delta > 0$, cartesian coordinates $(x_1, ..., x_{2n}) = (x', x_{2n})$ and a Lipschitz function $a(x')$, defined on $\Delta = \{x' : |x_i| < \mu \text{ for } i = 1, ..., 2n - 1\}$ such that

(1) $\quad x_0 = (0, a(0))$,

(2) $\quad$ if $x = (x', a(x'))$ for some x', then $x \in \partial\Omega$,

(3) $\quad U_\delta^+ = \{(x', x_{2n}) : a(x') < x_{2n} < a(x') + \delta\} \subseteq \Omega$,

(3') $\quad U_\delta^- = \{(x', x_{2n}) : a(x') - \delta < x_{2n} < a(x')\} \subseteq \mathbf{C}^n \setminus \overline{\Omega}$.

Theorem 13. *Let Ω be a domain in $\mathbf{C}^n$, $\gamma \subseteq \partial\Omega$ be a smooth closed manifold (without boundary) of real dimension $n-1$ which is homologically trivial in $\Omega\cup\gamma$ and $\Omega \cap CN(\gamma) = \emptyset$. Let $x_0 \in \gamma$ be such that $\partial\Omega$ is Lipschitz in x_0 and the following two conditions are satisfied:*

(i) $T_{x_0}\gamma \cap CN(x_0) = \{x_0\}$,
(ii) $\gamma \cap CN(x_0) = \{x_0\}$.

Then for each continuous function $f : \Omega \cup \gamma \to \mathbf{C}_n^C$ which is holomorphic and satisfies the equation $\mathbf{d}_C f = 0$ in Ω, there is a holomorphic function $\tilde{f}$ defined on $\tilde{\Omega}$ such that $\tilde{f} |_\Omega = f$ and $\mathbf{d}_C \tilde{f} = 0$ on $\tilde{\Omega}$.

Proof: We will proceed in this way: first, we shall prove that the surface consisting of straight lines passing through γ in the direction x_{2n} lies outside $CN(x_0)$ in a neighbourhood of x_0. Then we shall remove a small neighbourhood of x_0 from the set $\Omega\cup\gamma$ and in the remaining part we shall find a homologically trivial cycle Σ, which differs from γ only by the boundary of a manifold which does not contain null segments (part of the constructed line surface). Then we shall show that there is a point v of this manifold such that $\gamma \sim kS^{\epsilon}_{n-1}(v)$ and we shall apply Theorem 8.

Let $(x_1, ..., x_{2n})$ be cartesian coordinates from Definition 5. Let us denote by $\dot{x}_{2n}$ the unit vector in the direction of the last coordinate. Since a is Lipschitz, we have $\dot{x}_{2n} \notin T_{x_0}\gamma$. Let us consider a cylinder:

$$V_\epsilon = \{(x', x_{2n}) : \|x'\| \le \epsilon \}.$$

We shall prove the following statement

(S) There is $\epsilon_1 > 0$ such that ∂V_{ϵ_1} intersects γ transversally.

Let us define the angle between two sets A and B with $A \cap B = \{p\}$:

$$\angle(A, B) = \arccos \left[\sup_{\substack{a\in A\setminus\{p\} \\ b\in B\setminus\{p\}}} \left| \frac{(a-p, b-p)}{\|a-p\|\|b-p\|} \right| \right].$$

It follows from the condition (i) that there is $\epsilon_2 > 0$ such that for $U(x_0, \epsilon_2) \cap \gamma$ and the axis $< x_{2n} >$ of V_ϵ we have

$$\angle(U(x_0, \epsilon_2) \cap \gamma\ ,\ < x_{2n} >) = \alpha > 0. \tag{1}$$

Further it follows from the smoothness of γ that there is $\epsilon_1 < \epsilon_2$ such that for each $y \in U(x_0, \epsilon_1) \cap \gamma$ there is $t \in T_y\gamma$ such that

$$\angle(< t >, < y - x_0 >) < \alpha. \tag{2}$$

Let $v \in \partial V_{\epsilon_1} \cap \gamma$. From (1) we get

$$\angle(< v - x_0 >, < x_{2n} >) > \alpha$$

and also

$$\angle(< v - x_0 >, T_v \partial V_{\epsilon\ 1}) > \alpha. \tag{3}$$

Let us apply (2) to v. There is $t \in T_v\gamma$ such that

$$\angle(< t >, < v - x_0 >) < \alpha \tag{4}$$

It follows from (3) and (4) that

$$\angle(< t >, T_v \partial V_{\epsilon\ 1}) > 0,$$

so $t \notin T_v \partial V_{\epsilon\ 1}$, in other words $T_v \partial V_{\epsilon\ 1} \oplus T_v \gamma = \mathbf{R}^{2n}$, concluding the proof of (S).

By the assumptions, $< x_{2n} >$ and $T_{x_0}\gamma$ lie outside $CN(x_0)$. We shall prove that moreover

$$(T_{x_0}\gamma \oplus < x_{2n} >) \cap CN(x_0) = \{x_0\}. \tag{5}$$

It follows from the lipschitz condition that there is ξ, $0 < \xi < \delta$ such that the set

$$E = \bigcup_{x \in \gamma} \{y \in \mathbf{C}^n : \|x - y\| < \xi \ \ \& \ \ (y - x, \dot{x}_{2n}) > 0 \ \ \& \ \ \angle(< y - x >, < x_{2n} >) < \xi\}$$

lies in Ω. Take $t \in T_{x_0}\gamma$, $\|t\| = 1$ and $A \in R$ and prove that the vector $v = t + Ax_{2n}$ does not belong to CN. It suffices to find $\mu > 0$ such that $p = x_0 + \mu v \in \Omega$, for $\Omega \cap CN(x_0) = \emptyset$. Choose $\mu < \min(\frac{\xi}{A}, \frac{1}{2A})$ with the property that there is $y \in \gamma \cap U(x_0, \mu)$ such that

$$\|(x_0 + \mu t) - y\| < \mu \xi A.$$

Put $p' = y + \mu A \dot{x}_{2n}$. We have $p' \in U_\delta^+ \subseteq \Omega$. Further

$$\|p - p'\| < \mu \xi A,$$

so $\angle(< p - y >, < p' - y >) < \xi$ and $\mu v \in E \subseteq \Omega$, which proves (5).

It follows from (5) that there is $\epsilon_{\ 3}$ such that for every $y \in U(x_0, \epsilon_{\ 3}) \cap \gamma$, the line passing through the point y in direction x_{2n} does not intersect $CN(x_0)$. Finally, put $\epsilon = \min(\epsilon_{\ 1}, \epsilon_{\ 3})$. Let us denote

$$\begin{aligned} A &= (U_\delta^+ \cup \gamma\ ) \cap \operatorname{int} V_\epsilon\ , \\ B &= (\Omega \cup \gamma\ ) \setminus \overline{(U_{\frac{\delta}{3}}^+ \cap V_{\frac{\epsilon}{3}})}, \\ \beta &= \gamma\ \cap \partial V_{\frac{\epsilon}{2}}\ . \end{aligned}$$

By (S), β is a closed $(n-2)$ dimensional manifold with the orientation induced from $\gamma\ \cap V_{\frac{\epsilon}{2}}$. We define a chain

$$\gamma_{\ 1} = \bigcup_{x \in \beta} \{y = (y', y_{2n}) : \ \ y' = x' \ \ \& \ \ a(x') \leq y_{2n} \leq a(x') + \frac{\delta}{2}\},$$

so $\gamma_1 \simeq \beta \times [0, \frac{\delta}{2}]$. We define the orientation on γ_1 in such a way that the induced orientation on $\beta \times [0]$ is the same as on β. Further we define a new $(n-1)$-dimensional chain $\Sigma \subseteq B$:

$$\Sigma = (\gamma \setminus \operatorname{int} V_{\frac{\epsilon}{2}}) + \gamma_1 + (\gamma \cap V_{\frac{\epsilon}{2}})_1 \,,$$

where $(\gamma \cap V_{\frac{\epsilon}{2}})_1$ is $(\gamma \cap V_{\frac{\epsilon}{2}})$ shifted by the vector $\frac{\delta}{2}\dot{x}_{2n}$. Now, Σ is a cycle, because the boundaries of γ_1 and the other two chains cancel. We will show that Σ is trivial in B. Applying Maier-Vietoris sequence, since $A \cap B$ is trivial, there is an isomorphism

$$H_n(A) \oplus H_n(B) \to H_n(A \cup B)$$

sending the class $[\Sigma - \gamma] + [\Sigma]$ to $[\gamma]$. But $(\Sigma - \gamma) \sim 0$ in A, because A is homologically trivial, and $\gamma \sim 0$ in $A \cup B = \Omega \cup \gamma$. So also $\Sigma \sim 0$ in B. Let us denote by K_1 the complex in B for which $\partial K_1 = \Sigma$. The cycle $\Sigma - \gamma$ is the boundary of

$$K_2 = (V_{\frac{\epsilon}{2}} \cap \gamma) \times [0, \frac{\delta}{2}].$$

So for $K = K_1 + K_2$, we have $\partial K = \gamma$. From the assumption $CN(x_0) \cap \Omega = \emptyset$, and we get

$$\operatorname{dist}(K_1, CN(x_0)) > \nu > 0 \tag{6}$$

for some $\nu < \delta$. Let us denote $q = x_0 + \nu \dot{x}_{2n}$. We have

$$CN(q) \cap K_1 = \emptyset$$

by (1) and

$$CN(q) \cap K_2 = \{q\}$$

by (5), because K_2 consists of segments in the direction x_{2n} which do not intersect $CN(x_0)$, hence not $CN(q)$. The same argument as in the proof of Theorem 12 shows that

$$\gamma \sim k S^{\epsilon}_{n-1}(q) \;\text{ in }\; \Omega \setminus CN(q).$$

By Theorem 8, we can extend all solutions from $\Omega \cup \gamma$ to $\tilde{\Omega}$. □

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