Orthomodular lattices with almost orthogonal sets of atoms

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Abstract. The set A of all atoms of an atomic orthomodular lattice is said to be almost orthogonal if the set $\{b \in A : b \not\leq a'\}$ is finite for every $a \in A$. It is said to be strongly almost orthogonal if, for every $a \in A$, any sequence $b_1, b_2, \dots$ of atoms such that $a \not\leq b_1', b_1 \not\leq b_2', \dots$ contains at most finitely many distinct elements. We study the relation and consequences of these notions. We show among others that a complete atomic orthomodular lattice is a compact topological one if and only if the set of all its atoms is almost orthogonal.

Keywords: atomic orthomodular lattice, topological orthomodular lattice, almost orthogonal sets of atoms

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Let $L(\wedge, \vee, ', 0, 1)$ be an atomic orthomodular lattice (abbreviated OML; see [3] for definitions and notations). Let A denote the set of all atoms of L. We will say that A is *almost orthogonal* if, for every $a \in A$, the set $B_a = \{b \in A : b \not\leq a'\}$ is finite. We will say that A is *strongly almost orthogonal* if for every $a \in A$ there is a number $n_a \in N$ such that every sequence $b_1, b_2, \dots$ of atoms satisfying $a \not\leq b_1', b_1 \not\leq b_2', \dots$ contains at most n_a distinct elements. It is easy to check that a strongly almost orthogonal set of atoms is almost orthogonal. Indeed, let for some $a \in A, B_a \supset \{b_1, b_2, \dots\}$. Then $a, b_1, a, b_2, \dots$ contains infinitely many distinct elements.

Denote by $\mathcal{P}$ the relation on A defined by $a\mathcal{P}b$ if $a \not\leq b'$. Let $\overline{\mathcal{P}}$ denote the transitive closure of $\mathcal{P}$, i.e. $a\overline{\mathcal{P}}b$ if there are $e_1, e_2, \dots, e_n$ in A such that $a = e_1, b = e_n$, and $e_i\mathcal{P}e_{i+1}$, $i \leq n-1$. Then $\overline{\mathcal{P}}$ is an equivalence relation. It is easy to see that if A is strongly almost orthogonal, then every equivalence class of $\overline{\mathcal{P}}$ is finite.

Atomic orthomodular lattices with almost orthogonal sets of atoms were studied in [5]. It was shown that if the supremum of an equivalence class of $\overline{\mathcal{P}}$ exists, it is an atom of the center $C(L)$ of L. In particular, if A is strongly almost orthogonal, then L can be embedded into the complete OML $\overline{L}$, where $\overline{L} = \prod_{i\in I}[0, c_i]$, and $c_i = \bigvee T_i$, where $\{T_i : i \in I\}$ is the family of all equivalence classes of $\overline{\mathcal{P}}$. In addition, the sets of all atoms of L and $\overline{L}$ are isomorphic, which implies that $\overline{L}$ is the Mac Neille completion of L.

In [6], the notion of a topological OML was introduced as follows: an OML is a *topological OML* if there exists a Hausdorff topology τ on L such that the lattice

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operations $\wedge$ and $\vee$, and the orthocomplementation $'$ are continuous in τ. If, in addition, τ is compact, then L is called a *compact topological OML*. Every compact topological OML is complete and atomic (see [6]), and has an almost orthogonal set of all atoms (see [5]).

A compact topological OML is *profinite* if it is a projective limit of finite OML's with their discrete topologies. It was shown in [6] that a compact topological OML is profinite if and only if it is a product of finite OML's.

An OML L is called *residually finite* if it can be algebraically embedded into a product of finite OML's. It was shown in [5] that an atomic residually finite OML can be embedded (algebraically and topologically) into a profinite OML $\overline{L}$ such that $\overline{L}$ is isomorphic to the Mac Neille completion of L if and only if the set of all atoms of L is strongly almost orthogonal.

In the sequel, we will study the relations between almost orthogonal and strongly almost orthogonal sets of atoms. We will show, e.g., that every modular compact topological OML is profinite. We will also show that a complete, atomic OML is a topological OML if and only if the set of all its atoms is almost orthogonal. Finally, we give an example of an OML with an almost orthogonal but not strongly almost orthogonal set of all atoms.

We recall that two elements a, b in an OML L are said to be

- *in the position* P' if $a \wedge b' = a' \wedge b = 0$;
- *strongly perspective* if they have a common complement in $[0, a \vee b]$ (i.e. if there is a $c \in L$ such that $a \wedge c = b \wedge c = 0$ and $a \vee c = b \vee c = a \vee b$), we write $a \sim_s b$;
- *perspective* if they have a common complement in L (i.e. if there is a $c \in L$ such that $a \wedge c = b \wedge c = 0$ and $a \vee c = b \vee c = 1$), we write $a \sim b$.

Let $\approx$ denote the transitive closure of $\sim$.

Proposition 1. *Let A be the set of all atoms of an atomic OML L. Define the relation $\mathcal{P}$ on A by $a\mathcal{P}b$ if $a \not\leq b'$, and let $\overline{\mathcal{P}}$ denote the transitive closure of $\mathcal{P}$. Then $\mathcal{P} \subset \sim_s \subset \sim \subset \overline{\mathcal{P}} = \approx$.*

PROOF: If a, b are atoms and $a\mathcal{P}b$, then $a \wedge b' = a' \wedge b = 0$, i.e. a and b are in the position P'. It is well-known that $P' \subset \sim_s \subset \sim$. It remains to prove that $\sim \subset \overline{\mathcal{P}}$. Suppose that $a \sim b$, $a, b \in A$. Then there is a $c \in L$ with $a \wedge c = b \wedge c = 0$, $a \vee c = b \vee c = 1$. Therefore, $a' \wedge c' = b' \wedge c' = 0$. Let $p \leq c'$ be an atom. Then $a' \wedge p = b' \wedge p = 0$, hence $a\mathcal{P}p$ and $p\mathcal{P}b$, which implies that $a\overline{\mathcal{P}}b$. □

Janowitz [2] introduced the following relations in an OML L:

$$\begin{array}{lll} aS^0b & \text{if } c \leq a, d \leq b & \text{and } c \sim_s d \text{ imply } c = 0 = d; \\ aS^1b & \text{if } c \leq a, d \leq b & \text{and } c \sim d \text{ imply } c = 0 = d; \\ aS^nb & \text{if } c \leq a, d \leq b & \text{and } c \sim c_1 \sim \cdots \sim c_n \sim d \text{ imply } c = 0 = d; \\ aS^\infty b & \text{if } c \leq a, d \leq b & \text{and } c \approx d \text{ imply } c = 0 = d. \end{array}$$

Clearly, aS^mb ($m = \infty$ admitted) implies aS^nb for $n \leq m$.

Proposition 2. *Let L be an atomic OML and let A be the set of all atoms of L.*

(i) *If A is strongly almost orthogonal, then the condition $(*)$ holds, where*

$$(*) \qquad (\forall\, a \in A)(\exists\, n_a \in N) : (\forall\, b \in A)(aS^{n_a}b \Longleftrightarrow aS^{\infty}b).$$

(ii) *If A is almost orthogonal and $(*)$ holds, then A is strongly almost orthogonal.*

Corollary. *If A is almost orthogonal, then A is strongly almost orthogonal if and only if $(*)$ holds.*

PROOF: (i) Let A be strongly almost orthogonal. Assume that $a\overline{\mathcal{P}}b$, $a, b \in A$. We have $a\overline{\mathcal{P}}b \Leftrightarrow a \approx b \Leftrightarrow aS^{\infty}b$ does not hold if and only if $(\exists\, k_b \in N) : aS^{k_b}b$ does not hold. The strong almost orthogonality of A implies that all equivalence classes of $\overline{\mathcal{P}}$ (and, therefore, all equivalence classes of $\approx$) are finite. Let T_a be the equivalence class containing a. Put $n_a = \max\{k_b : b \in T_a\}$. Then $aS^{\infty}b \Leftrightarrow aS^{n_a}b$ for all $b \in A$.

(ii) Assume that A is almost orthogonal and that the condition $(*)$ holds. Let $a \in A$. Condition $(*)$ implies that, for any $b \in A$, $a \approx b$ implies that there exist $e_1, e_2, \dots, e_k \in A$, $a \sim e_1 \sim e_2 \sim \cdots \sim e_k \sim b$, $k \leq n_a$. By the proof of Proposition 1, if $p, q \in A$ and $p \sim q$, then there is an $r \in A$ such that $p\mathcal{P}r\mathcal{P}q$. This implies that there are $f_1, f_2, \dots, f_m$ in A such that $a\mathcal{P}f_1\mathcal{P}f_2\mathcal{P}\dots\mathcal{P}f_m\mathcal{P}b$ with $m \leq 2n_a$. Since A is almost orthogonal, we obtain that every equivalence class of $\overline{\mathcal{P}}$ is finite, i.e. A is strongly almost orthogonal. $\square$

For a, x in an OML L, let φ_x denote the Sasaki projection $x \wedge (x' \vee a)$.

Chevalier [1] introduced the following condition:

We say that an OML L *satisfies the condition* (C) if, for every $a \in L$, $\bigvee_{x\in L} \varphi_x(a)$ exists and is a central element.

We recall that the *central cover* of an element $a \in L$ (if it exists) is defined by $|a| = \bigwedge\{c \in C(L) : a \leq c\}$.

Proposition 3. *If L is an OML satisfying (C), then, for every element $a \in L$, the central cover $|a|$ exists, and $|a| = \bigvee_{x\in L} \varphi_x(a) = \bigvee\{x \in L : a' \wedge x = 0\}$.*
([1, Proposition 8, Examples and Remarks 4a].)

Proposition 4. *Let L be a complete OML. The following statements are equivalent:*

(i) *L satisfies (C);*
(ii) *$S^1 = S^{\infty}$.*

([1, Proposition 10].)

Theorem 1. *Let L be an atomic OML with almost orthogonal set of all atoms satisfying (C). Then L has a strongly almost orthogonal set of all atoms. In particular, L can be embedded onto a product of finite OML's.*

PROOF: Let A denote the set of all atoms of L. Let $a \in A$. By Proposition 3, $|a| = \bigvee\{x \in L : x \wedge a' = 0\}$. If $x \wedge a' = 0$, then for any atom $b \leq x$, we get $b \wedge a' = 0$, hence $b \not\leq a'$. Therefore,

$$|a| = \bigvee\{x \in L : x \wedge a' = 0\} \leq \bigvee\{b \in A : b \not\leq a'\} = \bigvee B_a\,.$$

Since B_a is finite, we have $|a| = b_1 \vee b_2 \vee \dots \vee b_n$. Let $b \in A$. If $b \leq b_i'$ for all $i \leq n$, then $b \leq |a|'$. Therefore, $b \leq |a|$ implies that $b \in \bigcup_{i \leq n} B_{b_i}$. This shows that $[0, |a|]$ is finite. Let T_a be the equivalence class of $\overline{\mathcal{P}}$ containing a. Let $b \in A$ and $b \not\leq |a|$. Then $b \leq |a|' \leq a'$, which shows that $b \overline{\mathcal{P}} a$ implies $b \leq |a|$. This shows that $T_a \subset [0, |a|]$, and hence T_a is finite. Therefore, $\bigvee T_a$ exists. On the other hand, $|a|$ is an atom of $C(L)$. Indeed, let $0 \neq d < |a|$, $d \in C(L)$. Then $a \not\leq d$, hence $a \leq d'$, which implies that $|a| \leq d'$, a contradiction. Therefore, $|a| = \bigvee T_a$. Let $\{T_i : i \in I\}$ be the family of all equivalence classes of $\overline{\mathcal{P}}$, and let $c_i = \bigvee T_i$, $i \in I$. Then $c_i \in C(L)$ and $\bigvee_{i \in I} c_i = 1$. Clearly, for every $x \in L, x = \bigvee_{i \in I} x \wedge c_i$, and the map $\psi : L \to \prod_{i \in I} [0, c_i]$, $\psi(x) = (x \wedge c_i)_{i \in I}$ is an embedding. □

An orthomodular lattice L has the *relative center property* if the center of any interval $[0, x]$ of L is $\{x \wedge c : c \in C(L)\}$.

Proposition 5. *Let L be an atomic OML with an almost orthogonal set of all atoms. Let the following condition be satisfied:*

(i) *Perspectivity of atoms in L is transitive.*

Then L has a strongly almost orthogonal set of all atoms.

Proof: If perspectivity of atoms is transitive, then $\overline{\mathcal{P}} = \sim$. By the proof of Proposition 1, $a \sim b$ implies $a \mathcal{P} c \mathcal{P} b$ for some $c \in A$. Due to the almost orthogonality of A, it follows that every equivalence class of $\overline{\mathcal{P}}$ is finite. □

Corollary 1. *Let L be an atomic OML with almost orthogonal set of all atoms satisfying the following condition:*

(i) *L is complete and has the relative center property.*

Then L has a strongly almost orthogonal set of all atoms.

Proof: By [3, §8, Theorem 14], (i) is equivalent to $S^0 = S^1$, and by [3, §8, Theorem 7], this implies $S^0 = S^n = S^\infty$ for all n. By Proposition 4, (C) is satisfied, which by Theorem 1 implies the desired result. Alternatively, the condition $S^1 = S^\infty$ implies that perspectivity of atoms is transitive, and we can apply Proposition 5. □

Corollary 2. *In a complete modular atomic OML, almost orthogonality implies strong almost orthogonality.*

Corollary 3. *Every modular compact topological OML is profinite.*

Recall that an OML L is $(\circ)$-continuous if for every nondecreasing net $(x_\alpha)_\alpha$ such that $\bigvee_\alpha x_\alpha = x$ and every $y \in L$ we have $\bigvee_\alpha (x_\alpha \wedge y) = x \wedge y$.

Proposition 6. *Let L be an OML with almost orthogonal set of all atoms. Then L is $(\circ)$-continuous.*

Proof: Let $(x_\alpha)_\alpha$ be a nondecreasing net such that $\bigvee_\alpha x_\alpha = x$. It suffices to prove that for every atom y, $y \leq x$, there is α such that $y \leq x_\alpha$. (Indeed, if $x \wedge y$ cannot be expressed as $\bigvee_\alpha (x_\alpha \wedge y)$, then there is an atom $z \leq x \wedge y$ such that $z \not\leq x_\alpha \wedge y$ for every α. But, as we shall prove, $z \leq x \wedge y \leq x$ implies that there is x_α with $z \leq x_\alpha$, and since $z \le y$, we get $z \le x_\alpha \wedge y$. Thus, $\bigvee_\alpha (x_\alpha \wedge y)$ exists and equals $x \wedge y$.) We denote by K_α the set of all atoms v contained in x_α such that $v \nleq y'$, and put $K_\infty = \bigcup_\alpha K_\alpha$. Since the set of the atoms of L is almost orthogonal, there are only finitely many atoms $v_1, v_2, \dots, v_n$ in K_∞. Hence, there is α_0 such that $v_i \le x_{\alpha_0}$ for $i = 1, 2, \dots, n$. For every $\alpha > \alpha_0$, let us put $z_\alpha = x_\alpha \wedge x'_{\alpha_0}$. Now, $x_\alpha = x_{\alpha_0} \vee z_\alpha$ for all $\alpha > \alpha_0$, and $x = \bigvee_\alpha x_\alpha = x_{\alpha_0} \vee \bigvee_{\alpha > \alpha_0} z_\alpha$. Since $z_\alpha \le y'$ whenever $\alpha > \alpha_0$, we obtain $\bigvee_{\alpha > \alpha_0} z_\alpha \le y'$. It follows from $y \le x$ that $y \le x_{\alpha_0}$. The proof is complete. □

Corollary 4. *A complete, atomic OML is a compact topological OML if and only if the set of all atoms is almost orthogonal.*

The proof follows from [5, Theorem 2.3 (ii)].

An example of a complete, atomic, (∘)-continuous OML which does not have an almost orthogonal set of all atoms is the OML of all closed subspaces of a finite dimensional Hilbert space.

Examples and remarks.

1. By [1], the condition (C) is strictly weaker than $S^0 = S^1$, a condition equivalent to the relative center property in a complete OML. As an example, see OML with Greechie diagram on Fig. 1: we have aS^0b, but xS^1y is never true for x, y both different from 0.

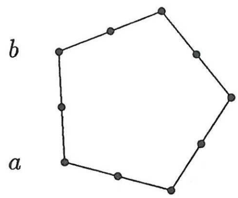

Fig. 1

2. An example of an OML with strongly almost orthogonal set of all atoms which does not satisfy (C) is the Dilworth lattice D_{16} (Fig. 2). We have $|a| = 1$ and $b = \bigvee_{x \in L} \varphi_x(a)$ (see [1]).

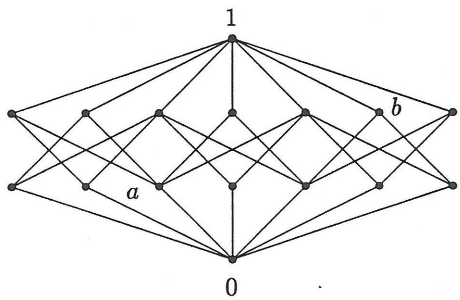

Fig. 2

3. In the rest of the paper, we will introduce an example of a complete, atomic, irreducible OML with almost orthogonal set of atoms. It is clear that its set of atoms is not strongly almost orthogonal. By Proposition 6, every atomic OML with almost orthogonal set of atoms is ($\circ$)-continuous. Hence, by [5], our example is a compact topological OML which is not profinite. The Greechie diagram of our example is given in Fig. 3.

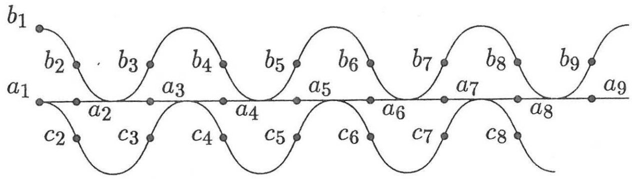

Fig. 3

Let us describe its construction.

Let $\{a_1, a_2, \dots, b_1, b_2, \dots, c_2, c_3, \dots\}$ be a set of atoms. Let $\mathcal{B}$ be the system of all Boolean σ-algebras constructed in the following way:

$B_D \in \mathcal{B}$ is the Boolean σ-algebra generated by a set of atoms $D = \{d_1, d_2, \dots\}$ such that for every $i \in N$

(i) $d_i = a_i$ or $d_i = b_i$ or $d_i = c_i$

(taking into account that c_1 does not exist);

(ii) $b_{2k-1} \in D \Longleftrightarrow b_{2k} \in D$ $(k \in N)$;
(iii) $c_{2k} \in D \Longleftrightarrow c_{2k+1} \in D$ $(k \in N)$.

The Boolean σ-algebras in the system $\mathcal{B}$ are "pasted" in the common elements. Denote the resulting structure by L. By Definition 3.1 in [4], L is a pasting of the system $\mathcal{B}$. By Theorem 3.5 in [4], L is an orthomodular poset. It is clear that L is σ-orthocomplete (i.e. the supremum of any orthogonal sequence exists in L) and, therefore, it is orthocomplete. (i.e. the supremum of any orthogonal subset exists in L).

For an atom $x \in L$, we denote by B_x the set of all atoms in L which are not compatible with x. It is easy to see that

$$\begin{aligned}
B_{a_k} &= \{c_{k-1}, c_k, b_k, b_{k+1}\} \quad \text{if } k \text{ is odd,}\\
B_{a_k} &= \{b_{k-1}, b_k, c_k, c_{k+1}\} \quad \text{if } k \text{ is even,}\\
B_{b_k} &= \{a_k, a_{k+1}, c_{k-1}, c_k, c_{k+1}, c_{k+2}\} \quad \text{if } k \text{ is odd,}\\
B_{b_k} &= \{a_{k-1}, a_k, c_{k-2}, c_{k-1}, c_k, c_{k+1}\} \quad \text{if } k \text{ is even,}\\
B_{c_k} &= \{a_{k-1}, a_k, b_{k-2}, b_{k-1}, b_k, b_{k+1}\} \quad \text{if } k \text{ is odd,}\\
B_{c_k} &= \{a_k, a_{k+1}, b_{k-1}, b_k, b_{k+1}, b_{k+2}\} \quad \text{if } k \text{ is even,}
\end{aligned}$$

where $k = 2, 3, \dots$ (taking into account that c_1 does not exist). It remains to prove that L is a lattice. It is not difficult to see that L contains many Dilworth lattices as sublattices, e.g. those on Fig. 4.

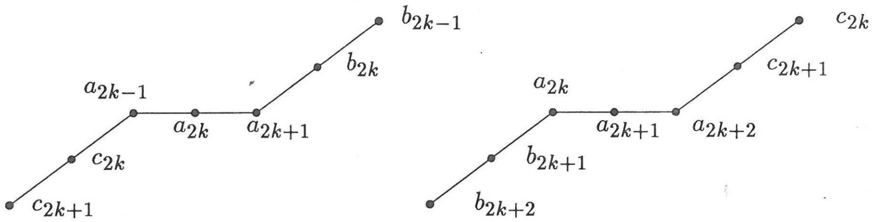

Fig. 4

Taking this into account, we see that the supremum of any two atoms exists in L, and moreover, the supremum of any two noncompatible atoms can be expressed as the supremum of compatible atoms a_i $(i \in N)$, e.g.

$$a_{2k} \vee c_{2k} = a_{2k} \vee a_{2k+1}, \quad c_{2k+1} \vee b_{2k-1} = a_{2k-1} \vee a_{2k} \vee a_{2k+1} \,.$$

Now we are able to express the supremum of any finite set of atoms as the supremum of a set of compatible atoms. Let D be any set of atoms in L. Since D is at most countable, we may write

$$D = \{d_1, d_2, \dots\},$$

and put $d = (d_1 \vee d_2) \vee (d_1 \vee d_2 \vee d_3) \vee \dots$. All the finite suprema on the right hand side exist, and they form an increasing sequence which is contained in a Boolean sub-σ-algebra of L, and therefore d exists. It is easy to see that $d = \bigvee d_i$. This proves that L is an OML.

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