On zero-dimensionality of subgroups of locally compact groups

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Abstract. Improving the recent result of the author we show that $\operatorname{ind} H = 0$ is equivalent to $\dim H = 0$ for every subgroup H of a Hausdorff locally compact group G.

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All topological groups considered in this note are assumed to be Hausdorff. A subset of a topological space is said to be *clopen* if it is both open and closed. A topological space X is *zero-dimensional* if it has a base consisting of clopen sets (i.e. if $\operatorname{ind} X = 0$), and X is *strongly zero-dimensional* if every (locally) finite open cover of X consisting of functionally open sets has a finite disjoint clopen refinement (i.e. if $\dim X = 0$). Strongly zero-dimensional spaces are zero-dimensional [1, Theorem 6.2.6] but not vice versa [1, Example 6.2.20]. However, the implication can be reversed for totally bounded groups: Recently, the author proved that a zero-dimensional subgroup of a compact group is strongly zero-dimensional [3, Corollary 3.4]. The aim of our note is to extend this result over subgroups of locally compact groups.

Theorem. *A zero-dimensional subgroup of a locally compact group is strongly zero-dimensional.*

In the proof of this theorem, we need the notion of $\mathbb{R}$-factorizable group [4]: A topological group G is $\mathbb{R}$-*factorizable* if for every real-valued continuous function $f : G \to \mathbb{R}$ defined on G there exist a topological group H with a countable base, a continuous homomorphism $\pi : G \to H$ and a continuous mapping $\varphi : H \to \mathbb{R}$ such that $f = \varphi \circ \pi$. The following proposition formally improves [3, Theorem 3.3]:

Proposition. *A zero-dimensional topological group having an open $\mathbb{R}$-factorizable subgroup is strongly zero-dimensional.*

Proof: Let H be an open $\mathbb{R}$-factorizable subgroup of a zero-dimensional topological group G. Being a subspace of a zero-dimensional space G, H is zero-dimensional, and since H is $\mathbb{R}$-factorizable, it is strongly zero-dimensional by [3, Theorem 3.3].

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Observe that H is clopen in G, as is every open subgroup of any topological group [2, Chapter 2, Theorem 5.5]. Since G can be covered by disjoint clopen copies of H (namely, by some translations of H) and H is strongly zero-dimensional, G is also strongly zero-dimensional [1, Theorem 6.2.13]. □

Proof of Theorem: Let H be a zero-dimensional subgroup of a locally compact group G, and let U be an open neighbourhood of the neutral element of G having compact closure $\overline{U}$ in G. Then G^*, the smallest subgroup of G that contains $\overline{U}$, is σ-compact. As a subgroup of a σ-compact group, $H^* = H \cap G^*$ is $\mathbb{R}$-factorizable [4, Corollary 1.13]. Since H^* contains the non-empty open set $U \cap H$ and is a subgroup of H, H^* is open in H [2, Chapter 2, Theorem 5.5]. Now, Proposition finishes the proof. □

In conclusion, let us mention that, quite surprisingly, the following question remains open:

Question. Is there a normal zero-dimensional group which is not strongly zero-dimensional?

Even if we drop "normal" here, the answer to this question seems to be unknown.

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