Biequivalence vector spaces in the alternative set theory

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Abstract. As a counterpart to classical topological vector spaces in the alternative set theory, biequivalence vector spaces (over the field Q of all rational numbers) are introduced and their basic properties are listed. A methodological consequence opening a new view towards the relationship between the algebraic and topological dual is quoted. The existence of various types of valuations on a biequivalence vector space inducing its biequivalence is proved. Normability is characterized in terms of total convexity of the monad and/or of the galaxy of 0. Finally, the existence of a rather strong type of basis for a fairly extensive area of biequivalence vector spaces, containing all the most important particular cases, is established.

Keywords: alternative set theory, biequivalence, vector space, monad, galaxy, symmetric Sd-closure, dual, valuation, norm, convex, basis

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Contents:

0. Introduction.

The aim of this paper is to lay a foundation to the investigation of topological (or perhaps also bornological) vector spaces within the framework of the alternative set theory (AST), which could enable a rather elementary exposition of some topics of functional analysis reducing them to the study of formally finite dimensional vector spaces equipped with some additional "nonsharp" or "hazy" first order structure representing the topology. Concerning the aspect of linear algebra, in this initial paper we will restrict our attention to vector spaces over the field Q of all rational numbers topologized by the usual biequivalence $\langle \doteq, \leftrightarrow \rangle$ (see Section 1). Nevertheless, we hope that the reader will find this restrictions inessential and ready for generalizations in various directions. The topological structure will be represented, as usual in AST, by a biequivalence (see [G-Z 1985a]) on (the underlying class of) the vector space, and it will be subject to some extremely natural and self-offering conditions warranting its compatibility with the operations of addition of vectors and multiplication of vectors by scalars. Such an attempt makes it possible to investigate simultaneously and in a uniform way indiscernibility and continuity phenomena on one hand in close connection with the phenomena of accessibility and boundness (playing an important role in classical topological vector spaces, too) on the other.

The first steps towards the problematics of topological vector spaces in AST were already made in the thesis by E. Rampas [Rm 1980], however, in our opinion, it was the lack of the explicit notion of accessibility which turned to be a serious obstacle to a more considerable progress. Also, the reader will probably find some connections between our approach to topological vector spaces and that of nonstandard analysis as presented e.g. in [H-Mr 1972]. But we would like to stress that it is not our aim to develop new technical proof tools for the classical functional analysis by means of ultrapowers, enlargements, nonstandard hulls and similar methods, as it usually is the case in nonstandard analysis. Our biequivalence vector spaces are treated as a primary subject of interest and study, independently, to a large extent, of their classical counterparts, and not as auxiliary constructs. Thus, e.g., the monad of the infinitesimal vectors and the galaxy of the bounded ones do not result as a product of an advanced set-theoretical construction, but they are the very starting point of our exposition included in the basic definition.

The plan of our paper has already been sketched in the Abstract and Contents. Let us add only that Sections 1, 2, 3 are of a preparatory character, while the “meat” of the paper starts with Section 4.

1. Notation and preliminaries.

The reader is assumed to be acquainted with the basic notions and results of the alternative set theory as presented in [V 1979], with the notion of biequivalence which has been introduced and developed in [G-Z 1985a], [G-Z 1985b], and, of course, with some fundamentals on linear algebra.

For reader’s convenience we will list some basic facts, the most frequently used symbols and notational conventions below.

V denotes the universe of sets, sets from V are denoted by small Latin letters, subclasses of V by capital ones. There is a canonical set-theoretical ordering $\leq$ of V such that each nonempty Sd-class has the least element with respect to $\leq$, and for each x, the cut $\{y;\, y < x\}$ is a set.

N, FN, Z and FZ denote the classes of all natural numbers, finite natural numbers, integers and finite integers, respectively. The characters i, j, k, m, n always denote elements of Z or N. Notice the difference to [V 1979], where they were used to denote elements of FN only.

Q denotes the class of all rational numbers; the small Greek letters α, β, γ, δ, ε, κ, λ, μ, ν (possibly indexed) always denote elements of Q.

$$IQ = \{\alpha;\, (\forall n \in FN)(|\alpha| < 2^{-n})\},$$
$$BQ = \{\alpha;\, (\exists n \in FN)(|\alpha| < 2^{n})\}$$

denote the classes of all infinitesimal (infinitely small) and bounded (finitely large) rationals, respectively. Then Q endowed with the canonical operations and order becomes an ordered field, BQ is its convex subring and IQ is a convex maximal ideal in BQ. The quotient BQ/IQ is then the ordered field of real numbers.

The codable class HR of all hyperreal numbers was constructed in [G-Z 1985b] on the basis of some revealment Sd* of the system of all Sd-classes (for the notion of revealment see [S-V 1980]). However, for the purpose of the present paper it is quite sufficient to deal with the extension of Q consisting only of all set-theoretically definable hyperreals, i.e., of Sd-cuts on Q without the last element. They form a subfield of the ordered field HR. Though HR is not a class from the extended universe, the pair of relations

$$\begin{aligned} p \doteq q \quad &\Leftrightarrow \quad (\forall n \in FN)(|p-q| < 2^{-n}), \\ p \leftrightarrow q \quad &\Leftrightarrow \quad (\exists n \in FN)(|p-q| < 2^{n}) \end{aligned}$$

behaves as a biequivalence on HR. We put

$$p <\cdot\, q \quad \Leftrightarrow \quad p < q \ \ \& \ \ p \mathrel{\dot{\neq}} q$$

for $p, q \in HR$. Then

$$\begin{aligned} IHR &= \{p \in HR;\, p \doteq 0\}, \\ BHR &= \{p \in HR;\, p \leftrightarrow 0\}, \\ HR^{+} &= \{p \in HR;\, 0 <\cdot\, p\} \end{aligned}$$

denote the classes of all infinitesimal, bounded and strictly positive hyperreals, respectively. Sometimes it will be found convenient to extend HR by adjoining the least element $-\infty$ and the largest element ∞ to it. For $p, q \in HR \cup \{-\infty, \infty\}$,

$$[p, q] = \{\alpha \in Q;\, p \leq \alpha \leq q\}$$

always denotes the interval of all rational numbers between p, q.

A function $\Phi : X \longrightarrow HR$ is simply a relation $\Phi \subseteq Q \times X$ such that $\Phi(x) = \Phi''\{x\} \in HR$ for each $x \in X$.

As each Sd*-class (and the more, each Sd-class) $A \subseteq Q$ with an upper bound has its supremum in HR, the decisive part of the usual functions used in the classical analysis defined on (some subclass of) Q with values in HR can be put into our framework and extended to (some subclass of) HR. In particular, we will use the functions $\sqrt{p}$, p^q and $\lg p = \log_2 p$. Details are left to the reader.

If X is a class and u is a set, then X^u denotes the class of all set-functions $f : u \longrightarrow X$. If $\operatorname{card}(u) = n \in N$, then X^u can be identified with the class X^n of all ordered n-tuples $\langle x_1, \dots, x_n \rangle$ such that $x_i \in X$ for each i. Thus whenever placing a sequence $x_1, \dots, x_n$ into the brackets $\langle\ \rangle$, it is understood that the corresponding function $\{\langle x_i, i\rangle; 1 \leq i \leq n\}$ is a set.

If $W = Q^m$ is regarded as a vector space over Q, then its elements will be represented as ordered m-tuples with lower indices, e.g., $x = \langle x_1, \dots, x_m \rangle$. Similarly, for ordered n-tuples of scalar coefficients $\langle \alpha_1, \dots, \alpha_n \rangle \in Q^n$ occurring in linear combinations, a lower indexation will be used. For ordered n-tuples of vectors $\langle x^1, \dots, x^n \rangle \in W^n$ (even in the case W is not of the form Q^m), the upper indexation is reserved.

Finally, we place here an important lemma on set choice which will be used several times throughout the paper. In [G-Z 1985b], it was proved for π-relations, however, its extension to σ-relations, too, presents no difficulty.

Lemma 1.1. *Let R be a relation which is a σ-class or a π-class, and $u \subseteq \mathrm{dom}(R)$ be a set. Then there is a set function f such that $\mathrm{dom}(f) = u$ and $f \subseteq R$, i.e., $\langle f(x), x \rangle \in R$ for each $x \in u$.*

2. Symmetric Sd-closures.

In the present section we will describe the common core of several closure operators which will occur throughout the paper. Many of the results which will be stated below, are rather analogous to those known from the classical theory of matroids (see e.g. [We 1976]). They are included mainly in order to fix the terminology and notation and to introduce some necessary modifications. On the other hand, this allows us to reduce most of the proofs to mere sketches or to omit them completely.

Let $\mathcal{C}$ be any operation assigning to each set u a class denoted by $\mathcal{C}(u)$ or, more briefly, $\mathcal{C}u$. Then $\mathcal{C}$ will be called an Sd-closure (on the universal class V) provided the following conditions are satisfied:

(0) there is a set-theoretical formula $\varphi(x, u)$ such that

$$(\forall u)(\mathcal{C}(u) = \{x;\, \varphi(x, u)\}),$$

(1) $\quad (\forall u)(u \subseteq \mathcal{C}(u))$,

(2) $\quad (\forall u, v)(u \subseteq v \;\Rightarrow\; \mathcal{C}(u) \subseteq \mathcal{C}(v))$,

(3) $\quad (\forall u)(\bigcup\{\mathcal{C}(v); v \subseteq \mathcal{C}(u)\} = \mathcal{C}(u))$.

An Sd-closure $\mathcal{C}$ will be called symmetric if it additionally satisfies the following exchange condition:

(4) $\quad (\forall x, y)(\forall u)(x \in \mathcal{C}(u \cup \{y\}) \setminus \mathcal{C}(u) \;\Rightarrow\; y \in \mathcal{C}(u \cup \{x\}))$.

Obviously, if the closure $\mathcal{C}$ is symmetric, then the binary relation $\{\langle x, y \rangle; x \in \mathcal{C}\{y\}\}$ is symmetric on the class $V \setminus \mathcal{C}(\emptyset)$.

An Sd-closure $\mathcal{C}$ can be extended to operate on all classes X by

$$\mathcal{C}(X) = \bigcup\{\mathcal{C}(u);\, u \subseteq X\}.$$

Now, the condition (3) can be rewritten into a more comprehensive and familiar form

(3') $\quad (\forall u)(\mathcal{C}\mathcal{C}(u) = \mathcal{C}(u))$.

Till the end of the section, $\mathcal{C}$ denotes a fixed but otherwise arbitrary Sd-closure.

Proposition 2.1. (a) *There is a normal formula $\psi(x, X)$ such that*

$$(\forall X)(\mathcal{C}(X) = \{x;\ \psi(x, X)\}).$$

(b) $(\forall X)(X \subseteq \mathcal{C}(X))$.

(c) $(\forall X, Y)(X \subseteq Y \ \Rightarrow\ \mathcal{C}(X) \subseteq \mathcal{C}(Y))$.

(d) *If X is an Sd-class (σ-class, π-class), then so is $\mathcal{C}(X)$ and it holds*

$$\mathcal{C}\mathcal{C}(X) = \mathcal{C}(X).$$

(e) *If $\mathcal{C}$ is symmetric, then*

$$(\forall x, y)(\forall X)(x \in \mathcal{C}(X \cup \{y\}) \setminus \mathcal{C}(X) \ \Rightarrow\ y \in \mathcal{C}(X \cup \{x\})).$$

PROOF: (a), (b), (c) and the first part of (d) are trivial. To complete (d), consider an $x \in \mathcal{C}\mathcal{C}(X)$. Then $x \in \mathcal{C}(v)$ for some $v \subseteq \mathcal{C}(X)$. This means that the relation

$$R = \{\langle u, y\rangle;\ y \in v \ \&\ u \subseteq X \ \&\ y \in \mathcal{C}(u)\},$$

which is an Sd-class (σ-class, π-class) if X has the corresponding property, has domain v. By the set-choice Lemma 1.1., there is a function $f \subseteq R$ with the domain v, i.e., $f(y) \subseteq X$ and $y \in \mathcal{C}(f(y))$ for each $y \in v$. Then for the set $w = \bigcup \mathrm{rng}(f) \subseteq X$ it holds $v \subseteq \mathcal{C}(w)$, hence

$$x \in \mathcal{C}(v) \subseteq \mathcal{C}\mathcal{C}(w) = \mathcal{C}(w) \subseteq \mathcal{C}(X).$$

(e) If $x \in \mathcal{C}(X \cup \{y\}) \setminus \mathcal{C}(X)$, then obviously $x \in \mathcal{C}(u \cup \{y\}) \setminus \mathcal{C}(u)$ for some $u \subseteq X$. As $\mathcal{C}$ is symmetric,

$$y \in \mathcal{C}(u \cup \{x\}) \subseteq \mathcal{C}(X \cup \{x\}).$$

□

Notice that the equality $\mathcal{C}\mathcal{C}(X) = \mathcal{C}(X)$ cannot be proved for arbitrary classes X. All the same, $\mathcal{C}(X)$ still will be called the closure of X (with respect to $\mathcal{C}$).

A class X will be called $\mathcal{C}$-independent if

$$X \cap \mathcal{C}(\emptyset) = \emptyset \quad \text{and} \quad (\forall x \in X)(\mathcal{C}\{x\} \cap \mathcal{C}(X \setminus \{x\}) \subseteq \mathcal{C}(\emptyset)).$$

It is routine to check that a class X is $\mathcal{C}$-independent iff each set $u \subseteq X$ is $\mathcal{C}$-independent.

Proposition 2.2. *If X is a $\mathcal{C}$-independent class, then*

$$(\forall x \in X)(x \notin \mathcal{C}(X \setminus \{x\})).$$

If $\mathcal{C}$ is symmetric, then also the reversed implication holds for each σ- or π-class X.

Proof: Let X be independent. Assume that there is an $x \in X$ such that $x \in \mathcal{C}(X \setminus \{x\})$. Then even

$$x \in X \cap \mathcal{C}\{x\} \cap \mathcal{C}(X \setminus \{x\}) \subseteq X \cap \mathcal{C}(\emptyset) = \emptyset.$$

Now, assume $x \notin \mathcal{C}(X \setminus \{x\})$ for each $x \in X$. Then obviously $x \notin \mathcal{C}(\emptyset)$ for each $x \in X$. If there were an $y \in (\mathcal{C}\{x\} \cap \mathcal{C}(X \setminus \{x\})) \setminus \mathcal{C}(\emptyset)$, then, by the symmetry of $\mathcal{C}$ and by 2.1 (d), it would follow

$$x \in \mathcal{C}\{y\} \subseteq \mathcal{C}\mathcal{C}(X \setminus \{x\}) = \mathcal{C}(X \setminus \{x\}).$$

□

A class X will be called $\mathcal{C}$-generating if $\mathcal{C}(X) = V$. A $\mathcal{C}$-independent $\mathcal{C}$-generating class will be called a $\mathcal{C}$-basis.

Theorem 2.3. *Let $\mathcal{C}$ be a symmetric Sd-closure. Then for each $\mathcal{C}$-independent Sd-class X_0 and each $\mathcal{C}$-generating Sd-class X_1 such that $X_0 \subseteq X_1$ there is a $\mathcal{C}$-basis X such that* $\mathrm{Sd}(X)$ *and* $X_0 \subseteq X \subseteq X_1$.

Proof: Using the canonical Sd-ordering of the universal class V, one can construct, by induction over N, an Sd-function F such that either $\mathrm{dom}(F) \in N$ or $\mathrm{dom}(F) = N$ and for each $n \in N$ either $F(n)$ is the first element of the Sd-class $X_1 \setminus \mathcal{C}(X_0 \cup F''n)$ if it is nonempty, or $F''\{n\} = \emptyset$ if $X_1 \setminus \mathcal{C}(X_0 \cup F''n) = \emptyset$. It can be easily seen that the Sd-class $X = X_0 \cup \mathrm{rng}(F) \subseteq X_1$ is $\mathcal{C}$-generating. Using the exchange condition and an induction argument, one can verify that it is $\mathcal{C}$-independent, too. □

Also, the following Steinitz inequality can be established in essentially the same way as in the classical case.

Proposition 2.4. *Let $\mathcal{C}$ be a symmetric Sd-closure and u, v be sets. If u is $\mathcal{C}$-independent and v is $\mathcal{C}$-generating, then* $\mathrm{card}(u) \leq \mathrm{card}(v)$.

Corollary 2.5. *Let $\mathcal{C}$ be as above and both X, Y be Sd-classes and $\mathcal{C}$-bases. If one of them is a set, then also the remaining one is a set with the same number of elements.*

Finally, let us remark, that all the notions and results of this section can directly be restated to Sd-closures on arbitrary Sd-classes (not just V).

3. Vector spaces over Q.

A vector space over the field Q of all rational numbers is an arbitrary class W endowed with the operations of addition $+ : W \times W \longrightarrow W$ and scalar multiplication $\cdot : Q \times W \longrightarrow W$, subject to the usual axioms. If in addition the class W and the operations $+$ and $\cdot$ are set-theoretically definable, then W will be called an Sd-vector space over Q. Unless otherwise said, the term "vector space" always means an Sd-vector space over Q.

Till the end of this section W denotes a fixed but otherwise an arbitrary vector space. Note that the fact of the set-theoretical definability of the basic operations in W enables us to define by induction expressions like

$$\sum_{i=1}^{n} \alpha_i x^i = \alpha_1 x^1 + \cdots + \alpha_n x^n,$$

where $\langle \alpha_1, \ldots, \alpha_n \rangle \in Q^n$, $\langle x^1, \ldots, x^n \rangle \in W^n$, for all natural numbers n, and not just for the finite ones.

For $A \subseteq Q, X, Y \subseteq W$ and $n \in N$, the following notation will frequently be used:

$$\begin{aligned}
X + Y &= \{x + y;\, x \in X,\ y \in Y\},\\
A \cdot X &= \{\alpha \cdot x;\, \alpha \in A,\ x \in X\},\\
X : A &= \{x \in W;\, A \cdot \{x\} \subseteq X\},\\
n \star X &= \{x^1 + \cdots + x^n;\, \langle x^1, \ldots, x^n \rangle \in X^n\},\\
[X] &= \Big\{\sum_{i=1}^{n} \alpha_i x^i;\, n \in N,\ \langle \alpha_1, \ldots, \alpha_n \rangle \in Q^n,\ \langle x^1, \ldots, x^n \rangle \in X^n\Big\},\\
\widehat{X} &= \{\alpha x + (1-\alpha) y;\, \alpha \in [0,1],\ x, y \in X\},\\
\langle X \rangle &= \Big\{\sum_{i=0}^{n} \alpha_i x^i;\, n \in N,\ \langle \alpha_0, \ldots, \alpha_n \rangle \in [0,1]^{n+1},\\
&\qquad\qquad \sum_{i=0}^{n} \alpha_i = 1,\ \langle x^0, \ldots, x^n \rangle \in X^{n+1}\Big\}.
\end{aligned}$$

If $A = \{\alpha\}$ is a singleton, then $\{\alpha\} \cdot X$ will be denoted simply by $\alpha \cdot X$. Obviously, for all $A \subseteq Q$, $X, Y \subseteq W$, it holds

$$A \cdot X \subseteq Y \quad \Leftrightarrow \quad X \subseteq Y : A.$$

If $n \in N$, then obviously $n \cdot X \subseteq n \star X$ for each X, but, in general, the inclusion is proper.

As it can easily be seen, both [ ], $\langle\ \rangle$ (regarded as operations on the sets $u \subseteq W$, only) are Sd-closures on (the underlying class of) the vector space W. $[X]$ will be called the linear span of X and $\langle X \rangle$ will be called the convex hull of X. Moreover, the Sd-closure [ ] is even symmetric, so that all the notions and results of the previous section directly apply. We will use the terms "algebraically independent", "algebraic basis", etc. instead of "[ ]-independent", "[ ]-basis", etc. In particular, 2.3 and 2.5 imply

Theorem 3.1. *For every algebraically independent Sd-class X_0 and every algebraically generating Sd-class X_1 such that $X_0 \subseteq X_1 \subseteq W$, there is an algebraic basis X such that $X_0 \subseteq X \subseteq X_1$ and* $\mathrm{Sd}(X)$. *Moreover, if $X, Y \subseteq W$ are both*

Sd-classes and algebraic bases and one of them is a set, then also the remaining one is a set with the same number of elements.

Assume that u is an algebraic basis in W with n elements and $u = \{x^1, \ldots, x^n\}$ is its fixed set enumeration. Then W can directly be identified with the vector space Q^n of all ordered n-tuples of rationals with componentwise defined operations.

If there is a proper Sd-class X forming an algebraic basis of W, then there is an Sd-bijection $F : N \approx X$. Writing x^n instead of $F(n)$ and regarding every linear combination $\alpha_0 x^0 + \alpha_1 x^1 + \cdots + \alpha_n x^n$, where $\langle \alpha_0, \ldots, \alpha_n \rangle \in Q^{n+1}$, as a polynomial in the variable x, one directly obtains an identification of W with the vector space $Q[x]$ of all polynomials (including those of an infinite degree) in one variable x over Q with operations defined coeficientwise. Thus we have proved the following theorem:

Theorem 3.2. *For every vector space W there is an Sd-function F which is a linear isomorphism of W either onto $Q[x]$ or onto Q^n for a uniquely defined $n \in N$.*

Consequently, the algebraic structure of Sd-vector spaces over Q (and over any set-theoretically definable field, as well) is rather a simple one. However, some vector spaces of more complex structure can, and also do, occur as (not set-theoretically definable) subspaces of $Q[x]$ or of the Q^n's. This fact is only welcome because it convinces us that our original definition is not too restrictive, and the remaining spaces which could form the counterpart of some classical ones, did not disappear as it could seem in view of the last theorem, but they still are included as subspaces in the spaces forming the main subject of our study. This even justifies the restriction of our initial investigation of basic vector spaces to spaces of the form Q^n. Indeed, if $C \subseteq N$ is any nonempty proper cut without the last element (see [K-Z 1988]) and $C \subseteq n \in N$, then the (not Sd-) vector space $Q_C[x]$ of those polynomials from $Q[x]$ whose degree belongs to C, can directly be identified with the subspace

$$Q^n|C = \{x \in Q^n;\ (\forall k \le n)(k \notin C \ \Rightarrow\ x_k = 0)\}$$

of Q^n. On the other hand, especially if C is a revealed and additive (or even multiplicative) cut, then $Q_C[x]$ and $Q^n|C$ reflect, in some sense, most of the properties of the space $Q[x]$.

In order to generalize the notion of subspace, let us consider a subring A of Q and an additive subgroup X of W. Then X will be called an A-submodule of W, or simply an A-module, if $A \cdot X \subseteq X$.

In the forthcoming sections, not only subspaces but in particular, BQ-modules will play a remarkable role.

Every set-theoretically definable linear map $F : Q^m \longrightarrow Q^n$ can be represented by a matrix $\mathbf{a} \in Q^{n \times m}$ in the obvious way. In particular, every set-theoretically definable linear functional $F : Q^m \longrightarrow Q$ is uniquely determined by a vector $x \in Q^m$. Thus to be able to start the study of the duals of the spaces Q^m, what remains is to fix the notation of the scalar or inner product

$$x \cdot y = x_1 y_1 + \cdots + x_m y_m$$

for $x, y \in Q^m$.

4. Biequivalence vector spaces.

A biequivalence vector space (BVS) is a triple $\langle W, M, G\rangle$, where W is a (set-theoretically definable) vector space (over Q), M is a π-class, G is a σ-class and the following conditions hold:

(0) $0 \in M \subseteq G \subseteq W$,
(1) $M + M \subseteq M$,
(2) $G + G \subseteq G$,
(3) $IQ \cdot G \subseteq M$.

The elements of M will be called infinitely small or infinitesimal vectors and the elements of G will be called finitely large or bounded vectors. The vectors from $W \setminus G$ are then called infinitely large.

Lemma 4.1. *Let $\langle W, M, G\rangle$ be a BVS. Then*

(a) $(\forall x \in M)(\exists \alpha \in Q \setminus BQ)(\alpha \cdot x \in M)$,
(b) $(\forall x \in W \setminus G)(\exists \alpha \in IQ)(\alpha \cdot x \in W \setminus G)$.

PROOF: (a) If $x \in M$, then $C = \{n \in N; nx \in M\}$ is a π-class and, by (1), $FN \subseteq C$. Since FN is not a π-class, $C \setminus FN \neq \emptyset$. (b) can be proved in a similar way. □

Proposition 4.2. *Let $\langle W, M, G\rangle$ be a BVS. Then also the following conditions hold:*

(4) $BQ \cdot M \subseteq M$,
(5) $BQ \cdot G \subseteq G$,

i.e., M and G are BQ-submodules of W.

PROOF: (4) Let $\alpha \in BQ$, $x \in M$. By 4.1 (a), $\beta \cdot x \in M$ for some $\beta \in Q \setminus BQ$. Then $\frac{\alpha}{\beta} \in IQ$ and, by (3), $\alpha \cdot x = \frac{\alpha}{\beta} \cdot (\beta \cdot x) \in M$.

(5) Let $\alpha \in BQ$, $x \in G$. Assume that $\alpha \cdot x \notin G$. By 4.1 (b), there is a $\beta \in IQ$ such that $\beta \cdot \alpha \cdot x \notin G$. But $\beta \cdot \alpha \in IQ$, hence $\beta \cdot \alpha \cdot x \in M \subseteq G$ by (0) and (3). □

The name "biequivalence vector space" is justified by the following obvious proposition.

Proposition 4.3. *Let $\langle W, M, G\rangle$ be a BVS. Then the pair of relations $\langle \doteq_M, \leftrightarrow_G\rangle$ defined by*

$$x \doteq_M y \quad \Leftrightarrow \quad x - y \in M,$$
$$x \leftrightarrow_G y \quad \Leftrightarrow \quad x - y \in G$$

is a biequivalence on W.

One can easily express the conditions (1)–(5) as a kind of continuity in terms of the biequivalence $\langle \doteq_M, \leftrightarrow_G\rangle$ instead of M, G, now. Obviously, M is the monad and G is the galaxy of the zero vector 0 with respect to $\langle \doteq_M, \leftrightarrow_G\rangle$. More generally, $\{x\} + M$ is the monad and $\{x\} + G$ is the galaxy of any $x \in W$. Similarly, $X + M$

is the figure and $X + G$ is the expansion of any class $X \subseteq W$ with respect to the biequivalence $\langle \doteq_M, \leftrightarrow_G \rangle$.

The conditions (0)–(5) also guarantee that the factorization BQ/IQ yielding the field of real numbers and the factorization G/M of the BQ-module G with respect to its submodule M are compatible, i.e., the multiplication $\cdot : BQ/IS \times G/M \longrightarrow G/M$ is correctly defined by

$$(\alpha + IQ)(x + M) = \alpha x + M$$

for $\alpha \in BQ$, $x \in G$. Thus G/M naturally becomes a topological vector space over the field BQ/IQ of reals, endowed with the metrizable topology induced by the π-equivalence $\doteq_M$ (cf. [M 1979], [G-Z 1985b]); it will be called the realization of $\langle W, M, G \rangle$.

The following lemma can easily be proved, even without the assumption that M is a π-class and G is a σ-class.

Lemma 4.4. *Let W be a vector space and $\langle W, M, G \rangle$ be a triple of classes satisfying the conditions* (0)–(5). *Then*

(a) $(Q \setminus IQ) \cdot (W \setminus M) \subseteq W \setminus M$,
(b) $(Q \setminus IQ) \cdot (W \setminus G) \subseteq W \setminus G$,
(c) $(Q \setminus BQ) \cdot (W \setminus M) \subseteq W \setminus G$.

Proposition 4.5. *Let $\langle W, M, G \rangle$ be a BVS. Then*

(a) $M = IQ \cdot G$,
(b) $G = M : IQ$,

in other words, each of the classes M, G uniquely determines the remaining one.

Proof: (a) $IQ \cdot G \subseteq M$ holds by (3). Let $x \in M$. By 4.1 (a), $\alpha x \in M \subseteq G$ for some $\alpha \in Q \setminus BQ$. Then $\frac{1}{\alpha} \in IQ$ and $x = \frac{1}{\alpha}(\alpha x)$.

(b) If $x \in G$, then $IQ \cdot \{x\} \subseteq M$ by (3) again. If $IQ \cdot \{x\} \subseteq M$, then, according to 4.1 (b), x cannot belong to $W \setminus G$. $\square$

Let us recall that a class $X \subseteq W$ in a vector space W is called balanced if $[-1, 1] \cdot X \subseteq X$.

A class S will be called a bounded neighbourhood of 0 in a BVS $\langle W, M, G \rangle$ if S is an Sd-class and $M \subseteq S \subseteq G$.

A sequence $\{S_n;\, n \in FZ\}$ of balanced Sd-classes $S_n \subseteq W$ in a vector space W is called a bigenerating sequence provided there is a $\lambda \in BQ$, $\lambda \geq 1$, such that for each $n \in FZ$ it holds

$$S_n + S_n \subseteq S_{n+1} \subseteq 2\lambda \ \cdot S_n.$$

Let $S \subseteq W$ be a balanced Sd-class in a vector space W and $\lambda \in BQ$, $\lambda \geq 1$. Then the pair $\langle S, \lambda \rangle$ will be called a generating pair in W if it holds

$$\widehat{S} \subseteq \lambda \cdot S.$$

Theorem 4.6. *Let X be a vector space and M, G be subclasses of W. Then the following conditions are equivalent:*

(a) *$\langle W, M, G\rangle$ is a biequivalence vector space;*
(b) *the triple $\langle W, M, G\rangle$ satisfies the conditions (0)–(3) from the definition of a BVS and there is an Sd-class S such that $M \subseteq S \subseteq G$;*
(c) *$\langle W, M, G\rangle$ is as in (b) and there exists a generating pair $\langle S, \lambda\rangle$ such that $M \subseteq S \subseteq G$;*
(d) *there is a bigenerating sequence $\{S_n;\, n \in FZ\}$ in W such that*

$$M = \bigcap\{S_n;\, n \in FZ\}, \qquad G = \bigcup\{S_n;\, n \in FZ\}.$$

Proof: (a)⇒(b) and (d)⇒(a) are trivial. To prove the remaining implications, let us state the following claim:

($*$) Let the triple $\langle W, M, G\rangle$ satisfy the conditions (0)–(3) from the definition of a BVS. Let S, T be any Sd-classes such that $M \subseteq S$ and $T \subseteq G$. Then there is a $\mu \in BQ$ such that $T \subseteq \mu \cdot S$.

Indeed, if this is not the case, then there is a sequence $\{x^k; 1 \le k \in FN\} \subseteq T$ such that $\frac{1}{k}x^k \notin S$ for each k. Hence there is an infinite $k \in N$ and an $x \in T$ such that $\frac{1}{k}x \notin S$. But $\frac{1}{k} \in IQ$ and $T \subseteq G$ imply $\frac{1}{k}x \in M \subseteq S$.

Now, let S be an Sd-class such that $M \subseteq S \subseteq G$. Then $S_0 = [-1, 1] \cdot S$ is a balanced Sd-class still satisfying $M \subseteq S_0 \subseteq G$. By ($*$) there is a $\lambda \in BQ$, $\lambda \ge 1$, such that $\widehat{S_0} \subseteq \lambda \cdot S_0$, hence $\langle S_0, \lambda\rangle$ is a generating pair. This proves (b)⇒(c). Concerning (c)⇒(d), let us start with a generating pair $\langle S, \lambda\rangle$ such that $M \subseteq S \subseteq G$. We put

$$S_n = (2\lambda)^n \cdot S$$

for each $n \in FZ$. Obviously $\{S_n;\, n \in FZ\}$ is a sequence of balanced Sd-classes and even

$$S_n + S_n = 2 \cdot (\frac{1}{2} \cdot S_n + \frac{1}{2} \cdot S_n) \subseteq 2 \cdot \widehat{S}_n \subseteq 2\lambda \cdot S_n = S_{n+1}$$

holds for each n. The inclusions

$$M \subseteq \bigcap\{S_n;\, n \in FZ\}, \qquad \bigcup\{S_n;\, n \in FZ\} \subseteq G$$

are obvious. The reversed inclusions easily follow from ($*$). □

In particular, a vector space W with subclasses M, G satisfying the conditions (0)–(3) is a BVS, i.e., M is a π-class and G is a σ-class, if and only if there is an Sd-class S between them, i.e., $M \subseteq S \subseteq G$.

Unless else explicitly said, till the end of this section, $\langle W, M, G\rangle$ denotes a fixed but otherwise arbitrary BVS.

Let us denote

$$\mathring{W} = M : Q, \qquad \widetilde{W} = Q \cdot G.$$

Intuitively, the class

$$\mathring{W} = \{x \in Q;\ (\forall \alpha \in Q)(\alpha x \in M)\}$$

consists of extremely small vectors which cannot be made visible or extracted out of the monad M by any scalar multiple. The class

$$\widetilde{W} = \{x \in W;\ (\exists \alpha \in Q \setminus \{0\})(\alpha x \in G)\}$$

consists of those vectors which, though perhaps infinitely large, still possess a certain imaginable size; the remaining vectors forming the class $W \setminus \widetilde{W}$ cannot be attracted into the galaxy G by any nonzero scalar multiple and are completely not attainable from the galaxy G. In this sense, the classes $\mathring{W} \setminus \{0\}$ and $W \setminus \widetilde{W}$, provided nonempty, could be interpreted as representing two different types of hidden parameters. However, we withstand the temptation to re-open the offering question for the present and will proceed in a less dramatic way.

Theorem 4.7. (a) *$\mathring{W}$ and $\widetilde{W}$ are subspaces (i.e., Q-submodules) of W and*

$$0 \in \mathring{W} \subseteq M \subseteq G \subseteq \widetilde{W} \subseteq W;$$

(b) $\mathring{W} = G : Q$ *and* $\widetilde{W} = Q \cdot M$;

(c) $\mathring{W}$ *and* $\widetilde{W}$ *are Sd-classes.*

Proof: (a) is trivial.

(b) We will prove only the nontrivial inclusion $\supseteq$ in the first assertion; the second one can be proved analogously. Let x be such that $Q \cdot \{x\} \subseteq G$. Suppose $x \notin \mathring{W}$, i.e., $\alpha x \in G \setminus M$ for some α. It suffices to take an arbitrary $\beta \in Q \setminus BQ$ and 4.4 (c) implies $\beta\alpha x \notin G$, contradicting the choice of x.

(c) From the definition, it follows that $\mathring{W}$ is a π-class and $\widetilde{W}$ is a σ-class; (b) implies that $\mathring{W}$ is a σ-class and $\widetilde{W}$ is a π-class. □

Using the facts just proved, we can "ignore" the classes $\mathring{W} \setminus \{0\}$ and $W \setminus \widetilde{W}$ restricting us to the "imaginably large" vectors, i.e., to the class $\widetilde{W}$, and identifying the "extremely small" ones, i.e., the class $\mathring{W}$, with the zero vector 0. More exactly, the triple $\langle \widetilde{W}/\mathring{W}, M/\mathring{W}, G/\mathring{W} \rangle$ obtained by the restriction and factorization can be represented (coded) via an appropriate set-theoretical choice as a BVS $\langle W_1, M_1, G_1 \rangle$ such that $W_1 \subseteq \widetilde{W} \subseteq W$, $W_1 \cap \mathring{W} = \{0\}$, $M_1 = M \cap W_1$ and $G_1 = G \cap W_1$, already satisfying $\mathring{W}_1 = \{0\}$, $\widetilde{W}_1 = W_1$. Owing to the possibility of such a construction, it would be quite sufficient for the purpose of the present paper, to deal with biequivalence vector spaces $\langle W, M, G \rangle$ satisfying $\mathring{W} = \{0\}$, $\widetilde{W} = W$, which will be called trim.

Now, revisiting the proof of Theorem 4.6, notice that the definition of the classes S_n makes sense for all $n \in Z$, not just for finite ones. We leave to the reader the easy proof of the following theorem:

Theorem 4.8. *Let $\langle W, M, G\rangle$ be a BVS, $\langle S, \lambda\rangle$ be a generating pair in W such that $M \subseteq S \subseteq G$ and $S_n = (2\lambda)^n \cdot S$ for each $n \in Z$. Then $\{\langle x, n\rangle \in W \times Z;\, x \in S_n\}$ is an Sd-class, $S_n + S_n \subseteq S_{n+1} = 2\lambda \cdot S_n$ holds for each n and*

$$M = \bigcap \{S_n;\, n \in FZ\}, \qquad G = \bigcup \{S_n;\, n \in FZ\},$$
$$\mathring{W} = \bigcap \{S_n;\, n \in Z\}, \qquad \widetilde{W} = \bigcup \{S_n;\, n \in Z\}.$$

5. Duals.

There is a rather unnatural schism between the algebraic and topological approach to duals (and more generally, to spaces of linear maps) in the study of topological vector spaces within the scope of the classical set-theoretical mathematics. The algebraic dual of a — no matter that topological — vector space consists of all its linear functionals. However, only the bounded — or if you wish — the continuous ones are admitted into its topological dual, and several topologies can be introduced on them (see e.g. [Rb-Rb 1964], [Wi 1978]). This schism seems to be surmounted by our approach based on the BVS concept and on the representation of vector spaces in the form Q^n, discussed in Section 3.

Proposition 5.1. *Let $\langle W_1, M_1, G_1\rangle$, $\langle W_2, M_2, G_2\rangle$ be two biequivalence vector spaces and $F : W_1 \longrightarrow W_2$ be a linear mapping. Then the following three conditions are equivalent:*

(a) $F''G_1 \subseteq G_2$,
(b) $F''M_1 \subseteq M_2$,
(c) $F''M_1 \subseteq G_2$.

Proof: As F is linear, for all $A \subseteq Q$, $X \subseteq W_1$ it obviously holds

$$F''(A \cdot X) = A \cdot (F''X), \qquad F''(X : A) \subseteq (F''X) : A.$$

Using this observation, (a)⇒(b) and (c)⇒(a) will be proved by the following computations:

If $F''G_1 \subseteq G_2$, then

$$F''M_1 = F''(IQ \cdot G_1) = IQ \cdot (F''G_1) \subseteq IQ \cdot G_2 = M_2.$$

If $F''M_1 \subseteq G_2$, then

$$F''G_1 = F''(M_1 : IQ) \subseteq (F''M_1) : IQ \subseteq G_2 : IQ \subseteq G_2.$$

The remaining implication (b)⇒(c) is trivial. □

Linear maps satisfying (a) are called bounded, and those satisfying (b) are called continuous. Thus we have established for BVS' an analogue of the well known classical result: a linear map is continuous iff it is bounded.

In our approach, all the set-theoretically definable linear functionals on the space Q^n, represented as vectors from Q^n using the inner product, fall into the dual space. The bounded (= continuous) ones are exactly those forming the galaxy of 0 in the dual. Then the monad of 0 in the dual is already determined uniquely. More precisely, the dual $\langle Q^n, M, G\rangle'$ of a BVS $\langle Q^n, M, G\rangle$ is a triple $\langle Q^n, M', G'\rangle$, where

$$M' = \{x \in Q^n;\ (\forall y \in G)(x \cdot y \in IQ)\}$$

and

$$\begin{aligned} G' &= \{x \in Q^n;\ (\forall y \in G)(x \cdot y \in BQ)\} \\ &= \{x \in Q^n;\ (\forall y \in M)(x \cdot y \in IQ)\} \\ &= \{x \in Q^n;\ (\forall y \in M)(x \cdot y \in BQ)\}. \end{aligned}$$

Since $\langle Q, IQ, BQ\rangle$ obviously is a BVS, the fact that all the three expressions for G' coincide follows from the previous proposition. Note that from the definition of M' it also follows that it is a π-class; the fact that G' is a σ-class is due to the last expression for G'. Also the satisfaction of the conditions (0)–(3) from the previous section is evident for $\langle Q^n, M', G'\rangle$. Thus we have proved the following result.

Theorem 5.2. *If $\langle Q^n, M, G\rangle$ is a BVS, then its dual $\langle Q^n, M', G'\rangle$ is a BVS, too.*

A BVS $\langle Q^n, M, G\rangle$ is called reflexive if $\langle Q^n, M, G\rangle'' = \langle Q^n, M, G\rangle$. Note that the inclusions $M \subseteq M''$, $G \subseteq G''$ are trivial.

In essentially the same way, given two biequivalence vector spaces $\langle Q^m, M_1, G_1\rangle$ and $\langle Q^n, M_2, G_2\rangle$, the vector space $Q^{n\times m}$ of all $n \times m$ matrices over Q can be converted into a BVS $\langle Q^{n\times m}, M, G\rangle$ putting

$$\begin{aligned} M &= \{\mathbf{a} \in Q^{n\times m};\ (\forall x \in G_1)(\mathbf{a} \cdot x \in M_2)\}, \\ G &= \{\mathbf{a} \in Q^{n\times m};\ (\forall x \in M_1)(\mathbf{a} \cdot x \in G_2)\}, \end{aligned}$$

where $\mathbf{a} \cdot x$ denotes the usual multiplication of matrices and the elements of Q^m, Q^n are regarded as column vectors.

We will close the section by a rather important and instructive example. The reader can compare our approach and results with those in [H-Mr 1983b]. Let p be a positive hyperreal number or the sign of infinity ∞ and n be a natural number (to avoid trivialities, we assume $n > 1$). For each $x \in Q^n$, we put

$$\|x\|_p = \begin{cases} \left(|x_1|^p + \cdots + |x_n|^p\right)^{1/p} & \text{if } 0 < p < \infty, \\ \max\{|x_1|, \ldots, |x_n|\} & \text{if } p = \infty. \end{cases}$$

Then, as it can easily be seen, $\|\cdot\|_p : Q^n \longrightarrow HR$,

$$\begin{aligned} &\|x\|_p = 0 \quad \Leftrightarrow \quad x = 0, \\ &\|\alpha \cdot x\|_p = |\alpha| \cdot \|x\|_p, \\ &\|x + y\|_p \le \max\{1, 2^{1/p-1}\} \cdot (\|x\|_p + \|y\|_p) \end{aligned}$$

hold for all $\alpha \in Q$, $x, y \in Q^n$, and the last quoted estimation is the best possible. Therefore, putting

$$M_p(n) = \{x \in Q^n;\ \|x\|_p \in IHR\},$$
$$G_p(n) = \{x \in Q^n;\ \|x\|_p \in BHR\},$$

the triple $\mathcal{L}_p(n) = \langle Q^n, M_p(n), G_p(n)\rangle$ becomes a BVS if and only if p is not infinitesimal.

Now, let q be another positive hyperreal or ∞. As the proof of the following assertion can be fulfilled by rather elementary means, we venture to omit it.

Proposition 5.3. *For all admitted n, p, q, each of the conditions*

$$M_p(n) = M_q(n), \qquad G_p(n) = G_q(n)$$

is equivalent to

$$\left(\frac{1}{p} - \frac{1}{q}\right) \cdot \lg n \in BHR.$$

Corollary 5.4. (a) *If n is finite, then for all $p, q \in HR^+ \cup \{\infty\}$, the biequivalence vector spaces $\mathcal{L}_p(n)$, $\mathcal{L}_q(n)$ coincide.*

(b) *If $p \in HR^+$, then $\mathcal{L}_p(n) = \mathcal{L}_\infty(n)$ iff $\lg n < kp$ for some $k \in FN$.*

It also follows that for each $n \in N$, $p \in HR^+ \cup \{\infty\}$, there is an $\alpha \in Q$, $\alpha \cdot\!> 0$, such that $\mathcal{L}_p(n) = \mathcal{L}_\alpha(n)$. If $p \neq \infty$, then obviously we can require $\alpha \doteq p$. But in general, the condition $p \doteq q$ is not sufficient for $\mathcal{L}_p(n) = \mathcal{L}_q(n)$.

For infinite n, the spaces $\mathcal{L}_p(n)$ remind of their classical counterparts, namely the ℓ_p and L_p spaces. Concerning their relationship to the latter ones, more precisely to the spaces $L_p[0,1]$ (with the usual Lebesgue measure on the real interval $[0,1]$), it probably would appear more transparent if we started with the definition

$$|\, x \,|_p = \begin{cases} \left(\frac{|x_1|^p + \cdots + |x_n|^p}{n}\right)^{1/p} & \text{if } 0 < p < \infty \\ \max\{|x_1|, \ldots, |x_n|\} & \text{if } p = \infty \end{cases}$$

instead of the original one. Indeed, the last definition can be viewed as an infinite sum representation of the classical p-norm given by the Lebesgue integral

$$\|f\|_p = \begin{cases} \left(\int_0^1 |f(t)|^p \, dt\right)^{1/p} & \text{if } 0 < p < \infty \\ \sup\{|f(t)|; t \in [0,1]\} & \text{if } p = \infty \end{cases}$$

for the classical functions $f \in L_p[0,1]$. However, as for each $x \in Q^n$ it holds

$$\|x\|_p = n^{1/p} \cdot |\, x \,|_p\ ,$$

the new biequivalence vector spaces obtained in this way are canonically isomorphic to our original $\mathcal{L}_p(n)$'s through an Sd-map. Thus everything established for the $\mathcal{L}_p(n)$'s immediately applies to the new BVS' defined by using $|\cdot|_p$ instead of $\|\cdot\|_p$.

One remarkable thing is the behavior of the duals of the spaces $\mathcal{L}_p(n)$ which is much nicer than that of their classical analogues. Let us put for each $p \in HR^+ \cup \{\infty\}$

$$p' = \begin{cases} \frac{p}{p-1} & \text{if } 1 < p < \infty \\ \infty & \text{if } 0 <\cdot\, p \leq 1 \\ 1 & \text{if } p = \infty. \end{cases}$$

Then the following result can be proved in a fairly standard way.

Proposition 5.5. *For all admitted n, p, it holds*

$$\mathcal{L}_p(n)' = \mathcal{L}_{p'}(n)\,.$$

Note that the stated equality holds not only if $1 \leq p < \infty$, as in the classical situation, but also for $p = \infty$, and even if $0 <\cdot\, p < 1$. On the other hand, as proved in [Dy 1940], the classical spaces L_p have no continuous functionals except the zero constant map if $0 < p < 1$.

To translate the last proposition to the biequivalence vector spaces defined through the "integral" norm $|\cdot|_p$, one only has to substitute the more "integral reminding" inner product

$$x \bullet y = \frac{1}{n}\big(x_1 y_1 + \cdots + x_n y_n\big)$$

on Q^n into the place of the original $x \cdot y$.

Propositions 5.3 and 5.5 have the following consequence.

Proposition 5.6. *The biequivalence vector space $\mathcal{L}_p(n)$ is reflexive if and only if*

$$p \geq 1 \qquad \text{or} \qquad \Big(\frac{1}{p} - 1\Big) \cdot \lg n \in BHR.$$

6. Valuations on vector spaces.

Perhaps the most important ones among the classical topological vector spaces are the Banach spaces. Also in the alternative set theory (where every π-equivalence automatically induces the structure of a complete metrizable space), it is important to have some valuations of the size of vectors from a given vector space W. The prescribed conditions, valuations should be subject to, ought to be strong enough to enable smooth computations and to guarantee that any valuation on W induced the structure of a BVS on W. On the other hand, they may not be too restrictive, as it would be desirable, any BVS $\langle W, M, G\rangle$ could be obtained in this way from a suitable valuation on W. As it will be shown in this section, the solution of the raised problem lies in the following definition.

Let W be a vector space, $p, q \in BHR \setminus IHR$, $p > 0$, $q > 0$. An Sd-function Φ will be called a (p,q)-valuation on W provided $\mathrm{dom}(\Phi)$ is a subspace of W, $\mathrm{rng}(\Phi) \subseteq HR$ and for all $\alpha \in Q$, $x, y \in \mathrm{dom}(\Phi)$ the following two conditions hold:

$$\Phi(\alpha x) = |\alpha|^p \cdot \Phi(x),$$
$$\Phi(x+y) \le q \cdot (\Phi(x) + \Phi(y)).$$

Then obviously also

$$\Phi(0) = 0 \quad \text{and} \quad \Phi(x) \ge 0$$

for each $x \in \mathrm{dom}(\Phi)$. Φ will be called a valuation on W if it is a (p,q)-valuation for some pair of admitted parameters p, q. A $(1,1)$-valuation will be called a norm.

Let Φ be a valuation on a vector space W. We put

$$\ker(\Phi) = \{x \in \mathrm{dom}(\Phi);\ \Phi(x) = 0\},$$
$$M(\Phi) = \{x \in \mathrm{dom}(\Phi);\ \Phi(x) \in IHR\},$$
$$G(\Phi) = \{x \in \mathrm{dom}(\Phi);\ \Phi(x) \in BHR\}.$$

The following proposition is an immediate consequence of the definition of valuation.

Proposition 6.1. *Let W be a vector space and Φ be a valuation on W. Then the triple $\langle W, M(\Phi), G(\Phi)\rangle$ is a BVS satisfying*

$$\mathring{W} = \ker(\Phi), \qquad \widetilde{W} = \mathrm{dom}(\Phi).$$

A valuation Φ on a vector space W will be called total if $\ker(\Phi) = \{0\}$ and $\mathrm{dom}(\Phi) = W$. According to the last proposition a valuation Φ on W is total iff the BVS $\langle W, M(\Phi), G(\Phi)\rangle$ is trim.

A valuation Φ will be called trivial if $\ker(\Phi) = \mathrm{dom}(\Phi)$. A trivial valuation is a (p,q)-valuation for all possible choices of p, q. However, for a nontrivial valuation Φ, the parameter p, such that Φ is a (p,q)-valuation for some q, is determined uniquely. On the other hand, if Φ is a (p,q_1)-valuation and $q_1 \le q_2$, then Φ obviously is a (p,q_2)-valuation, as well. But even in this case there is the least

$$q_0 = \inf\bigl\{\lambda \in Q;\ (\forall x, y \in \mathrm{dom}(\Phi))\bigl(\Phi(x+y) \le \lambda(\Phi(x) + \Phi(y))\bigr)\bigr\}$$

with this property. Finally, the computation

$$2^p \cdot \Phi(x) = \Phi(2x) \le 2q \cdot \Phi(x)$$

shows that if there is a nontrivial (p,q)-valuation, then it holds

$$2^{p-1} \le q, \quad \text{or equivalently,} \quad p \le 1 + \lg q.$$

Now let us state the converse of Proposition 6.1.

Theorem 6.2. *Let $\langle W, M, G\rangle$ be a BVS. Then there are numbers $p, q \in BHR$ such that $0 <\cdot\, p \le 1 \le q$, a $(1,q)$-valuation Φ and a $(p,1)$-valuation Ψ on W such that*

$$\ker(\Phi) = \ker(\Psi) = \mathring{W}, \qquad \mathrm{dom}(\Phi) = \mathrm{dom}(\Psi) = \widetilde{W},$$
$$M(\Phi) = M(\Psi) = M, \qquad G(\Phi) = G(\Psi) = G.$$

Proof: First we will construct a $(1,q)$-valuation Φ. Let S be any balanced bounded neighbourhood of 0 in $\langle W, M, G\rangle$. For each $x \in \widetilde{W}$, we put

$$\Phi(x) = \inf\{\alpha \in Q;\ \alpha \ge 0,\ x \in \alpha \cdot S\}.$$

Then, by a rigorous checking of suitable set formulas, we obtain that $\Phi : \widetilde{W} \longrightarrow HR$ is an Sd-function and for all $\alpha \in Q$, $x \in \widetilde{W}$, it holds

$$\Phi(\alpha \cdot x) = |\alpha| \cdot \Phi(x),$$

and also the equalities $\ker(\Phi) = \mathring{W}$, $\mathrm{dom}(\Phi) = \widetilde{W}$, $M(\Phi) = M$ and $G(\Phi) = G$ are satisfied. Moreover,

$$\{x \in \widetilde{W};\ \Phi(x) < 1\} \subseteq S \subseteq \{x \in \widetilde{W};\ \Phi(x) \le 1\}.$$

By the claim $(*)$ from the proof of 4.6, there is a $\lambda \in BQ$, $\lambda \ge 1$, such that $\widehat{S} \subseteq \lambda \cdot S$. If we put

$$q_0 = \inf\{\lambda \in Q;\ \lambda \ge 1,\ \widehat{S} \subseteq \lambda \cdot S\},$$

then it is routine to check that $q_0 \in BHR$, $q_0 \ge 1$ and for each $q \in BQ, q \ge q_0$, and all $x, y \in \widetilde{W}$, it holds

$$\Phi(x + y) \le q \cdot (\Phi(x) + \Phi(y)).$$

Now, we will construct a $(p,1)$-valuation Ψ using the already constructed valuation Φ. We put

$$T = \{x \in \widetilde{W};\ \Phi(x) < 1\}.$$

Then T again is a bounded balanced neighbourhood of 0 in $\langle W, M, G\rangle$, and

$$\Phi(x) = \inf\{\alpha \in Q;\ \alpha \ge 0,\ x \in \alpha \cdot T\}$$

for $x \in \widetilde{W}$. By the already used claim $(*)$, there is a $\kappa \in BQ$, $\kappa \ge 3$, such that $T + T + T \subseteq \kappa \cdot T$. For each $n \in Z$, we put

$$T_n = \kappa^n \cdot T.$$

Then the Sd-sequence $\{T_n;\ n \in Z\}$ satisfies all the conditions of 4.8, and even

$$T_n + T_n + T_n \subseteq \kappa \cdot T_n = T_{n+1}$$

holds for each n. Let us denote $p = 1/\lg\kappa$, and put

$$\Gamma(x) = \Phi(x)^p$$

for $x \in \widetilde{W}$. Then obviously $0 <\cdot\, p <\cdot\, 1$ and the fact that Γ is a (p, q^p)-valuation for each $q \geq q_0$ can be verified by two straightforward computations which are left to the reader. According to the choice of κ and p, for all $x \in \widetilde{W}$, $n \in Z$, we have

$$\Gamma(x) < 2^n \qquad \text{iff} \qquad \Phi(x) < \kappa^n \qquad \text{iff} \qquad x \in T_n,$$

hence $\ker(\Gamma) = \mathring{W}$, $\operatorname{dom}(\Gamma) = \widetilde{W}$, $M(\Gamma) = M$ and $G(\Gamma) = G$. Finally we put for $x \in \widetilde{W}$

$$\Psi(x) = \inf\Big\{\sum_{i=0}^{k} \Gamma(x^i)\,;\; k \in N,\; \langle x^0, ..., x^k\rangle \in \widetilde{W}^{k+1},\; \sum_{i=0}^{k} x^i = x\Big\}.$$

It is clear immediately from the construction that Ψ is a $(p, 1)$-valuation on W and $\operatorname{dom}(\Psi) = \widetilde{W}$. The remaining conditions will follow from the inclusions

$$T_n \subseteq \{x \in \widetilde{W};\; \Psi(x) < 2^n\} \subseteq T_{n+1}$$

holding for each $n \in Z$. In view of the inequality $\Psi(x) \leq \Gamma(x)$, the first inclusion is trivial, since $\Gamma(x) < 2^n$ for $x \in T_n$. To establish the second inclusion, it is enough to show that for all $k \in N$, $\langle x^0, ..., x^k\rangle \in \widetilde{W}^{k+1}$ and $n \in Z$, it holds

$$\Gamma(x^0) + \cdots + \Gamma(x^k) < 2^n \;\Rightarrow\; x^0 + ... + x^k \in T_{n+1}.$$

This can be done by induction over k. For $k = 0$, it is trivial. Let $k \geq 1$ and

$$\mu = \Gamma(x^0) + \cdots + \Gamma(x^k) < 2^n.$$

Without loss of generality we can assume that $\mu > 0$ and $\Gamma(x^0) \leq \Gamma(x^1) \leq \cdots \leq \Gamma(x^k)$. Let $j \leq k$ be the least number such that

$$\Gamma(x^0) + \cdots + \Gamma(x^j) > \frac{\mu}{2}.$$

Then obviously $0 < j \leq k$ and

$$\Gamma(x^0) + ... + \Gamma(x^{j-1}) < 2^{n-1},$$

hence $x^0 + ... + x^{j-1} \in T_n$ by the induction argument. If $j = k$, then $\Gamma(x^k) < 2^n$ and $x^k \in T_n$, hence

$$x^0 + \cdots + x^{k-1} + x^k \in T_n + T_n \subseteq T_{n+1}.$$

If $j < k$, then $\Gamma(x^j) < 2^{n-1}$ and also

$$\Gamma(x^{j+1}) + \cdots + \Gamma(x^k) < 2^{n-1}.$$

Again, by the induction presumption, we have $x^j \in T_n$, $x^{j+1} + \cdots + x^k \in T_n$, hence

$$x^0 + \cdots + x^{j-1} + x^j + x^{j+1} + \cdots + x^k \in T_n + T_n + T_n \subseteq T_{n+1}.$$

□

If Ψ is a $(p, 1)$-valuation, then the function $\Psi(x - y)$ becomes a metric on $\widetilde{W}$ invariant with respect to translations (with the possible exception that $\Psi(x-y) = 0$ iff $x - y \in \mathring{W}$, and not only if $x = y$).

Let us record the following technical result for the future.

Lemma 6.3. *Let $\langle W, M, G\rangle$ be a BVS. Then for each $n \in N$, there is an $m \in N$ such that*

$$n \star M \subseteq m \cdot M, \qquad n \star G \subseteq m \cdot G.$$

Proof: Let Ψ be the $(p,1)$-valuation just constructed. Then for each $n \in N$, $\langle x^1, \dots, x^n\rangle \in \widetilde{W}^n$, it holds

$$\Psi(x^1 + \dots + x^n) \le n \cdot \max\{\Psi(x^1), \dots, \Psi(x^n)\}.$$

Thus it suffices to take any $m \ge n^{1/p}$. □

Let us recall that a class X in a vector space W is called convex if $\widehat{X} = X$, i.e., if $\alpha x + (1-\alpha)y \in X$ for all $x, y \in X$, $\alpha \in [0,1]$. X will be called totally convex if $\langle X\rangle = X$, i.e., if $\sum_{i=0}^n \alpha_i x^i \in X$ for all $n \in N$, $\langle x^0, \dots, x^n\rangle \in X^{n+1}$, $\langle \alpha_0, \dots, \alpha_n\rangle \in [0,1]^{n+1}$ such that $\alpha_0 + \dots + \alpha_n = 1$. As it can easily be proved by induction, an Sd-class is totally convex if and only if it is convex. However, as we shall see within short, this result cannot be generalized even to σ- and π-classes.

A biequivalence vector space $\langle W, M, G\rangle$ is called locally convex if there is a convex bounded neighbourhood of 0 in $\langle W, M, G\rangle$.

A BVS $\langle W, M, G\rangle$ is called normable if there is a norm (i.e., a $(1,1)$-valuation) Φ on W such that $\mathrm{dom}(\Phi) = \widetilde{W}$, $\ker(\Phi) = \mathring{W}$, $M(\Phi) = M$ and $G(\Phi) = G$.

Besides the classical characterization of normability we have

Theorem 6.4. *Let $\langle W, M, G\rangle$ be a BVS. Then the following conditions are equivalent:*

(a) *$\langle W, M, G\rangle$ is locally convex;*
(b) *$\langle W, M, G\rangle$ is normable;*
(c) *M is totally convex;*
(d) *G is totally convex.*

Proof: (a)⇒(b) If T is a convex bounded neighbourhood of 0 in $\langle W, M, G\rangle$, then

$$S = T : [-1,1] = \{x \in W;\ [-1,1] \cdot \{x\} \subseteq T\}$$

obviously is a convex balanced bounded neighbourhood of 0 in $\langle W, M, G\rangle$. The rest follows from the first part of the proof of 6.2.

(b)⇒(c) Let Φ be a norm on $\langle W, M, G\rangle$ satisfying the conditions required. Let $n \in N$, $\langle x^0, \dots, x^n\rangle \in M^{n+1}$, $\langle \alpha_0, \dots, \alpha_n\rangle \in [0,1]^{n+1}$ and $\alpha_0 + \dots + \alpha_n = 1$. We denote

$$x = \sum_{i=0}^n \alpha_i x^i\ .$$

Then

$$\Phi(x) \le \sum_{i=0}^n \alpha_i \Phi(x^i) \le \Big(\sum_{i=0}^n \alpha_i\Big) \cdot \max\{\Phi(x_i);\ 0 \le i \le n\} \in IHR.$$

Hence $x \in M$.

(c)$\Rightarrow$(d) is an immediate consequence of the equality $G = M : IQ$.

(d)$\Rightarrow$(a) Let S be any bounded neighbourhood of 0 in $\langle W, M, G\rangle$. As G is totally convex, $S \subseteq \langle S\rangle \subseteq G$. Thus the Sd-class $\langle S\rangle$ is a convex bounded neighbourhood. □

It could be of some interest that the not normable BVS' have the following rather pathological property.

Proposition 6.5. *A BVS $\langle W, M, G\rangle$ is not normable iff for some $n \in N$ there are $\langle x^0, \dots, x^n\rangle \in M^{n+1}$, $\langle \alpha_0, \dots, \alpha_n\rangle \in Q^{n+1}$ such that $|\alpha_0| + \cdots + |\alpha_n| \in IQ$ and*

$$\sum_{i=0}^{n} \alpha_i x^i \notin G.$$

Proof: If $\langle W, M, G\rangle$ is not normable, then by the preceding Theorem there is an $n \in N$ and $\langle y^0, \dots, y^n\rangle \in M^{n+1}$, $\langle \beta_0, \dots, \beta_n\rangle \in [0,1]^{n+1}$ such that $\beta_0 + \cdots + \beta_n = 1$ and $\sum \beta_i y^i \notin M$. Then one can find a $\gamma \in Q \setminus BQ$ such that $\gamma^2 \cdot y^i \in M$ for each i. It suffices to put

$$x^i = \gamma^2 \cdot y_i, \qquad \alpha_i = \frac{\beta_i}{\gamma}\ .$$

□

For $p \geq 1$, $n \geq 1$, the function $\|\cdot\|_p$ from Section 5 is a norm on the BVS $\mathcal{L}_p(n)$. Thus for p satisfying

$$p \geq 1 \qquad \text{or} \qquad \left(\frac{1}{p} - 1\right) \cdot \lg n \in BHR,$$

$\mathcal{L}_p(n)$ is normable. If $p < 1$, then $\|\cdot\|_p$ is not a norm, though it still is a $\left(1, 2^{1/p-1}\right)$-valuation. On the basis of 6.3, it can easily be verified that for $p < 1$, $\left(\frac{1}{p} - 1\right) \cdot \lg n \notin BHR$, the BVS $\mathcal{L}_p(n)$ is not normable, as the monad $M_p(n)$ (and hence the galaxy $G_p(n)$, as well) is not totally convex. On the other hand, in every BVS, both the monad and the galaxy of 0 are convex.

7. The envelope operation.

As a motivational example let us consider the BVS $\langle Q^2, M, G\rangle$ where

$$M = \{x \in Q^2;\ x_1 + x_2\sqrt{2} \in IHR\},$$
$$G = \{x \in Q^2;\ x_1 + x_2\sqrt{2} \in BHR\}.$$

It can easily be seen that $\langle Q^2, M, G\rangle$ is a trim BVS with the two-element algebraic basis $\{\langle 1, 0\rangle, \langle 0, 1\rangle\}$. However, from a topological point of view, $\langle Q^2, M, G\rangle$ is only one-dimensional, as both the vectors $\langle 1, 0\rangle$, $\langle 0, 1\rangle$ "lie on the same line".

To be able to apprehend the phenomenon of vectors "lying in a subspace $[u]$ generated by a set u though not necessarily belonging to $[u]$", which can occur in a BVS over Q, we will introduce the notion of the envelope of a set u defined by

$$\mathcal{E}(u) = \{x \in W;\ [\{x\}] \subseteq [u] + M\}$$

in every BVS $\langle W, M, G\rangle$.

In the above example, as one can directly verify, $\langle 0,1\rangle \in \mathcal{E}\{\langle 1,0\rangle\}$ and vice versa.

Till the end of the section $\langle W, M, G\rangle$ denotes a fixed but otherwise arbitrary BVS. By the way, observe that

$$\mathcal{E}(\emptyset) = \mathcal{E}(\{0\}) = \mathring{W}.$$

Theorem 7.1. *$\mathcal{E}$ is a symmetric Sd-closure on W.*

Proof: It can easily be shown that for each $u \subseteq W$

$$\mathcal{E}(u) = \{x \in W;\ [\{x\}] \subseteq [u] + G\}.$$

As a consequence

$$\mathcal{E}(u) = \{x \in W;\ [\{x\}] \subseteq [u] + S\}$$

for every bounded neighbourhood S of 0. Thus $\mathcal{E}(u)$ can be defined by a set-theoretical formula. Also the following conditions are trivial for any $u, v \subseteq W$:

$$u \subseteq [u] \subseteq \mathcal{E}(u),$$
$$u \subseteq v \ \Rightarrow\ \mathcal{E}(u) \subseteq \mathcal{E}(v),$$

and additionally

$$[u] = [v] \ \Rightarrow\ \mathcal{E}(u) = \mathcal{E}(v).$$

In order to prove the nontrivial inclusion in the equality

$$\bigcup\{\mathcal{E}(v);\ v \subseteq \mathcal{E}(u)\} = \mathcal{E}(u),$$

let us consider an x belonging to the left side. Take an arbitrary $\alpha \in Q$. Then for some n there are $\langle y^1, \dots, y^n\rangle \in \mathcal{E}(u)^n$, $\langle \beta_1, \dots, \beta_n\rangle \in Q^n$ such that

$$\alpha x - \sum \beta_i y^i \in M.$$

By 6.3, there is an $m \in N$ such that $\frac{1}{m} \cdot (n \star M) \subseteq M$. As for each $i \leq n$ it holds $y^i \in \mathcal{E}(u)$, there is a $z^i \in [u]$ such that

$$m\beta_i y^i - z^i \in M.$$

By Lemma 1.1, the elements z^i can be chosen in such a way that $\langle z^1, \dots, z^n\rangle \in [u]^n$. We put

$$z = \frac{1}{m} \cdot \sum z^i.$$

Then $z \in [u]$ and, by the choice of m, it holds

$$\sum \beta_i y^i \ - z = \frac{1}{m} \cdot \sum (m\beta_i y^i - z^i) \in M.$$

Consequently

$$\alpha x - z = \big(\alpha x - \sum \beta_i y^i\big) + \big(\sum \beta_i y^i \ - z\big) \in M,$$

hence $[\{y\}] \subseteq [u] + M$. This proves that for each $u \subseteq W$,

$$\mathcal{E}\mathcal{E}(u) = \mathcal{E}(u).$$

It remains to prove the exchange condition

$$x \in \mathcal{E}(u \cup \{y\}) \setminus \mathcal{E}(u) \ \Rightarrow \ y \in \mathcal{E}(u \cup \{x\})$$

for all $x, y \in W$, $u \subseteq W$. If $x \notin \mathcal{E}(u)$, then $\alpha x \notin [u] + M$ for some α. Take an arbitrary $\beta \neq 0$. We will prove

$$\beta y \in [u \cup \{x\}] + M.$$

If $y \notin \widetilde{W}$, then even $y \in [u \cup \{x\}]$. If $y \in \widetilde{W}$, then there is a $\gamma \in IQ$, $\gamma \neq 0$, such that $\gamma\beta y \in M$. As $x \in \mathcal{E}(u \cup \{y\})$, there is a $z \in [u]$ and a $\delta \in Q$ such that

$$(*) \qquad \frac{\alpha}{\gamma} x - z - \delta y \in M.$$

As $\gamma \in IQ$, it follows

$$\alpha x - \gamma z - \gamma\delta y \in M.$$

Since $\alpha x \notin [u] + M$, $\gamma z \in [u]$, we conclude $\gamma\delta y \notin M$. Therefore $|\beta| < |\delta|$, in other words, $\left|\frac{\beta}{\delta}\right| < 1$. Thus multiplying $(*)$ by $\frac{\beta}{\delta}$ one obtains

$$\frac{\alpha\beta}{\gamma\delta} x - \frac{\beta}{\delta} z - \beta y \in M.$$

Hence $\beta y \in [u \cup \{x\}] + M$ and $y \in \mathcal{E}(u \cup \{x\})$. □

The theorem just proved, together with 2.3 and 2.5, yields the following consequence.

Theorem 7.2. *Let $\langle W, M, G\rangle$ be a BVS. Then for every $\mathcal{E}$-independent Sd-class X_0 and every $\mathcal{E}$-generating Sd-class X_1, such that $X_0 \subseteq X_1 \subseteq W$, there is an $\mathcal{E}$-basis X such that $\mathrm{Sd}(X)$ and $X_0 \subseteq X \subseteq X_1$. Moreover, if $X, Y \subseteq W$ are both Sd-classes and $\mathcal{E}$-bases and one of them is a set, then also the remaining one is a set with the same number of elements.*

8. Bases in biequivalence vector spaces.

Through the entire section $\langle W, M, G\rangle$ denotes a fixed but otherwise arbitrary BVS.

A class $X \subseteq W$ will be called independent (in $\langle W, M, G\rangle$) if

$$X \cap M = \emptyset \quad \text{and} \quad (\forall x \in X)([\{x\}] \cap ([X \setminus \{x\}] + M) \subseteq M).$$

Obviously, a class X is independent iff each its subset is independent.

Proposition 8.1. *If $X \subseteq W$ is an independent class, then for each $n \in N$ and all $\langle \alpha_1, \dots, \alpha_n\rangle \in Q^n$, $\langle x^1, \dots, x^n\rangle \in X^n$, such that $x^i \neq x^j$ for $1 \leq i < j \leq n$,*

$$\alpha_1 x^1 + \dots + \alpha_n x^n \in M$$

implies $\alpha_i \in IQ$ for each $i \leq n$. If $X \subseteq G$, then this necessary condition is also sufficient.

Proof: Let $X \subseteq W$ be independent, and $\langle \alpha_1, \dots, \alpha_n\rangle$, $\langle x^1, \dots, x^n\rangle$ satisfy the corresponding presumptions. Let us denote $z = \sum \alpha_j x^j \in M$. Assume that $\alpha_i \notin IQ$, i.e., $\frac{1}{\alpha_i} \in BQ$. Then

$$x^i = \frac{1}{\alpha_i}\Big(z - \sum_{j\neq i} \alpha_j x^j\Big) \in [X \setminus \{x^i\}] + M,$$

hence $x^i \in M$, contradicting $X \cap M = \emptyset$.

Now, let $X \subseteq G$ satisfy the condition of the proposition. $X \cap M = \emptyset$ is obvious. Assume that $x \in X$, $x \notin u \subseteq X$ and $\alpha x \in [u] + M$ for some α. Then there is an n-tuple of distinct elements $\langle x^1, \dots, x^n\rangle \in u^n$ and an n-tuple of scalars $\langle \alpha_1, \dots, \alpha_n\rangle \in Q^n$ such that

$$\alpha x - \sum \alpha_i x^i \in M.$$

Then $\alpha \in IQ$, and, as $x \in G$, also $\alpha x \in M$. □

A class $X \subseteq W$ is called generating (in $\langle W, M, G\rangle$) if

$$[X] + M = W.$$

Observe that a set u is generating iff it is $\mathcal{E}$-generating. An independent generating class $X \subseteq W$ will be called a basis of $\langle W, M, G\rangle$.

The main result of this section is that every BVS possessing an $\mathcal{E}$-generating set, in particular, every BVS of the form $\langle Q^n, M, G\rangle$, has a set basis. In view of the long period for which the problem of existence of a Schauder basis in any classical separable Banach space had remained open, until it was solved negatively by P. Enflo [En 1973], this perhaps might occur rather surprising. On the other hand, in our case a slight modification of the Auerbach method (see e.g. [Sn 1970]) yields the proof of the result.

Preliminarily we will introduce some notation and state two auxiliary results. Let $u \subseteq \widetilde{W}$ be an $\mathcal{E}$-independent set which will be specified more precisely later, and $u = \{x^1, \ldots, x^n\}$, $n > 0$, be its fixed set-enumeration. Then the subspace $[u]$ of W can be identified with Q^n, i.e., each $y \in [u]$ can be identified with the n-tuple $\langle y_1, \ldots, y_n \rangle \in Q^n$ of its co-ordinates, uniquely determined by the equation

$$y = \sum_{i=1}^{n} y_i x^i .$$

If $\langle y^1, \ldots, y^n \rangle \in [u]^n$, then $D(y^1, \ldots, y^n)$ denotes the determinant of the $n \times n$-matrix $\left(y^i_j\right)$ formed by the co-ordinates of the column vectors y^i.

Lemma 8.2. *There is a number $\kappa \in Q$, $\kappa > 0$, such that for each $y \in [u] \cap G$ and each $i \leq n$, it holds*

$$|y_i| \leq \kappa.$$

Proof: It suffices to show an apparently weaker statement, namely

$$(*) \qquad (\forall i \leq n)(\exists \kappa \in Q, \kappa > 0)(\forall y \in [u] \cap G)(|y_i| \leq \kappa).$$

The needed conclusion is then a consequence of $(*)$ and Lemma 1.1. Assume that $(*)$ does not hold, i.e., there is an $i \leq n$ such that

$$(\forall \kappa \in Q)(\exists y \in [u] \cap G)(|y_i| > \kappa).$$

We will obtain a contradiction by proving $x^i \in \mathcal{E}(u \setminus \{x^i\})$. Take an arbitrary κ and a $y \in [u] \cap G$ such that

$$\left| \frac{\kappa}{y_i} \right| < 1.$$

Obviously $y = \sum y_j x^j \in G$. Hence also

$$\frac{\kappa}{y_i} y = \kappa x^i + \sum_{j \neq i} \frac{\kappa y_j}{y_i} x^j \in G.$$

By 7.1, $x^i \in \mathcal{E}(u \setminus \{x^i\})$. □

An immediate consequence of Lemma 8.2 is the following

Lemma 8.3. *There is a number $\lambda \in Q$, $\lambda > 0$, such that for each n-tuple $\langle y^1, \ldots, y^n \rangle \in ([u] \cap G)^n$, it holds*

$$|D(y^1, \ldots, y^n)| \leq \lambda.$$

Theorem 8.4. *Assume that there is a set $s \subseteq W$ such that $\widetilde{W} \subseteq \mathcal{E}(s)$ holds in $\langle W, M, G\rangle$. Then there is an Sd-class $X \subseteq W$ such that $X \cap \widetilde{W} \subseteq G$ and X is a basis of $\langle W, M, G\rangle$. Moreover, for any two Sd-classes $X, Y \subseteq W$ which are bases of $\langle W, M, G\rangle$, $X \cap \widetilde{W}$, $Y \cap \widetilde{W}$ are sets with the same number of elements, and if one of the Sd-classes $X \setminus \widetilde{W}$, $Y \setminus \widetilde{W}$ is a set, then also the remaining one is a set with the same number of elements.*

PROOF: There is an Sd-subspace U of W such that $W = \widetilde{W} + U$ and $\widetilde{W} \cap U = \{0\}$. Then the BVS $\langle U, U \cap M, U \cap G\rangle = \langle U, \{0\}, \{0\}\rangle$ is trivial and satisfies $\widetilde{U} = \{0\}$. Hence every algebraic basis X_1 of U already is a basis of $\langle U, U \cap M, U \cap G\rangle$. Thus, if X_0, X_1 are Sd-classes such that X_0 is a basis of $\langle \widetilde{W}, M, G\rangle$ and X_1 is an algebraic basis of U, then the Sd-class $X = X_0 \cup X_1$ is a basis of $\langle W, M, G\rangle$. It remains to prove the following special case of our Theorem.

Theorem 8.5. *Assume that there is a set $s \subseteq W$ such that $\widetilde{W} = W = \mathcal{E}(s)$ holds in $\langle W, M, G\rangle$. Then there is a set $v \subseteq G$ which is a basis of $\langle W, M, G\rangle$. Moreover, if u is an arbitrary $\mathcal{E}$-basis and v is an arbitrary basis of $\langle W, M, G\rangle$, then u, v have the same number of elements.*

PROOF: As each basis v of $\langle W, M, G\rangle$ at the same time is an $\mathcal{E}$-basis of $\langle W, M, G\rangle$, the last assertion is trivial. Let us prove the existence of the basis v. By 7.2, there is an $\mathcal{E}$-basis $u \subseteq s$ of $\langle W, M, G\rangle$. This will be the set u to which Lemmas 8.2 and 8.3 and the preliminarily introduced notation will be applied (the case $u = \emptyset$ obviously can be excluded as trivial). Let us consider the Sd-class

$$A = \{|D(y^1, \dots, y^n)|;\ \langle y^1, \dots, y^n\rangle \in ([u] \cap S)^n\} \subseteq Q,$$

where S is a fixed bounded balanced neighbourhood in $\langle W, M, G\rangle$. By 8.3, there is a $\lambda > 0$ such that $\alpha \le \lambda$ for each $\alpha \in A$. Therefore for each $\delta > 0$, there is a $\langle z^1, \dots, z^n\rangle \in ([u] \cap S)^n$ such that

$$\alpha \le (1 + \delta)\, |D(z^1, \dots, z^n)|$$

for each $\alpha \in A$. We will prove that whenever $\delta \in BQ$ (in particular, we can choose $\delta \in IQ$), then the corresponding set $v = \{z^1, \dots, z^n\} \subseteq G$ is a basis of $\langle W, M, G\rangle$. First notice that using an appropriate $(1, q)$-valuation Φ on $\langle W, M, G\rangle$, a $\langle \gamma_1, \dots, \gamma_n\rangle \in Q^n$ can be found, such that $\langle \gamma_1 x^1, \dots, \gamma_n x^n\rangle \in (S \setminus \frac{1}{2} \cdot S)^n$. Then, as

$$D(\gamma_1 x^1, \dots, \gamma_n x^n) \ne 0,$$

also

$$D(z^1, \dots, z^n) \ne 0,$$

so that the set v is algebraically independent, and having the same number of elements as u, it follows $[u] = [v]$ and $\mathcal{E}(u) = \mathcal{E}(v) = W$, i.e., v is generating. It remains to prove its independence. As $v \subseteq G$, by 8.1 it is enough to show that whenever $\langle \alpha_1, \dots, \alpha_n\rangle \in Q^n$ is such that

$$\alpha_1 z^1 + \dots + \alpha_n z^n \in M,$$

then $\alpha_i \in IQ$ for each i. Assume that $\alpha_k \notin IQ$ for some $k \leq n$. Then there is a $\beta \in Q \setminus BQ$ such that $\beta \cdot \sum \alpha_i z^i \in M \subseteq S$. Then

$$\begin{aligned} |D(z_1, \ldots, z_n)| &= \frac{1}{|\beta\alpha_k|} \left| D(z^1, \ldots, \beta \sum \alpha_i z^i, \ldots, z^n) \right| \\ &\leq \frac{1+\delta}{|\beta\alpha_k|} |D(z^1, \ldots, z^k, \ldots, z^n)|, \end{aligned}$$

which is a contradiction, as $\frac{1+\delta}{\beta\alpha_k} \in IQ$. □

Let us conclude the article with two immediate consequences.

Corollary 8.6. *Every BVS of the form $\langle Q^n, M, G \rangle$ has a set basis. All set bases of the BVS $\langle Q^n, M, G \rangle$ have the same number of elements $\leq n$.*

Let us recall that the biequivalence $\langle \doteq_M, \leftrightarrow_G \rangle$ on $\langle W, M, G \rangle$ was defined in 4.3. For the notion of compatible biequivalence see [G-Z 1985a].

Corollary 8.7. *For any BVS $\langle W, M, G \rangle$ the following conditions are equivalent:*
- (a) *the biequivalence $\langle \doteq_M, \leftrightarrow_G \rangle$ is compatible;*
- (b) *$\langle W, M, G \rangle$ has a finite basis;*
- (c) *$\langle W, M, G \rangle$ has a finite $\mathcal{E}$-basis.*

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