Comment.Math.Univ.Carolin. 32,4 (1991)615–625

On Liouville theorem and the regularity of weak solutions to some nonlinear elliptic systems of higher order

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Abstract. The aim of this paper is to show that Liouville type property is a sufficient and necessary condition for the regularity of weak solutions of nonlinear elliptic systems of the higher order.

Keywords: regularity of weak solutions, nonlinear elliptic systems

Classification: 35J60, 35D10

Introduction.

In this paper, we shall deal with nonlinear elliptic systems. More precisely, we shall consider the following problem.
Let Ω be a bounded domain with Lipschitz boundary in $\mathbb{R}^n$, $n \geq 2$. Let us denote $\sigma(n,k) = \binom{n+k-1}{k}$ and $\varrho(n,k) = \binom{n+k}{k}$, $n,k \in \mathbb{N}$. We shall study the weak solutions $u \in [H^{k,\infty}(\Omega)]^N$ to the system

$$
\begin{gathered}
\sum_{|\alpha|\leq k} (-1)^{|\alpha|} D^\alpha (a^i_\alpha(x, \gamma(u))) = \sum_{|\alpha|\leq k} (-1)^{|\alpha|} D^\alpha f^i_\alpha, \\
i = 1, \dots, N; \quad x \in \Omega, \quad k \geq 1, \\
\gamma(u) = \{D^\alpha u^i : i = 1, \dots, N; \quad |\alpha| \leq k\}.
\end{gathered}
\tag{0.1}
$$

By a weak solution of (0.1) we mean a function $u \in [H^k(\Omega)]^N$ ($H^k(\Omega) \equiv H^{k,2}(\Omega)$ — Sobolev space, $u = (u^1, \dots, u^N)$ — see [4]) such that

$$
\sum_{i=1}^N \sum_{|\alpha|\leq k} \int_\Omega a^i_\alpha(x, \gamma(u)) D^\alpha \varphi^i \, dx = \sum_{i=1}^N \sum_{|\alpha|\leq k} f^i_\alpha D^\alpha \varphi^i \, dx, \quad \varphi \in [\mathcal{D}(\Omega)]^N. \tag{0.2}
$$

The authors are indebted to Professor J. Nečas for the very valuable advice concerning the paper

We shall suppose that:

$$a^i_\alpha \in C^1(\overline{\Omega} \times \mathbb{R}^{N\varrho(n,k)}), \quad i = 1, \dots, N, \quad |\alpha| \le k, \tag{0.3}$$

$$\sum_{i,j=1}^{N} \sum_{|\alpha|=|\beta|=k} \frac{\partial a^i_\alpha}{\partial \eta^j_\beta}(x,\eta)\xi^i_\alpha \quad \xi^j_\beta > 0, \tag{0.4}$$

$$(x,\eta) \in (\overline{\Omega} \times \mathbb{R}^{N\varrho(n,k)}), \quad \xi \in \mathbb{R}^{N\sigma(n,k)}, \quad \xi \ne 0.$$

$$f^i_\alpha \in H^{1,p_\alpha}(\Omega), \quad p_\alpha = \frac{p}{k - |\alpha| + 1}, \quad \text{where} \quad p > n, \tag{0.5}$$

$$p \ge 2(k+1), \quad i = 1, \dots, N, \quad |\alpha| \le k.$$

Let us denote for $M > 0, G > 0$:

$$\begin{aligned} [M] =& \{u \in [H^{k,\infty}(\Omega)]^N : u \text{ solves (0.1) and } \|u\|_{[H^{k,\infty}(\Omega)]^N} \le M\}, \\ [G] =& \{f^i_\alpha \in H^{1,p_\alpha}(\Omega) : \sum_{i=1}^{N} \sum_{|\alpha| \le k} \|f^i_\alpha\|_{H^{1,p_\alpha}(\Omega)} \le G\}. \end{aligned}$$

We shall use the notations

$$\begin{aligned} \gamma_1(u) =& \{D^\alpha u^i : i = 1, \dots, N; |\alpha| \le k-1\}, \\ \gamma_2(u) =& \{D^\alpha u^i : i = 1, \dots, N; |\alpha| = k\}, \\ P^N_m =& \{(P_1, \dots, P_N) : P_i \text{ — polynomial with } \deg(P_i) \le m, \\ & i = 1, \dots, N\}, \quad m \ge 0, \\ B(x^0, R) =& \{x \in \mathbb{R}^n : |x - x^0| < R\}. \end{aligned}$$

Definition 0.6. The condition (L) of the Liouville type is satisfied for the system (0.1) if for $\forall\, x^0 \in \Omega$, $\xi \in \mathbb{R}^{N\varrho(n,k-1)}$, the only weak solutions in $\mathbb{R}^n$ to the system

$$\sum_{|\alpha|=k} (-1)^{|\alpha|} D^\alpha(a^i_\alpha(x^0, \xi, \gamma_2(v))) = 0, \quad i = 1, \dots, N \tag{0.7}$$

with bounded derivatives of k-th order are polynomials of at most k-th degree (i.e. $v \in P^N_k$).

Definition 0.8. We say that the system (0.1) has a property of regularity (R) if for $\forall\, x^0 \in \Omega$, $\xi \in \mathbb{R}^{N\varrho(n,k-1)}$, $M > 0$ there exist $\eta > 0, c > 0$ and $\mu \in (0,1)$ such that every weak solution u (in $\mathbb{R}^n$) of the system (0.7) with $|D^\alpha u^i| \le M$, $|\alpha| = k$, $i = 1, \dots, N$, belongs to the space $[C^{k,\mu}(\overline{B(0,\eta)})]^N$ and

$$\|u\|_{[C^{k,\mu}(\overline{B(0,\eta)})]^N} \le c.$$

It will be proved in this paper that the property (L) implies the interior regularity, i.e. if u is a weak solution to (0.1) then $u \in [C^{k,\mu}(\overline{\Omega}')]^N$, where $\Omega' \subset\subset \Omega, \mu \in (0, 1-\frac{n}{p})$.

It will be also shown that $(R) \Rightarrow (L)$.

These results generalize the results of [2], [3]. In [2], [3], the analogous assertions are proved for nonlinear elliptic systems of the second order. The history of the regularity problem and the Liouville's property is described in [2], [3], [5].

1. Some lemmas.

By standard arguments (see [6]), we could prove

Lemma 1.1. *Let $u \in [H^{k,\infty}(\Omega)]^N$ be a weak solution to the system* (0.1) *and let* (0.3), (0.4), (0.5) *be satisfied. Then $u \in [H^{k+1}_{\mathrm{loc}}(\Omega)]^N$.*

In our next considerations, we shall use the result from [1]. This result concerns the solutions $u \in [H^k(\Omega)]^N \cap [H^{k-1,\infty}(\Omega)]^N$ of quasilinear elliptic systems of the type

$$\sum_{j=1}^N \sum_{\substack{|\alpha|\le k\\ |\beta|=k}} (-1)^{|\alpha|} D^\alpha (A^{i\,j}_{\alpha\beta}(x,\gamma_1(v)) D^\beta v^j) = \sum_{|\alpha|\le k} (-1)^{|\alpha|} D^\alpha g^i_\alpha, \tag{1.2}$$
$$x \in \Omega, \quad i = 1, \dots, N,$$

with the following assumptions

$$A^{i\,j}_{\alpha\beta} \in C(\overline{\Omega} \times \mathbb{R}^{N\varrho(n,k-1)}), \tag{1.3}$$

$$g^i_\alpha \in L^{p_\alpha}(\Omega), \quad p_\alpha = \frac{p}{k-|\alpha|+1}, \quad p > n, \quad p \ge 2(k+1), \tag{1.4}$$

$$\sum_{i,j=1}^N \sum_{|\alpha|=|\beta|=k} A^{i\,j}_{\alpha\beta}(x,\xi)\, \zeta^i_\alpha\, \zeta^j_\beta > 0, \tag{1.5}$$

for all $x \in \overline{\Omega}$, $\xi \in \mathbb{R}^{N\varrho(n,k-1)}$, $\zeta \in \mathbb{R}^{N\sigma(n,k)}$, $\zeta \neq 0$.

If we denote

$$\begin{aligned} [M'] =& \{u \in [H^k(\Omega)]^N \cap [H^{k-1,\infty}(\Omega)]^N : u \text{ is a solution to (1.2)} \\ & \qquad \text{and } u_{[H^{k-1,\infty}(\Omega)]^N} \le M'\}, \quad M' > 0, \\ [G'] =& \{g^i_\alpha \in L^{p_\alpha}(\Omega) : \sum_{i=1}^N \sum_{|\alpha|\le k} \|g^i_\alpha\|_{L^{p_\alpha}(\Omega)} \le G'\}, \quad G' > 0, \\ A' =& \sup_{\substack{|\xi|\le M'\\ x\in\Omega}} \{\sum_{i,j,\alpha,\beta} |A^{\alpha\beta}_{i\,j}(x,\xi)|\}, \\ U(x^0,R) =& R^{-n} \int_{B(x^0,R)} (\sum_{i=1}^N \sum_{|\alpha|=k-1} |D^\alpha u^i(x) - (D^\alpha u^i)_{x^0,R}|^2)\, dx, \end{aligned}$$

where $u \in [H^{k-1}(\Omega)]^N$, $(D^\alpha u^i)_{x^0,R}$ means the integral mean value $D^\alpha u^i$ in $B(x^0,R)$, we shall state the following

Lemma 1.6. *Suppose that $u \in [M']$ and the right-hand sides of the system* 1.2 *belong to $[G']$. Let* (1.3), (1.4), (1.5) *be satisfied. Let Ω' be a domain such that $\Omega' \subset\subset \Omega$. Let*

$$\lim_{R\to 0^+} \inf U(x,R) = 0 \tag{1.7}$$

uniformly with respect to $x \in \overline{\Omega}', u \in [M']$. Then $u \in [C^{k-1,\mu}(\overline{\Omega}')]^N, \mu \in (0, 1-\frac{n}{p})$ and the a-priori estimate

$$\|u\|_{[C^{k-1,\mu}(\overline{\Omega}')]^N} \le c(M', G', A', \overline{\Omega}'), \quad c > 0,$$

holds uniformly with respect to the class $[M'] \cup [G']$.

This lemma (in a slightly generalized form) is proved in [1]. In [1], the problems analogous to those in this paper are solved for quasilinear elliptic systems of higher order. In both papers, the methods of proofs are based on the same idea. The crucial point is to show that the Liouville property implies the assumption (1.7). But the methods are technically different.

2. Main results.

Theorem 2.1. *Let $u \in [M]$ and the right-hand sides of the system* (0.1) *belong to $[G]$. Let Ω' be a domain such that $\Omega' \subset\subset \Omega$. Suppose that* (0.3), (0.4), (0.5) *and the condition* (L) *be satisfied. Then there exists a constant $c = c(\Omega', M, G)$ such that*

$$\|u\|_{[C^{k,\mu}(\overline{\Omega}')]^N} \le c, \quad \mu \in (0, 1 - \frac{n}{p}).$$

Proof: For all $x^0 \in \overline{\Omega}'$ and $R > 0$, we define the transformation $T_{x^0R} : y = T_{x^0R}(x) = \frac{x-x^0}{R}$. For $u \in [M]$ $(u = (u^1, \dots, u^N))$, we define on $O_{x^0R} = T_{x^0R}(\Omega)$:

$$u^i_{x^0R}(y) = \frac{u^i(x^0 + Ry)}{R^k} - \sum_{|\gamma|<k} \frac{D^\gamma u^i(x^0)}{R^{k-|\gamma|}\gamma\,!}\, y^\gamma , \tag{2.2}$$
$$i = 1, \dots, N.$$

From (2.2), it follows that for $i = 1, \dots, N$

$$\tag{2.3} \begin{cases} D^\alpha u^i_{x^0R}(0) = 0, \quad |\alpha| \le k-1, \\ D^\alpha u^i(x^0Ry) = R^{k-|\alpha|} D^\alpha u^i_{x^0R}(y) + \sum\limits_{\substack{|\gamma|<k \\ \alpha\le\gamma}} R^{|\gamma-\alpha|} B_{\gamma,\alpha} \frac{D^\gamma u^i(x^0)}{\gamma\,!} y^{\gamma-\alpha}, \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad |\alpha| \le k, \\ D^\alpha u^i(x^0 + Ry) = D^\alpha u^i_{x^0R}(y) \quad \text{a.e. in } O_{x^0R},\ |\alpha| = k. \end{cases}$$

$B_{\gamma,\alpha}$ — constants which are related to the derivatives of "y^γ". Let us choose a number $a > 0$. Then there exists $R_0 > 0$ such that for $\forall\, x^0 \in \overline{\Omega}'$ and $0 < R \le R_0$ $B(0,2a) \subset O_{x^0 R}$. From (2.3) it follows that $D^\alpha u^i_{x^0 R}, |\alpha| \le k, i = 1, \dots, N$, are bounded uniformly with respect to $x^0 \in \overline{\Omega}'$ and $0 < R \le R_0$. Clearly there exists a constant $t > 0$ such that for all $x^0 \in \overline{\Omega}'$ and $0 < R \le R_0$

$$\|u_{x^0 R}\|_{[H^k(B(0,2a))]^N} \le t. \tag{2.4}$$

Putting $R^k\, \varphi^i(\frac{x-x^0}{R}) \in \mathcal{D}(\Omega)$ in (0.2) as a test function and using the transformation $x = x^0 + Ry$ we have

(2.5)
$$\sum_{i=1}^{N} \sum_{|\alpha|\le k} \int_{O_{x^0 R}} R^{k-|\alpha|} a^i_\alpha(x^0 + Ry,\, \gamma_1(u(x^0+Ry)), \gamma_2(u(x^0+Ry)))\, D^\alpha \varphi^i(y)\, dy =$$
$$= \sum_{i=1}^{N} \sum_{|\alpha|\le k} \int_{O_{x^0 R}} R^{k-|\alpha|} f^i_\alpha(x^0 + Ry)\, D^\alpha \varphi^i(y)\, dy.$$

From Lemma 1.1 it follows that $u_{x^0 R} \in [H^{k+1}_{\mathrm{loc}}(O_{x^0 R})]^N$ and for $v_{x^0 R} = \frac{\partial u_{x^0 R}}{\partial y_r}$, the following equation in variations holds:

$$\begin{aligned}
&\sum_{i=1}^{N} \sum_{|\alpha|\le k} \int_{O_{x^0 R}} R^{k+1-|\alpha|} \frac{\partial a^i_\alpha}{\partial x_r} \cdot D^\alpha \varphi^i(y)\, dy + \\
&+ \sum_{i,j=1}^{N} \sum_{\substack{|\alpha|\le k\\ |\beta|\le k}} \int_{O_{x^0 R}} R^{k-|\alpha|} \frac{\partial a^i_\alpha}{\partial \eta^j_\beta} \cdot \big[R^{k-|\beta|} D^\beta v^j_{x^0 R}(y) + \\
&+ \sum_{\substack{|\gamma|<k\\ \beta+\lambda_r\le\gamma}} R^{|\gamma-\beta|} B_{\gamma,\beta+\lambda_r} \frac{D^\gamma u(x^0)}{\gamma!} y^{\gamma-\beta-\lambda_r}\big]\, D^\alpha \varphi^i(y)\, dy = \\
&= \sum_{i=1}^{N} \sum_{|\alpha|\le k} \int_{O_{x^0 R}} R^{k+1-|\alpha|} \frac{\partial f^i_\alpha}{\partial x_r}(x^0 + Ry) D^\alpha \varphi^i(y)\, dy.
\end{aligned} \tag{2.6}$$

(We omitted the arguments in $\frac{\partial a^i_\alpha}{\partial x_r}, \frac{\partial a^i_\alpha}{\partial \eta^j_\beta}, \lambda_r = (0, \dots, 0, 1, 0, \dots, 0) \in \mathbb{R}^n$.)

From (0.5) it follows that

$$R^{k+1-|\alpha|} \|\frac{\partial f^i_\alpha}{\partial x_r}(x^0 + Ry)\|_{L^2(B(0,2a))} \le c_1(a) R^{\frac{p-n}{p_\alpha}} \cdot G. \tag{2.7}$$

Now putting in (2.6) $\varphi^i = v^i_{x^0 R}\chi^{2k}$, $i = 1, \dots, N$, where $\chi \in \mathcal{D}(B(0,2a)), 0 \le \chi \le 1, \chi = 1$ on $B(0,a)$ and using the standard argumentation ((0.4), uniform boundedness of $\frac{\partial a^i_\alpha}{\partial x_r}$, $\frac{\partial a^i_\alpha}{\partial \eta^j_\beta}$, Hölder inequality and (2.7)), we obtain the estimates

$$J := \sum_{i=1}^{N} \sum_{|\alpha|=k} \int_{B(0,2a)} (D^\alpha v^i_{x^0R} \cdot \chi^k)^2 \, dy \leq$$

$$\leq c_2 \sum_{i,j=1}^{N} \sum_{|\alpha|=|\beta|=k} \int_{B(0,2a)} \frac{\partial a^i_\alpha}{\partial \eta^j_\beta} (D^\beta v^j_{x^0R} \cdot \chi^k)\,(D^\alpha v^i_{x^0R} \cdot \chi^k) \, dy$$

and $J \leq c_3\, J^{\frac{1}{2}} + c_4$, where $c_3, c_4 > 0$ are some constants.

The last inequality implies that there exists a constant $c_5 > 0$ such that

$$\sum_{i=1}^{N} \sum_{|\alpha|=k} \int_{B(0,a)} (D^\alpha v^i_{x^0R}(y))^2 \, dy \leq J \leq c_5 \tag{2.8}$$

for all $x^0 \in \overline{\Omega}', u \in [M], R \in (0, R_0]$. (2.4) and (2.8) imply that there exists a constant c_6 such that

$$\|u_{x^0R}\|_{[H^{k+1}(B(0,a))]^N} \leq c_6 \tag{2.9}$$

for all $x^0 \in \overline{\Omega}', u \in [M]$ and $R \in (0, R_0]$.

Now we shall prove that

$$\liminf_{R\to 0^+} \Big(\sum_{i=1}^{N} \sum_{|\alpha|=k} R^{-n} \int_{B(x^0,R)} |D^\alpha u^i(x) - (D^\alpha u^i)_{x^0,R}|^2 \, dx \Big) = 0 \tag{2.10}$$

holds uniformly with respect to $x^0 \in \overline{\Omega}'$ and $u \in [M]$. Let us suppose the contrary. Then there exists $\{x^s\}_{s=1}^\infty \subset \overline{\Omega}', x^s \to \overline{x} \in \overline{\Omega}', \{R_s\}_{s=1}^\infty \subset \mathbb{R}^+, R_s \to 0, \{u_s\}_{s=1}^\infty \subset [M]$ and $\varepsilon > 0$ such that

$$\sum_{i=1}^{N} \sum_{|\alpha|=k} R_s^{-n} \int_{B(x^s,R_s)} |D^\alpha u^i_s(x) - (D^\alpha u^i_s)_{x^s,R_s}|^2 \, dx \geq \varepsilon. \tag{2.11}$$

Putting $a = m, m \in \mathbb{N}$ ($\mathbb{N} = \{1, 2, \dots\}$) and using (2.9) and the diagonalization process, we obtain a function $P \in [H^{k+1}(B(0,m))]^N$ such that for all $m \in \mathbb{N}$:

$$u_{sx^sR_s} \to P \text{ in } [H^{k+1}(B(0,m))]^N \text{ weakly,} \tag{2.12}$$

$$u_{sx^sR_s} \to P \text{ in } [H^k(B(0,m))]^N, \tag{2.13}$$

$$D^\alpha u^i_{sx^sR_s} \to D^\alpha P^i \text{ a.e. in } B(0,m), \quad |\alpha| \leq k, \quad i = 1, \dots, N. \tag{2.14}$$

(2.3), (2.14) imply that there exists a constant $\tau > 0$ such that

$$|D^\alpha P^i| \le \tau, \quad |\alpha| = k, \quad i = 1, \dots, N. \tag{2.15}$$

Now let $\psi \in [\mathcal{D}(\mathbb{R}^n)]^N$. It is clear that there exist m, R_1 such that $\operatorname{supp}\psi \subset B(0,m) \subset O_{x^0 R}$ for all $x^0 \in \overline{\Omega}'$ and $0 < R \le R_1$.

Putting $\varphi = \psi$ in (2.5), we have

(2.16)
$$\sum_{i=1}^N \sum_{|\alpha|\le k} \int_{B(0,m)} R_s^{k-|\alpha|} a_\alpha^i(x^s + R_s y, \gamma_1(u_s(x^s + R_s y)), \gamma_2(u_s(x^s + R_s y)))\cdot$$
$$\cdot D^\alpha \psi^i(y)\, dy =$$
$$= \sum_{i=1}^N \sum_{|\alpha|\le k} \int_{B(0,m)} R_s^{k-|\alpha|}\ {}^s f_\alpha^i(x^s + R_s y) D^\alpha \psi^i(y)\, dy.$$

For $|\alpha| < k$, $i = 1, \dots, N$

$$|R_s^{k-|\alpha|} a_\alpha^i D^\alpha \psi^i| \le c_7\ R_s^{k-|\alpha|} \to 0, \quad \text{if} \quad s \to \infty.$$

Using the Lebesgue's dominated convergence theorem, we have

$$\lim_{s\to\infty} \int_{B(0,m)} R_s^{k-|\alpha|} a_\alpha^i D^\alpha \psi^i\, dy = 0. \tag{2.17}$$

From the imbedding $H^{k,\infty}(\Omega) \hookrightarrow H^{k,p}(\Omega) \hookrightarrow\hookrightarrow C^{k-1}(\overline{\Omega})$ it follows that

$$\gamma_1(u_s(x^s + R_s y)) \to \gamma_1(P(\overline{x})), \quad \text{in} \quad B(0,m), s \to \infty,$$
$$\gamma_2(u_s(x^s + R_s y)) = \gamma_2(u_{s x^s R_s}(y)) \to \gamma_2(P(y)) \text{ a.e. in } \ B(0,m), \ \ s \to \infty.$$

Then for $i = 1, \dots, N$, $|\alpha| < k$, the continuity of a_α^i and Lebesgue's theorem imply

(2.18)
$$\lim_{s\to\infty} \int_{B(0,m)} a_\alpha^i(x^s + R_s y, \gamma_1(u_s(x^s + R_s y)), \gamma_2(u_s(x^s + R_s y)))\cdot$$
$$\cdot D^\alpha \psi^i(y)\, dy =$$
$$= \int_{B(0,m)} a_\alpha^i(\overline{x}, \gamma_1(P(\overline{x})), \gamma_2(P(y))) D^\alpha \psi^i(y)\, dy.$$

Now for $i = 1, \dots, N$, if $|\alpha| = k$, then $p_\alpha = p$ and $H^{1,p}(\Omega) \hookrightarrow\hookrightarrow C(\overline{\Omega})$. Using this fact and Lebesgue's theorem, we have ($k \ge 1$)

$$\lim_{s\to\infty} \int_{B(0,m)} {}^s f_\alpha^i(x^s + R_s y) D^\alpha \psi^i(y)\, dy = f_\alpha^i(\overline{x}) \int_{B(0,m)} D^\alpha \psi^i(y)\, dy = 0. \tag{2.19}$$

If $|\alpha| < k, p_\alpha > n$ then using the same argument we have

$$\lim_{s\to\infty} \int_{B(0,m)} R_s^{k-|\alpha|}\ {}^s\!f_\alpha^i(x^s + R_s y) D^\alpha \psi^i(y)\, dy = 0. \tag{2.20}$$

If $|\alpha| < k, p_\alpha = n$, then $H^{1,p_\alpha}(\Omega) \hookrightarrow L^q(\Omega)$ for all $q \geq 1$.

Choosing q so that $k - |\alpha| - \frac{n}{q} > 0$ and using Hölder inequality, we obtain

$$\begin{aligned} &\Big|\int_{B(0,m)} R_s^{k-|\alpha|}\ {}^s\!f_\alpha^i(x^s + R_s y) D^\alpha \psi^i(y)\, dy\Big| \leq c_8 R_s^{k-|\alpha|-\frac{n}{q}} \|\ {}^s\!f_\alpha^i\|_{L^q(\Omega)} \leq \\ &\qquad \leq c_9\, R_s^{k-|\alpha|-\frac{n}{q}} \cdot G \to 0 \quad \text{for} \quad s \to \infty. \end{aligned} \tag{2.21}$$

If $|\alpha| < k, p_\alpha < n$, then $H^{1,p_\alpha}(\Omega) \hookrightarrow L^{q_\alpha}(\Omega)$, where $\frac{1}{q_\alpha} = \frac{1}{p_\alpha} - \frac{1}{n}$.

Using Hölder inequality, we have

$$\begin{aligned} &\Big|\int_{B(0,m)} R_s^{k-|\alpha|}\ {}^s\!f_\alpha^i(x^s + R_s y) D^\alpha \psi^i(y)\, dy\Big| \leq \\ &\leq c_{10} R_s^{(k-|\alpha|+1)(1-\frac{n}{p})} \cdot \|\ {}^s\!f_\alpha^i\|_{L^{q_\alpha}(\Omega)} \leq \\ &\leq c_{11} R_s^{(k-|\alpha|+1)(1-\frac{n}{p})} \cdot G \to 0 \quad \text{for} \quad s \to \infty. \end{aligned} \tag{2.22}$$

These facts imply that

$$\sum_{i=1}^{N} \sum_{|\alpha|=k} \int_{\mathbb{R}^n} a_\alpha^i(\overline{x}, \gamma_1(P(\overline{x})), \gamma_2(P(y))) D^\alpha \psi^i(y)\, dy = 0$$
$$\text{for all} \quad \psi \in [\mathcal{D}(\mathbb{R}^n)]^N.$$

From (2.15) and the condition (L) it follows that $P \in P_k^N$. (2.3), (2.11) and (2.13) imply

$$\begin{aligned} \varepsilon &\leq \liminf_{s\to\infty} R_s^{-n} \sum_{i=1}^{N} \sum_{|\alpha|=k} \int_{B(x^s,R_s)} |D^\alpha u_s^i(x) - (D^\alpha u_s^i)_{x^s,R_s}|^2\, dx \leq \\ &\leq \liminf_{s\to\infty} \sum_{i=1}^{N} \sum_{|\alpha|=k} \int_{B(0,1)} |D^\alpha u_{s x^s R_s}(y) - D^\alpha P^i|^2\, dy = 0. \end{aligned}$$

From this it follows that (2.10) holds uniformly with respect to $x^0 \in \overline{\Omega}'$ and $u \in [M]$.

Now from (0.2), we obtain an equation in variations which has the following form

$$
\begin{aligned}
&\sum_{i,j=1}^{N}\sum_{s,t=1}^{n}\sum_{\substack{|\alpha|\le k\\|\beta|=k}}\int_\Omega \delta_{st}\frac{\partial a^i_\alpha}{\partial\eta^j_\beta}(x,\gamma(u))D^\beta\Big(\frac{\partial u^j}{\partial x_t}\Big)D^\alpha\varphi^i_s\,dx = \\
&=\sum_{i=1}^{N}\sum_{s=1}^{n}\sum_{|\alpha|\le k}\int_\Omega\Big[\frac{\partial f^i_\alpha}{\partial x_s}-\frac{\partial a^i_\alpha}{\partial x_s}(x,\gamma(u))- \\
&-\sum_{j=1}^{N}\sum_{|\beta|<k}\frac{\partial a^i_\alpha}{\partial\eta^j_\beta}(x,\gamma(u))D^\beta\Big(\frac{\partial u^j}{\partial x_s}\Big)\Big]D^\alpha\varphi^i_s\,dx,
\end{aligned}
\tag{2.23}
$$

where δ_{st} — the symbol of Kronecker delta.

Putting

$$
g^{\,i}_{\substack{s\\\alpha}}=\frac{\partial f^i_\alpha}{\partial x_s}-\frac{\partial a^i_\alpha}{\partial x_s}(x,\gamma(u))-\sum_{j=1}^{N}\sum_{|\beta|<k}\frac{\partial a^i_\alpha}{\partial\eta^j_\beta}(x,\gamma(u))D^\beta\Big(\frac{\partial u^j}{\partial x_s}\Big)
\tag{2.24}
$$

and using the fact that $u\in[M]$, we obtain the assertion:
There exists a constant $G'>0$ such that

$$
\sum_{i=1}^{N}\sum_{s=1}^{n}\sum_{|\alpha|\le k}\|g^{\,i}_{\substack{s\\\alpha}}\|_{L^{p_\alpha}(\Omega)}\le G'.
\tag{2.25}
$$

For $i,j=1,\dots,N$; $s,t=1,\dots,n$; $|\alpha|\le k$, $|\beta|=k$, we define

$$
A^{\,ij}_{\substack{s\,t\\\alpha\beta}}(x,\kappa)=\delta_{st}\frac{\partial a^i_\alpha}{\partial\eta^j_\beta}(x,\gamma_1(u),\overline{\kappa}),
$$

$x\in\overline{\Omega}, \kappa\in\mathbb{R}^{Nn\varrho(n,k-1)}, \overline{\kappa}\in\mathbb{R}^{N\sigma(n,k)}$.

It is clear that ($u\in[C^{k-1}(\overline{\Omega})]^N$)

$$
A^{\,ij}_{\substack{s\,t\\\alpha\beta}}\in C(\overline{\Omega}\times\mathbb{R}^{Nn\varrho(n,k-1)}).
\tag{2.26}
$$

Putting $U^j_t=\frac{\partial u^j}{\partial x_t}$ for $j=1,\dots,N$, $t=1,\dots,n$ in (2.23), we obtain a quasilinear system

$$
\begin{aligned}
&\sum_{i,j=1}^{N}\sum_{s,t=1}^{n}\sum_{\substack{|\alpha|\le k\\|\beta|=k}}\int_\Omega A^{\,ij}_{\substack{s\,t\\\alpha\beta}}(x,\gamma_1(U))D^\beta U^j_t D^\alpha\varphi^i_s\,dx = \\
&=\sum_{i=1}^{N}\sum_{s=1}^{n}\sum_{|\alpha|\le k}\int_\Omega g^{\,i}_{\substack{s\\\alpha}}D^\alpha\varphi^i_s\,dx.
\end{aligned}
\tag{2.27}
$$

It is a matter of routine calculation to verify the assumptions of Lemma 1.6 for the system (2.27). (These assumptions hold uniformly with respect to all $u \in [M]$.)

Now from Lemma 1.6 it follows that there exists a constant $c > 0$ such that for $u \in [M], u \in [C^{k,\mu}(\overline{\Omega}')]^N$ and $\|u\|_{C^{k,\mu}(\overline{\Omega}')]^N} \leq c$. □

By the standard method from [2], we shall prove

Theorem 2.28. *Suppose that the system* (0.1) *has the property of regularity* (R)*. Then Liouville's property* (L) *holds.*

Proof: Let $x^0 \in \Omega, \xi \in \mathbb{R}^{N\varrho(n,k-1)}$ and u be a solution (in $\mathbb{R}^n$) to the system

$$\sum_{i=1}^{N} \sum_{|\alpha|=k} \int_{\mathbb{R}^n} a_\alpha^i(x^0, \xi, \gamma_2(u(x))) D^\alpha \varphi^i(x)\, dx = 0, \qquad \varphi \in [\mathcal{D}(\mathbb{R}^n)]^N, \tag{2.29}$$

such that for $M > 0$

$$|D^\alpha u^i| \leq M, \quad |\alpha| = k, \quad i = 1, \dots, N. \tag{2.30}$$

For $R > 0$, we define

$$u_R^i(y) = \frac{u^i(Ry)}{R^k}, \quad i = 1, \dots, N.$$

Putting $\varphi(\frac{x}{R})$ as a test function in (2.29) and using transformation $x = Ry$, we obtain

$$\sum_{i=1}^{N} \sum_{|\alpha|=k} \int_{\mathbb{R}^n} a_\alpha^i(x^0, \xi, \gamma_2(u_R(y))) D^\alpha \varphi^i(y)\, dy = 0.$$

(2.30) and the property (R) imply

$$|D^\alpha u_R^i(y) - D^\alpha u_R^i(0)| \leq c|y|^\mu, \tag{2.31}$$

$|\alpha| = k$, $i = 1, \dots, N$, $R > 0$, $y \in \overline{B(0,\eta)}$, $\eta > 0$, $\mu \in (0,1)$.

Now let us choose $x \in \mathbb{R}^n$. Then there exists $R_0 > 0$ such that $y_R = \frac{x}{R} \in \overline{B(0,\eta)}$ for all $R \geq R_0$. Using (2.31), we obtain

$$|D^\alpha u^i(x) - D^\alpha u^i(0)| \leq c\,\frac{|x|^\mu}{R^\mu}, \ |\alpha| = k, \ R \geq R_0, \qquad i = 1, \dots, N.$$

For R tending to infinity, we have that $u \in P_k^N$. □

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(*Received* May 31, 1991)