Making factorizations compositive

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Abstract. The main aim of this paper is to obtain compositive cone factorizations from non-compositive ones by itereration. This is possible if and only if certain colimits of (possibly large) chains exist. In particular, we show that (strong-epi, mono) factorizations of cones exist if and only if joint coequalizers and colimits of chains of regular epimorphisms exist.

Keywords: (locally) orthogonal $\mathcal{E}$-factorization, (local) factorization class, colimit of a chain, cointersection, regular epimorphism, joint coequalizer, (familially) strong epimorphism, decomposition number

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Introduction.

Factorizations of morphisms have been investigated for a long time (see e.g. [11]); later several authors became interested in factorizations of sources — or cones (cf. [2], [6], [8], [9], [14], [17], [18], [21], [22]). The main examples are decompositions $f = me$ of a morphism f, where either e is epic and m is strongly monic, or e is strongly epic and m is monic. If one tries to find a "canonical" decomposition with e regularly epic, some problems arise, due to the fact that regular epimorphisms are in general not closed under composition. Other examples are monotone-light factorizations for certain notions of connectedness, where the monotone morphisms need not compose. Such factorizations were first investigated by Ehrbar and Wyler (cf. [5]) and later by Tholen [20] for morphisms; generalizations to sources have been considered in [1], [2], and [21]. In Section 1 of the present paper we repeat the basic theory. Moreover, we show that a reasonable class $\mathcal{E}$ of morphisms is *right cancellable* (i.e. $ee' \in \mathcal{E}$ implies $e \in \mathcal{E}$ if $\mathcal{E}$ is closed under composition with split-epimorphisms).

Our main question is when orthogonal (="composable") factorizations can be obtained from locally orthogonal (="not necessarily composable") ones by iteration. This was already studied in [16]; we give a more systematic treatment in Section 2.

In order to handle large chains, we work in a model $\mathcal{S}$ of Zermelo–Fraenkel set theory containing a universe $\mathcal{U}$. The elements of $\mathcal{S}$ are called *conglomerates*, and the elements of $\mathcal{U}$ are called *sets*. The sub-conglomerates of $\mathcal{U}$ are called *classes*. The cardinality of $\mathcal{U}$ is called ∞. A *hyperordinal* is an ordinal $\alpha < \infty^+$ in $\mathcal{S}$, i.e. an ordinal of cardinality $\leq \infty$. The name *"ordinal"* is reserved for ordinals in $\mathcal{U}$, i.e. hyperordinals $< \infty$.

Since every hyperordinal is in bijection with a class, we can use well-ordered classes rather than hyperordinals. Then all results can be re-formulated in *von Neumann–Gödel–Bernays set-theory*, which is weaker than $ZF + \{\exists \text{ universe}\}$. We stress this fact, but we shall not give this formulation because it is technically more complicated.

For any category, we require that it has $\leq \infty$ objects and small hom-sets. "Small" always means "of cardinality $< \infty$", i.e. "in bijection with a set".

In Section 3 we apply the framework to the class of regular epimorphisms. Under "reasonable" conditions, the composites of chains of regular epimorphisms are just strong epimorphisms. An example due to MacDonald and Stone [15] shows that it does not suffice to consider only colimits of chains indexed by an ordinal or the class of all ordinals; a simple generalization shows that the length of the chain cannot even be bounded by a fixed hyperordinal. Extending a notion given by Gabriel and Ulmer [7], we define the *decomposition number* $\delta(\mathcal{A})$ of a "reasonable" category $\mathcal{A}$. The generalized MacDonald–Stone example shows that indeed every hyperordinal appears as $\delta(\mathcal{A})$ for some $\mathcal{A}$. Note that there may be proper chains of regular epimorphisms of length $> \delta(\mathcal{A})$, but the composite of such a chain can always be represented as a composite of a shorter chain.

1. Local factorization classes.

1.1 (1) We start by recalling some basic facts about locally orthogonal factorizations. By a *source factorization* in a category $\mathcal{A}$ we mean a pair $(e, (m_i)_{i\in I})$, where I is some class and e, m_i are $\mathcal{A}$-morphisms such that the domains of all m_i coincide with the codomain of e. If e is an identity morphism 1_A, we call $(1_A, (m_i)_i)$ a *source* and usually abbreviate it by $(m_i)_i$, keeping in mind that A has to be given additionally in case $I = \emptyset$. We say that $(e, (m_i)_{i\in I})$ is a *factorization of a source* $(f_i)_{i\in I}$ if $m_i e_i = f_i$ for all $i \in I$.

An $\mathcal{A}$-morphism p is called *orthogonal* to a source factorization if, for all morphisms g, h_i $(i \in I)$ with $m_i eg = h_i p$ $(i \in I)$, there is a unique morphism t with $tp = eg$ and $m_i t = h_i$ $(i \in I)$. In this case we write $p \perp (e, (m_i))$. For a class $\mathcal{E}$ of $\mathcal{A}$-morphisms, we call a source factorization $(e, (m_i))$ an *$\mathcal{E}$-factorization* if $e \in \mathcal{E}$. This $\mathcal{E}$-factorization is called *locally orthogonal* if $p \perp (e, (m_i)_i)$ for all $p \in \mathcal{E}$; it is called *orthogonal* if $p \perp (1, (m_i))$ for all $p \in \mathcal{E}$. $\mathcal{E}$ is called a (*local*) *factorization class* if $\mathcal{E}$ is closed under composition with isomorphisms, and if every source in $\mathcal{A}$ admits a (locally) orthogonal $\mathcal{E}$-factorization. If (locally) orthogonal $\mathcal{E}$-factorizations exist just for morphisms, i.e. for source indexed by singletons, we call $\mathcal{E}$ a (*local*) *factorization class for morphisms*.

(2) From [21] we recall the basic properties of locally orthogonal factorizations. First, any orthogonal $\mathcal{E}$-factorization is locally orthogonal. $\mathcal{E}$ is a factorization class (for morphisms) if and only if $\mathcal{E}$ is both a locally orthogonal factorization class (for morphisms) and closed under composition. A local factorization class consists only of epimorphisms; this is no longer true for local factorization classes for morphisms. Indeed, even the class of all $\mathcal{A}$-morphisms is a factorization class for morphisms.

$\mathcal{E}$ is a local factorization class if and only if the following two conditions hold:

(i) The *pushout*

$$\begin{tikzcd} \cdot \arrow[r, "f"] \arrow[d, "e"'] & \cdot \arrow[d, "e'"] \\ \cdot \arrow[r, "f'"'] & \cdot \end{tikzcd}$$

always exists for $e \in \mathcal{E}$, and $e' \in \mathcal{E}$ holds.

(ii) For any (possibly large or empty) source (e_i) with all $e_i \in \mathcal{E}$ the *cointersection* (=generalized pushout)

$$\begin{tikzcd} \cdot \arrow[d, "e_i"'] \arrow[dr, "d"] & \\ \cdot \arrow[r, "p_i"'] & \cdot \end{tikzcd}$$

exists, and $d \in \mathcal{E}$ holds.

If $\mathcal{E}$ is a local factorization class for morphisms, the above colimits need not exist, but in case of existence we can conclude $e' \in \mathcal{E}$, and $d \in \mathcal{E}$.

A locally orthogonal $\mathcal{E}$-factorization $(e, (m_i)_{i\in I})$ is always *rigid* in the sense that for any t with $te = e$ and $m_i t = m_i$ for all $i \in I$ it follows that $t = 1$. From this one easily concludes that a locally orthogonal $\mathcal{E}$-factorization of a given source is unique up to a canonical isomorphism.

We are now interested in cancellation properties of local factorization classes. We say that a class $\mathcal{E}$ of morphisms is *right cancellable* if $ee' \in \mathcal{E}$ always implies $e \in \mathcal{E}$.

Proposition 1.2. *The following assertions hold whenever $\mathcal{E}$ is a local factorization class for morphisms:*

(i) *If $ee' \in \mathcal{E}$, then there are $e'' \in \mathcal{E}$ and a split-epimorphism q with $qe'' = e$.*
(ii) *If $qe \in \mathcal{E}$ for all $e \in \mathcal{E}$ and all split-epimorphisms q, then $\mathcal{E}$ is right cancellable.*
(iii) *If $ee' \in \mathcal{E}$ and if e' belongs to $\mathcal{E}$ or is epic, then $e \in \mathcal{E}$.*

Proof: (i) Consider a locally orthogonal $\mathcal{E}$-factorization (e'', q) of e. Now $ee' \perp (e'', q)$ yields a unique l with $lee' = e''e'$ and $ql = 1$; thus q is split-epic.

(ii) is an immediate consequence of (i).

(iii): Start as in (i). If $e' \in \mathcal{E}$, we can apply the uniqueness part of $e' \perp (e'', q)$ to conclude $le = e''$ (cf. [20]). If e' is epic, the latter equation follows immediately from $lee' = e''e'$. Now rigidity of (q, e'') yields $lq = 1$. Hence q is an isomorphism and $e = qe'' \in \mathcal{E}$. □

Note that in (i) we even obtained a "canonical" decomposition in the sense that (e'', q) is orthogonal. But this can always be achieved, if an $\mathcal{E}$-(split-epic) factorization exists at all, by the following

Proposition 1.3. *Let $\mathcal{E}$ be a class of morphisms closed under composition with isomorphisms and let (e', q') be some locally orthogonal $\mathcal{E}$-factorization of a morphism qe with $e \in \mathcal{E}$ and q split-epic. Then q' is also split-epic.*

Proof: Since q is split-epic, there exists some s with $qs = 1$. From $e \perp (e', q')$ we get a unique t with $te = e'$ and $q't = q$, hence $q'ts = qs = 1$. Thus q' is split-epic. □

2. Iterated factorizations.

If $(e, (m_i))$ is a locally orthogonal $\mathcal{E}$-factorization of some source (f_i) for some local factorization class $\mathcal{E}$, we can continue by taking a locally orthogonal $\mathcal{E}$-factorization $(e', (m'_i))$ of (m_i). Since $(e'e, (m'_i))$ is again a factorization of (f_i), we can take a locally orthogonal $\mathcal{E}$-factorization of (m'_i) and iterate the construction as often as possible. We shall see that we can obtain an orthogonal $\mathcal{E}$-factorization for some factorization class $\mathcal{E}'$ by transfinite iteration of this process if and only if certain (possibly large) colimits of chains exist.

For any hyperordinal (see introduction) $\alpha \geq 1$ we consider the category $\mathcal{K}_\alpha$ whose objects are all hyperordinals $< \alpha$ and with exactly one morphism $k_{\mu\nu} : \mu \longrightarrow \nu$ for $\mu \leq \nu$ and no such morphism otherwise. By an *α-chain* in a category $\mathcal{A}$ we mean a functor $\mathcal{K}_\alpha \longrightarrow \mathcal{A}$; we usually write it as a family $(e_{\mu\nu} : A_\mu \longrightarrow A_\nu)_{\mu \leq \nu < \alpha}$ of $\mathcal{A}$-morphisms.

If $\mathcal{E}$ is a class of morphisms containing all isomorphisms and closed under composition with them, we call the above chain an *$\mathcal{E}$-admissible* chain, if $e_{\mu\mu+1} \in \mathcal{E}$ for all μ with $\mu + 1 < \alpha$ and if, moreover, the chain, considered as a functor, preserves colimits of chains. The latter condition means that, for each limit hyperordinal $\lambda < \alpha$, A_λ is the colimit of the λ-chain $(e_{\mu\nu})_{\mu \leq \nu < \lambda}$ with colimit morphisms $e_{\mu\lambda} : A_\mu \longrightarrow A_\lambda$. For any $\alpha < \infty^+$ and any $(\alpha + 1)$-chain $(e_{\mu\nu})$, we call $e_{0\alpha}$ the *composite* of the $(\alpha + 1)$-chain. For $\alpha > 0$ we denote by $\mathcal{E}^\alpha$ the class of all composites of $\mathcal{E}$-admissible $(\alpha + 1)$-chains. We define $\mathcal{E}^0$ to be the class of all isomorphisms. Then we have $\mathcal{E}^1 = \mathcal{E}$ and $\mathcal{E}^\beta \subset \mathcal{E}^\alpha$ for $\beta < \alpha$. If all elements of $\mathcal{E}$ are epic, it follows by transfinite induction that all $\mathcal{E}^\alpha$ consist only of epimorphisms.

If α, β, γ are hyperordinals with $\alpha = \beta + \gamma$ and $\gamma \geq 1$, then for any $\mathcal{E}$-admissible α-chain $(e_{\mu\nu})_{\mu \leq \nu < \alpha}$ the γ-chain $(e_{\beta+\mu\ \beta+\nu})_{\mu \leq \nu < \gamma}$ is also $\mathcal{E}$-admissible. Moreover, a colimit of $(e_{\beta+\mu\ \beta+\nu})_{\mu \leq \nu < \gamma}$ exists if and only if $(e_{\mu\nu})_{\mu \leq \nu < \alpha}$ has a colimit, and in this case both colimits coincide. In particular, if all colimits of $\mathcal{E}$-admissible γ-chains exist, then every $\mathcal{E}$-admissible α-chain admits a colimit.

In particular, for given $\alpha \geq 1$ we can choose the smallest $\gamma \geq 1$ such that $\alpha = \beta + \gamma$ for some hyperordinal β. Then γ is indecomposable [13], i.e. γ admits no decomposition $\gamma = \zeta + \eta$ with $\zeta, \eta < \gamma$, because then $\eta \geq 1$ (since $\zeta < \gamma$) and $\alpha = (\beta + \zeta) + \eta$, $1 \leq \eta < \gamma$, contradicting the choice of γ. The question of whether all colimits of $\mathcal{E}$-admissible α-chains exist is reduced to the case of indecomposable α. Any indecomposable hyperordinal is either $= 1$ or a limit hyperordinal.

Moreover, for $\alpha \geq 1$ we can consider the cofinality type of α, i.e. the smallest β with the property that there exists an unbounded order-preserving map $\varphi : \beta \longrightarrow \alpha$. Then β is always a regular cardinal, and a colimit of an α-chain $(e_{\mu\nu})_{\mu\leq\nu<\alpha}$ exists if and only if the β-chain $(e_{\varphi(\mu)\varphi(\nu)})_{\mu\leq\nu<\beta}$ admits a colimit and in this case both colimits coincide. But if $(e_{\mu\nu})_{\mu\leq\nu<1}$ is $\mathcal{E}$-admissible, we cannot conclude that $(e_{\varphi(\mu)\varphi(\nu)})_{\mu\leq\nu<\beta}$ is $\mathcal{E}$-admissible.

If $\alpha > 1$ is indecomposable and if all colimits of $\mathcal{E}$-admissible γ-chains exist for all $\gamma < \alpha$, then colimits of $\mathcal{E}$-admissible α-chains need not exist. Indeed, in $\mathcal{K}_\alpha$ let $\mathcal{E}$ be the conglomerate of all identities and all $k_{\zeta,\zeta+1}$ for $0 \leq \zeta < \alpha$. Then the identity functor $\mathcal{K}_\alpha \longrightarrow \mathcal{K}_\alpha$ is an $\mathcal{E}$-admissible α-chain without a colimit. On the other hand, if $1 \leq \gamma < \alpha$, then we shall see that every $\mathcal{E}$-admissible γ-chain admits a colimit in $\mathcal{K}_\alpha$. Since the domain β of e_{00} belongs to $\mathcal{K}_\alpha$, we have $\beta < \alpha$. Induction gives that for any $\zeta < \gamma$ the codomain of $e_{0\zeta}$ is $\leq \beta + \zeta$, hence the supremum σ of all those codomains is $\leq \beta + \gamma$. But if we had $\beta + \gamma \geq \alpha$, then for the smallest γ' with $\beta + \gamma' \geq \alpha$ we should get $1 \leq \gamma' \leq \gamma < \alpha$ and $\alpha = \beta + \gamma'$, contradicting indecomposability of α. Thus $\sigma \leq \beta + \gamma < \alpha$, and σ is a colimit of $(e_{\mu\nu})_{\mu\leq\nu<\gamma}$ in $\mathcal{K}_\alpha$.

For any hyperordinal $\alpha \geq 1$ one easily sees that $\alpha\omega$ (i.e. the order type of $\omega \times \alpha$ in lexicographic order) is an indecomposable hyperordinal $> \alpha$. In particular, if colimits of $\mathcal{E}$-admissible chains α-chains exist for all $\alpha \leq \infty$, then colimits of $\mathcal{E}$-admissible chains need not exist. This is the reason why we use hyperordinals.

Theorem 2.2. *For any local factorization class $\mathcal{E}$ and any hyperordinal α the following statements are equivalent:*

(i) *$\mathcal{E}^\alpha$ is local factorization class.*
(ii) *For any limit hyperordinal $\lambda \leq \alpha$, any $\mathcal{E}$-admissible λ-chain admits a colimit.*

Proof: (i)$\Rightarrow$(ii) For an $\mathcal{E}$-admissible λ-chain $(e_{\mu\nu})$, any $e_{0\nu}$ is a composite of the $(\nu+1)$-chain obtained by truncation, therefore $e_{0\nu} \in \mathcal{E}^\nu \subset \mathcal{E}^\lambda \subset \mathcal{E}^\alpha$. Now the local factorization class $\mathcal{E}$ consists of epimorphisms and admits cointersections, so the cointersection of all $e_{0\nu}$ exists, and all $e_{0\nu}$ are epimorphic. Thus the above cointersection is also a colimit of the given chain.

(ii) $\Rightarrow$ (i) Start with an arbitrary source $(f_i)_i$. By transfinite induction we shall construct an $\mathcal{E}$-admissible $(\alpha+1)$-chain $(e_{\mu\nu})$ and arrows $m_{\nu i}$ $(\nu \leq \alpha,\ i \in I)$ with $m_{\nu i}e_{\mu\nu} = m_{\mu i}$ $(\mu \leq \nu \leq \alpha,\ i \in I)$ such that $(e_{0\nu}, (m_{\nu i})_{i\in I})$ is a locally orthogonal $\mathcal{E}^\nu$-factorization of (f_i) for all $\nu \leq \alpha$. Then for $\nu = \alpha$ we get the conclusion.

First, define $e_{00} := 1$ and $m_{0i} := f_i$ for all $i \in I$. Now assume $1 \leq \kappa \leq \alpha$ and let all $e_{\mu\nu}, m_{\nu i}$ be defined for all $\mu \leq \nu < \kappa,\ i \in I$. Then we distinguish two cases:

Case I: $\kappa = \sigma + 1$ for some hyperordinal σ. By induction hypothesis, $(e_{0\sigma}, (m_{\sigma i}))$ is a locally orthogonal $\mathcal{E}^\nu$-factorization. Take a locally orthogonal $\mathcal{E}$-factorization $(e_{\sigma\kappa}, (m_{\kappa i}))$ of $(m_{\sigma i})$ and define $e_{\mu\kappa} := e_{\sigma\kappa}e_{\mu\sigma}$ for $\mu < \sigma$. Now by straightforward computations we see that $(e_{0\kappa}, (m_{\kappa i}))$ is a locally orthogonal $\mathcal{E}^\kappa$-factorization of (f_i).

Case II: κ is a limit hyperordinal. By (ii), the κ-chain $(e_{\mu\nu})_{\mu\leq\nu<\kappa}$ admits a colimit and hence can be extended to an $\mathcal{E}$-admissible $(\kappa+1)$-chain $(e_{\mu\nu})_{\mu\leq\nu\leq\kappa}$, thus

$e_{0\kappa} \in \mathcal{E}^\kappa$. For $i \in I$, the colimit property yields a morphism $m_{\kappa i}$ with $m_{\kappa i} e_{\mu\kappa} = m_{\mu i}$ for all $\mu < \kappa$, in particular $m_{\kappa i} e_{0\kappa} = m_{0i} = f_i$. Therefore $(e_{0\kappa}, (m_{\kappa i}))$ is an $\mathcal{E}^\kappa$-factorization of (f_i), which can easily been seen to be locally orthogonal.

□

2.3 (1) If $f \in \mathcal{E}^\alpha$ for some local factorization class $\mathcal{E}$ and some $\alpha < \infty^+$, then there may be many non-isomorphic representations of f as a composite of an $\mathcal{E}$-admissible α-chain. But the proof of 2.2 gives us the possibility to choose a canonical one, at least up to a unique natural isomorphism. Indeed if we apply the construction to the morphism (i.e. singleton-indexed source) f, the construction gives as an $\mathcal{E}$-admissible $(\alpha+1)$-chain $(e_{\mu\nu})$ and a morphism m_α such that $(m_\alpha, e_{0\alpha})$ is a locally orthogonal $\mathcal{E}^\alpha$-factorization of f which is then canonically isomorphic to $(1, f)$. Hence m_α is invertible, and this gives a canonical representation of f.

(2) Note that 2.2 (ii) is trivially satisfied for $\alpha < \omega$. In particular, we obtain 5.1 of [1] as the special case $\alpha = 2$ of 2.2.

Lemma 2.4. *Let $\mathcal{E}$ be a class of morphisms containing all isomorphisms and closed under composition with them. Then there exists a hyperordinal α with $\mathcal{E}^\beta = \mathcal{E}^\alpha$ for all hyperordinals $\beta \geq \alpha$.*

Proof: Assume the contrary. Then the conglomerate $K := \{\beta < \infty^+ \mid \bigcup\{\mathcal{E}^\alpha \mid \alpha < \beta\} \not\subseteq \mathcal{E}^\beta\}$ cannot have an upper bound $< \infty^+$. Since ∞^+ is a regular cardinal in $\mathcal{U}^+$, K must have cardinality ∞^+. Now $\{\mathcal{E}^\beta \setminus \bigcup\{\mathcal{E}^\alpha \mid \alpha < \beta\} \mid \beta \in K\}$ is a conglomeraste of ∞^+ pairwise disjoint nonempty classes of morphisms. But this is impossible, because the category has cardinality $\leq \infty$. □

Now we show how to extend local factorization classes to factorization classes:

Theorem 2.5. *For a hyperordinal α and a local factorization class $\mathcal{E}$, the following statements are equivalent:*

(i) *$\mathcal{E}^\alpha$ is a factorization class.*
(ii) *$\mathcal{E}^\beta = \mathcal{E}^\alpha$ for all hyperordinals $\beta \geq \alpha$, and every $\mathcal{E}$-admissible chain admits a colimit.*
(iii) *$\mathcal{E}^{\alpha+1} = \mathcal{E}^\alpha$, and any $\mathcal{E}$-admissible λ-chain admits a colimit for any limit hyperordinal $\lambda \leq \alpha$.*

Proof: (i)⇒(ii) For $\beta \geq \alpha$, consider a $(\beta+1)$-chain $(e_{\mu\nu})$. By transfinite induction we obtain $e_{0\nu} \in \mathcal{E}^\alpha$ for all $\nu \leq \beta$. In particular, we have $e_{0\beta} \in \mathcal{E}^\alpha$, proving $\mathcal{E}^\beta \subset \mathcal{E}^\alpha$ and therefore $\mathcal{E}^\beta = \mathcal{E}^\alpha$.

Now consider any $\mathcal{E}$-admissible chain $(e_{\mu\nu})_{\mu \leq \nu < \lambda}$. If λ is not a limit hyperordinal, the colimit of this chain trivially exists. For λ a limit hyperordinal, we already know that $\mathcal{E}^\alpha = \mathcal{E}^\beta$ for $\beta := \max\{\alpha, \lambda\}$, and we conclude the existence of the colimit from 2.2. (ii)⇒(iii) is trivial.

(iii)⇒(i) By 2.2, $\mathcal{E}^\alpha$ is a local factorization class, therefore any given source $(f_i)_i$ admits a locally orthogonal $\mathcal{E}^\alpha$-factorization $(e, (m_i))$. For a locally orthogonal $\mathcal{E}$-factorization $(p, (u_i))$ of (m_i) we have $pe \in \mathcal{E}^{\alpha+1} = \mathcal{E}^\alpha$, hence $pe \perp (e, (m_i))$. Now it follows easily that p is invertible and that $(e, (m_i))$ is an orthogonal $\mathcal{E}^\alpha$-factorization, proving (i). □

Corollary 2.6. *For a local factorization class $\mathcal{E}$, the following statements are equivalent:*

(i) *There exists a factorization class $\mathcal{E}'$ with $\mathcal{E} \subset \mathcal{E}'$.*
(ii) *Every $\mathcal{E}$-admissible chain admits a colimit.*
(iii) *There exists a hyperordinal α such that $\mathcal{E}^\alpha = \mathcal{E}^{\alpha+1}$, and every $\mathcal{E}$-admissible λ-chain admits a colimit for any limit hyperordinal $\lambda \leq \alpha$.*

Proof: (i)$\Rightarrow$(ii) For an $\mathcal{E}$-admissible chain $(e_{\mu\nu})_{\nu<\lambda}$, it follows by transfinite induction that $e_{0\nu} \in \mathcal{E}'$ for all $\nu < \lambda$. If λ is a limit hyperordinal, the colimit of $(e_{\mu\nu})$ can be constructed as a cointersection of all $e_{0\nu}$, $\nu < \lambda$; otherwise the existence of the colimit is trivial.

(ii)$\Rightarrow$(iii) The existence of α with $\mathcal{E}^\alpha = \mathcal{E}^{\alpha+1}$ follows from 2.4. The rest is trivial.

(iii)$\Rightarrow$(i) follows from 2.5 by choosing $\mathcal{E}' := \mathcal{E}^\alpha$. $\square$

2.7 (1) In a cocomplete category, all colimits of α-chains exist for $\alpha < \infty$. Hence, by 2.6, a local factorization class $\mathcal{E}$ can be extended to a factorization class, if $\mathcal{E}^{\alpha+1} = \mathcal{E}^\alpha$ for some $\alpha < \infty$. But α cannot be chosen uniformly, as the following example shows: For a hyperordinal α, the category $\mathcal{K}_{\alpha+1}$ (as in 2.1) admits all (even large) colimits, particularly all colimits of chains. The class $\mathcal{E}$ of all identity morphisms and all $k_{\mu\mu+1}$ (for $\mu < \alpha$) turns out to be a local factorization class. Now $k_{\mu\nu} \in \mathcal{E}^\beta$ is always equivalent to $\nu \leq \mu + \beta$. In particular, we have $k_{0\alpha} \in \mathcal{E}^\alpha$, but $k_{0\alpha} \notin \mathcal{E}^\beta$ for $\beta < \alpha$.

3. Regular and strong epimorphisms.

We want to use the above results to clarify the relationship between *regular* and *strong epimorphisms*. (See [12] for definitions and notice that most results remain valid if we define regular epimorphisms in the narrower sense of being the coequalizer of a pair of morphisms.)

If a regular epimorphism p admits a *kernel pair* (i.e. a pullback of p with itself), then p is the coequalizer of its kernel pair. If a category has kernel pairs and coequalizers of kernal pairs, then the regular epimorphisms form a local factorization class for morphisms. Indeed the decomposition (e, m) of a morphism f can be constructed by taking e to be the *codominion* of f, i.e. the coequalizer of the kernel pair of f (cf. [10]).

Now consider an object A and a family of pairs $x_i, y_i : B_i \longrightarrow A$ of parallel morphisms, I being an arbitrary class. A morphism $p : A \longrightarrow C$ is called a *joint coequalizer* of $(x_i, y_i)_i$ if $px_i = py_i$ for all $i \in I$ and if for any $f : A \longrightarrow D$ with $fx_i = fy_i$ for all $i \in I$ there exists a unique $h : C \longrightarrow D$ with $hp = f$. In particular, a morphism is a *regular epimorphism* if and only if it is a joint coequalizer of some family of pairs (and then also of the family of all pairs (x, y) with $px = py$).

Proposition 3.2. *For any category $\mathcal{A}$, the following statements are equivalent:*

(i) *$\mathcal{A}$ has coequalizers, and the regular epimorphism form a local factorization class in $\mathcal{A}$.*
(ii) *$\mathcal{A}$ has coequalizers and cointersections of regular epimorphisms.*
(iii) *$\mathcal{A}$ has all joint coequalizers.*

Proof: (i)$\Rightarrow$(ii) follows immediately from the characterization of local factorization classes by pushouts and cointersections in 1.1.

(ii)$\Rightarrow$(iii) The joint coequalizer of a family $(x_i, y_i)_{i\in I}$ can be constructed as the cointersection of all coequalizers e_i of (x_i, y_i) $(i \in I)$.

(iii)$\Rightarrow$(i) A locally orthogonal regularly epic factorization $(e, (m_i))$ of a source $(f_i)_{i\in I}$ can be constructed by taking e as the joint coequalizer of all pairs (x, y) with $f_i x = f_i y$ for all $i \in I$. □

3.3 (1) Note that in 2.2(ii) we do not require the existence of pushouts of pairs (e, f) with common domain and e regularly epic. Of course, the existence of such pushouts also follows from (i) and therefore from (ii). One can construct such pushouts directly from (iii):

$$\begin{tikzcd} {} \arrow[r, "e"] \arrow[d, "f"'] & {} \arrow[d, "f'"] \\ {} \arrow[r, "e'"'] & {} \end{tikzcd}$$

just take e' as the joint coequalizer of all pairs (fx, fy) with $ex = ey$.

(2) In 3.2 (i) and (ii) the existence of coequalizers is crucial. Indeed, let $\mathcal{A}$ be a groupoid, i.e. a category with all morphisms invertible and therefore regularly epic. Then even all morphisms form a factorization class, and all cointersections exist in $\mathcal{A}$. However, a pair of distinct parallel morphisms admits no coequalizer; hence 2.2 (iii) does not hold unless $\mathcal{A}$ is equivalent to a discrete category.

(3) Now we try to extend the class of regular epimorphisms to a factorization class. First note that every regular epimorphism p is *familially strong* (cf. [18]), i.e. $p\perp(1, (m_i))$ for any mono-source $(m_i)_{i\in I}$. (Here (m_i) is called a monosource, if $x = y$ holds whenever $m_i x = m_i y$ for all $i \in I$.) In a category with coequalizers, we have some kind of converse: if $p\perp(1, (m_i))$ for every regular epimorphism p, then (m_i) is a mono-source. On the other hand, even in "nice" categories, familially strong epimorphisms may be far from being regular (cf. [11], [12] and see 3.5 below.)

Every familially strong epimorphism is *strong*, and every strong epimorphism p is *extremal* in the sense that $p = me$ with m monic implies that m is an isomorphism.

Theorem 3.4. *In any category with joint coequalizers, the following statements are equivalent:*

(i) *There exists a factorization class, which contains all regular epimorphisms.*
(ii) *The strong epimorphisms form a factorization class.*
(iii) *Every regular-epi-admissible chain admits a colimit.*

If these properties hold, then strong epimorphisms coincide with familially strong and with extremal epimorphisms.

Proof: (i)$\Longrightarrow$(ii) Let $\mathcal{E}$ be a factorization class containing all regular epimorphisms, and let p be an extremal epimorphism. Then for an orthogonal $\mathcal{E}$-factorization (e, m) of p we have $q\perp(1, m)$ for all $q \in \mathcal{E}$, particularly for q regularly epic.

But then m is monic and even an isomorphism by extremality of $p = me$. Therefore we have $p = me \in \mathcal{E}$.

This proves that $\mathcal{E}$ contains all extremal epimorphisms, in particular all familially strong ones. Therefore all pushouts of familially strong epimorphisms along arbitrary morphisms and all cointersections of familially strong epimorphisms exist. Moreover, the class of familially strong epimorphisms is stable under pushouts and cointersections, hence it is a local factorization class. Since it is also closed under composition, it is even a factorization class.

If we replace our original $\mathcal{E}$ by the class of familially strong epimorphisms, the beginning of the proof shows that every extremal epimorphism is familially strong. Therefore extremal, strong, and familially strong epimorphisms coincide, and they form a factorization class.

(ii)$\Rightarrow$(i) is trivial, and (i)$\Rightarrow$(iii) is an immediate consequence of 2.6. □

3.5 (1) From 2.6 and 3.4 we see that, in a category with coequalizers, the strong epimorphisms coincide with composites of chains of regular epimorphisms, provided they form a factorization class. In this case, we define the *decomposition number* $\delta(f)$ of a morphism f as the smallest ordinal α with the property that, for $\mathcal{E}$ the class of regular epimorphisms, the locally orthogonal $\mathcal{E}^{\alpha}$-factorization of f is orthogonal; or, equivalently, coincides with the locally orthogonal $\mathcal{E}^{\alpha+1}$-factorization; or, equivalently, the morphism $e_{\alpha\alpha+1}$ in the proof of 2.2 is an isomorphism. This is a slight modification of the decomposition number (=*Zerlegungszahl*) ζ defined by Gabriel and Ulmer [7] under stronger conditions. Indeed, if $\delta(f) < \infty$ is a limit ordinal, then $\delta(f) = \zeta$; if $\delta(f)$ is a successor ordinal, then $\delta(f) = \zeta + 1$; if $\delta(f) \geq \infty$, then $\zeta = \infty$. Moreover, we define the decomposition number $\delta(\mathcal{A})$ of a category $\mathcal{A}$ to be the supremum of all decomposition numbers of $\mathcal{A}$-morphisms, i.e. the smallest hyperordinal α with $\mathcal{E}^{\alpha+1} = \mathcal{E}^{\alpha}$.

(2) Gabriel and Ulmer [7] prove $\delta(\mathcal{A}) < \infty$ for any locally presentable category $\mathcal{A}$; they even give a better estimate in a more general situation. In any cocomplete category $\mathcal{A}$ which is cowellpowered with respect to strong epimorphisms, one easily sees that $\delta(f) < \infty$ for all f; hence $\delta(\mathcal{A}) \leq \infty$. Then, of course, all chains of regular (even of strong) epimorphisms admit colimits. Examples of cowellpowered categories with $\delta(\mathcal{A}) = \infty$ are the duals of the category of semigroups and of rings [7]. In these categories, all $\mathcal{E}^{\alpha}$ are local factorization classes and $\mathcal{E}^{\alpha} \neq \mathcal{E}^{\beta}$ for $\alpha \neq \beta,\ \alpha, \beta \leq \infty$. Moreover, even a total category (in the sense of [19]) need not admit colimits of ∞-chains of regular epimorphisms [3].

(3) The main difference between our definition of decomposition number and the one given by Gabriel and Ulmer is that we admit hyperordinals rather than just ordinals and ∞. By a modification of an example due to MacDonald and Stone [15], we shall see that, indeed, every hyperordinal appears as a decomposition number of a morphism as well as of a category. For $\alpha < \infty^{+}$, let $\mathcal{C}_{\alpha}$ be the category of objects $(A, (\varphi_{\nu})_{\nu\leq\alpha})$, where A is a set and each φ_{ν} is a partial unary operation on A, defined on the set $\{x \in A \mid \forall \mu < \nu : \varphi_{\mu}(x) = x\}$; in particular, φ_0 is everywhere defined. For $\nu \leq \alpha$, define $A_{\nu} = (\mathbb{N} \cup \{0\}, (\varphi^{\nu}_{\mu})_{\mu\leq\alpha})$ by $\varphi^{\nu}_{0}(n) := n + 1$ for $n \geq 1$, $\varphi^{\nu}_{\mu}(0) := 0$ for $\mu < \nu$ and $\varphi^{\nu}_{\nu}(0) := 1$. Now $(e_{\mu\nu} : A_{\mu} \longrightarrow A_{\nu})_{\mu\leq\nu\leq\alpha}$

with $e_{\nu\nu}(n) := n$ and $e_{\mu\nu}(n) := 0$ for $\mu < \nu$ is a (regular-epi)-admissible chain; hence $e_{0\alpha} \in \mathcal{E}^{\alpha}$ is a strong epimorphism. But for $\nu < \alpha$ we have $e_{0\alpha} \notin \mathcal{E}^{\nu}$, because for $q \in \mathcal{E}^{\nu}$, $q : A_0 \longrightarrow (B, (\psi_\mu)_{\mu \leq \alpha})$, we can conclude that $\psi_{\nu+1}$ (and hence ψ_α) is nowhere defined. Hence, we have $\delta(\mathcal{C}_\alpha) \geq \delta(e_{0\alpha}) = \alpha$. Since we can also prove $\delta(\mathcal{C}_\alpha) \leq \alpha$, we get $\delta(\mathcal{C}_\alpha) = \alpha$.

For a morphism f we have $\delta(f) = 0$ if and only if f is monic. A category $\mathcal{A}$ admits joint coequalizers and has $\delta(\mathcal{A}) = 0$ if and only if for all $A, B \in |\mathcal{A}|$ there is at most one morphism $A \longrightarrow B$.

(4) Finally, we mention some properties of regular epimorphisms. Let $\mathcal{A}$ be a category, in which regular epimorphisms form a local factorization class for morphisms; for instance a category with kernel pairs and coequalizers of kernel pairs, or a category with joint coequalizers. For all $\mathcal{A}$-morphisms e, e' for which ee' is defined, we can consider the following implications:

(i) e, e' regularly epic $\Longrightarrow ee'$ regularly epic,
(ii) e split epic, e' regularly epic $\Longrightarrow ee'$ regularly epic,
(iii) ee' regularly epic $\Longrightarrow e$ regularly epic,
(iv) ee' regularly epic, e' epic $\Longrightarrow e$ regularly epic,
(v) e regularly epic, e' split epic $\Longrightarrow ee'$ regularly epic.

Obviously (i) implies (ii). If the implication (ii) holds for all e, e', then (iii) is also always satisfied by 1.2 (ii); but (iii) does not hold in general, see [12] and [4] for counterexamples.

On the other hand, (iv) is always true by 1.2 (iii); it can even be proved without the regular epimorphisms forming a local factorization class for morphisms [12]. One also easily sees that (v) holds in any category [12].

Nothing seems to be known about the implication (iii)⇒(ii). That (ii)⇒(i) does not hold in general is shown by the following

Example 3.6. Let $\mathcal{L}$ be a category of all (X, i, a, b) with X a set, $i \in \{0, 1\}$, $a, b \in X$; if $i = 0$ we also require $a \neq b$. A morphism $f : (X, i, a, b) \longrightarrow (Y, j, c, d)$ is given by a map $f : X \longrightarrow Y$ such that either $i = j$, $f(a) = c$, $f(b) = d$ or $i = 0$, $j = 1$, $f(a) = f(b) = c$. For the constant map $s : \{0, 1\} \longrightarrow \{0, 1\}$ with value 0 and the unique map $t : \{0, 1\} \longrightarrow \{0\}$ we see that the composite

$$(\{0,1\}, 0, 0, 1) \xrightarrow{s} (\{0,1\}, 1, 0, 1) \xrightarrow{t} (\{0\}, 1, 0, 0)$$

is not regularly epic in $\mathcal{L}$, but s and t are; thus $\mathcal{L}$ does not satisfy (i). On the other hand, consider a composite

$$(X, i, a, b) \xrightarrow{e'} (Y, j, c, d) \xrightarrow{e} (Z, k, u, v)$$

with e' regularly epic and e split-epic. Since e has a right-inverse l, we have $i \leq j = k$. Moreover, e is surjective, and we have $l(u) = c$, $l(v) = d$. In particular, we can have $u = v$ only if $j = k = 1$ and $c = d$, and we see that in this case e' can only be regularly epic if $i = 1$. In all cases we easily conclude that ee' is regular epic; hence $\mathcal{L}$ satisfies (ii). □

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