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Totality of colimit closures

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Abstract. Adámek, Herrlich, and Reiterman showed that a cocomplete category $\mathcal{A}$ is cocomplete if there exists a small (full) subcategory $\mathcal{B}$ such that every $\mathcal{A}$-object is a colimit of $\mathcal{B}$-objects. The authors of the present paper strengthened the result to totality in the sense of Street and Walters. Here we weaken the hypothesis, assuming only that the colimit closure is attained by transfinite iteration of the colimit closure process up to a fixed ordinal. This requires some investigations on generalized notions of generators.

Keywords: cocomplete category, (almost-)$\mathcal{E}$-generator, colimit closure, cointersection, total category

Classification: 18A20, 18A35

Introduction.

The main aim of this paper is to give sufficient criteria for totality (in the sense of Street and Walters [13]) of a (cocomplete) colimit closure of a small category. Adámek, Herrlich, and Reiterman [1] showed that such a category is complete if the closure can be obtained in one step. The authors of the present paper proved that such a category is even total (cf. [3]). Totality is a strong property of a category, it always implies completeness and cocompleteness as well as compactness in the sense of Isbell [7]. Here we generalize this result: a cocomplete category is total if it is the colimit closure of a small subcategory and if this closure is attained after some (small) ordinal number of steps (see 3.3 below). The latter condition is essential: In 3.6 we give an example of a cocomplete colimit closure $\mathcal{A}$ of a small category $\mathcal{B}$ such that $\mathcal{A}$ is not complete (hence not total); $\mathcal{A}$ even fails to have a terminal object.

Our investigations clarify the relationship between different notions. Unfortunately, we know no example of a category which satisfies the hypothesis of 3.3, but not of our previous criterion. But this phenomenon is related to the following fact of independent interest: if $\mathcal{A}$ is a colimit closure of some subcategory $\mathcal{B}$ and if every extremal epimorphism in $\mathcal{A}$ is a composite of a chain of length α of regular epimorphisms, then the colimit closure is attained at step $\alpha + 1$. In particular, if every extremal epimorphism is regular, then the closure is reached in two steps, i.e. every $\mathcal{A}$-object is a colimit of colimits of $\mathcal{B}$-objects. But 3.3 can at least be used to simplify totality proofs in some situations. For instance, it is easy to see that every group is a codomain of a coequalizer of two homomorphisms of free groups, hence the category of groups is the two-step cocompletion of $\{\mathbb{Z}\}$. It seems more

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complicated to show that it is even the one-step cocompletion of $\{F\}$, where F is the free group generated by two elements.

These investigations are closely related to some types of generators. In particular, under mild conditions, a category $\mathcal{A}$ is a colimit closure of $\mathcal{B} \subset \mathcal{A}$ if and only if the object set $|\mathcal{B}|$ is a strong generator of $\mathcal{A}$ (see 2.8 below). If $\mathcal{E}$ is a class of morphisms, a set $\mathcal{G}$ of objects is called an almost-$\mathcal{E}$-generator if for every $\mathcal{A}$-object A there exists an $\mathcal{E}$-morphism from some coproduct of $\mathcal{G}$-objects to A. Similarly, $\mathcal{G}$ is called an $\mathcal{E}$-generator if this morphism can even be chosen as the "canonical" one, i.e. the counit of a certain adjunction. Under a cancellation condition on $\mathcal{E}$, both notions coincide, but even in general there is a close relationship (see 1.2 below). Moreover, almost-$\mathcal{E}$-generators for certain classes $\mathcal{E}$ of strong epimorphisms are closely related to colimit closures (see 2.2, 2.4, and 2.6 below). In 1.4 we give sufficient conditions for the possibility of joining an $\mathcal{E}$-generator to a single-object generator either by coproduct or by direct product.

1. Generators.

In this section we shall investigate the relationship between certain types of generators and categories which are colimit closures of small sets.

Since later we shall need coproducts anyway, we shall always assume existence of small coproducts; sometimes one can weaken the hypothesis by assuming only existence of certain coproducts.

Now consider a small set $\mathcal{G} \subset |\mathcal{A}|$ of objects of some category $\mathcal{A}$ with coproducts. For any $A \in |\mathcal{A}|$, we form the coproduct

$$TA := \coprod_{G \in \mathcal{G}} \mathcal{A}(G, A) \cdot G,$$

where $\mathcal{A}(G, A)$ denotes the set of $\mathcal{A}$-morphisms $G \longrightarrow A$, and $\mathcal{A}(G, A) \cdot G$ is the coproduct of $\mathcal{A}(G, A)$ copies of G. By $\varepsilon_A : TA \longrightarrow A$ we denote the unique morphism with $\varepsilon_A \cdot u_f = f$ for all coproduct injections $u_f : G \longrightarrow TA$ (where $G \in \mathcal{G}, f \in \mathcal{A}(G, A)$). Note that ε_A is a counit of an adjunction (cf. [5], 3.4).

Now let $\mathcal{E}$ be a class of $\mathcal{A}$-morphisms containing all isomorphisms and closed under composition with them. We call the above $\mathcal{G}$ an *$\mathcal{E}$-generator* of $\mathcal{A}$ if $\varepsilon_A \in \mathcal{E}$ for all $A \in |\mathcal{A}|$. For $\mathcal{E}$ the class of all epimorphisms, an $\mathcal{E}$-generator is simply a *generator*; similarly, we use the term *strong* (*regular* resp.) *generator* when $\mathcal{E}$ is the class of all extremal (regular resp.) epimorphisms. Note that our notion of generator coincides with the usual one, i.e. $\mathcal{G}$ is a generator if and only if for all pairs of parallel morphisms $x, y : A \longrightarrow B$ we have $x = y$ whenever $xf = yf$ for all $f : G \longrightarrow A,\ G \in \mathcal{G}$.

There are other descriptions of $\mathcal{E}$-generators when $\mathcal{E}$ has additional properties. We particularly look at the following conditions on $\mathcal{E}$:

(A) If $ee' \in \mathcal{E}$, then $e \in \mathcal{E}$.
(B) If $ee' \in \mathcal{E}$ and $e' \in \mathcal{E}$, then $e \in \mathcal{E}$.
(C) If $e \in \mathcal{E}$ and q is split-epic, then $eq \in \mathcal{E}$.

Obviously (A) implies (B); it also implies (C) because, for $qs = 1$, from $eqs = e \in \mathcal{E}$ we can conclude $eq \in \mathcal{E}$ by (A). The class of (extremal) epimorphisms always satisfies (A) (and hence (B) and (C)). For the class of regular epimorphisms we always have (B) and (C), but not necessarily (A) (cf. [8]).

We call a set $\mathcal{G}$ of objects an *almost-$\mathcal{E}$-generator*, if for every $A \in |\mathcal{A}|$ there exists an $\mathcal{E}$-morphism $e : \coprod_{i\in I} G_i \longrightarrow A$ with all G_i in $\mathcal{G}$.

Proposition 1.2. *Let $\mathcal{A}$ be a category with coproducts, and let $\mathcal{E}$ be a class of $\mathcal{A}$-morphisms containing all isomorphisms and closed under composition with them. If $\mathcal{E}$ satisfies* (B) *and* (C) *above, then for any set $\mathcal{G} \subset |\mathcal{A}|$ and the statements* (i),(ii),(iii) *below the implications* (i) $\Longleftrightarrow$(ii) $\Longrightarrow$(iii) *hold. If $\mathcal{E}$ satisfies even* (A), *then all three statements are equivalent:*

(i) *$\mathcal{G}$ is an $\mathcal{E}$-generator.*
(ii) *An $\mathcal{A}$-morphism e belongs to $\mathcal{E}$ whenever all $G \in \mathcal{G}$ are projective with respect to e.*
(iii) *$\mathcal{G}$ is an almost-$\mathcal{E}$-generator.*

Proof: (i)$\Longrightarrow$(ii) For any $\mathcal{A}$-morphism $e : A \longrightarrow B$, consider the unique morphism $Te : TA \longrightarrow TB$ with $Te \cdot u_f = v_{ef}$ for all $f : G \longrightarrow A, G \in \mathcal{G}$, and the coproduct injections u_f as above and $v_h : G \longrightarrow TB$ (for $h : G \longrightarrow B,\ G \in \mathcal{G}$). If all $G \in \mathcal{G}$ are projective with respect to e, then Te is split-epic, hence $e \cdot \varepsilon_A = \varepsilon_B \cdot Te \in \mathcal{E}$ by (C), and we get $e \in \mathcal{E}$ from (B).

(ii)$\Longrightarrow$(i) follows immediately from the fact that all $G \in \mathcal{G}$ are projective with respect to ε_A (for $A \in |\mathcal{A}|$). (i)$\Longrightarrow$(iii) is trivial.

(iii)$\Longrightarrow$(i) For $e : C \longrightarrow A,\ C = \coprod_{i\in I} G_i$ with $G_i \in \mathcal{G}$, there always exists a $t : C \longrightarrow TA$ with $e = \varepsilon_A t$. Thus $e \in \mathcal{E}$ implies $\varepsilon_A \in \mathcal{E}$, if $\mathcal{E}$ satisfies (A). $\square$

1.3 From 1.2 we see that (i), (ii), (iii) are equivalent in the case of (extremal) epimorphisms. For $\mathcal{E}$ the class of regular epimorphisms, we call an almost-$\mathcal{E}$-generator also an *almost-regular generator*.

We can use 1.2 to show that an $\mathcal{E}$-generator often gives, in two different ways, rise to an $\mathcal{E}$-generator with just one object (only the first of which seems to be generally known):

Proposition 1.4. *In a category $\mathcal{A}$ with coproducts, let $\mathcal{G}$ be an $\mathcal{E}$-generator with $\mathcal{A}(G, G') \neq \emptyset$ for all $G, G' \in \mathcal{G}$, where $\mathcal{E}$ satisfies (B) and (C). Then $C := \coprod_{G\in\mathcal{G}} G$ is a (single-object) $\mathcal{E}$-generator of $\mathcal{A}$. If the product $P := \prod_{G\in\mathcal{G}} G$ exists and $\mathcal{E}$ satisfies even (A), then $\{P\}$ is also an $\mathcal{E}$-generator.*

Proof: For each $A \in |\mathcal{A}|$, the diagram

$$\begin{tikzcd}
\coprod_{G\in\mathcal{C}} \mathcal{A}(C,A)\cdot G \arrow[r, "g"] \arrow[d, "\cong"'] & TA \arrow[d, "\varepsilon_A"] \\
\mathcal{A}(C,A)\cdot C \arrow[r, "e"'] & A
\end{tikzcd}$$

commutes; here g is induced by the coproduct injections $G \longrightarrow C$, and e is the canonical morphism. With the existence of arrows $G \longrightarrow G'$ one has that each injection is split-monic, hence g is split-epic. Therefore, if ε_A belongs to $\mathcal{E}$, also e does. So $\{C\}$ is an $\mathcal{E}$-generator.

All product projections $P \longrightarrow G$ are split-epic, hence also the first (canonical) arrow in the diagram

$$\coprod_{G\in\mathcal{G}} \mathcal{A}(G,A)\cdot P \longrightarrow TA \xrightarrow{\varepsilon_A} A.$$

Hence, by 1.2, $\{P\}$ is an $\mathcal{E}$-generator. $\square$

Note that in the above proof P can even be replaced by any object having all $G \in \mathcal{G}$ as retracts, regardless whether the product exists.

2. Colimit closures.

For a full subcategory $\mathcal{B}$ of a cocomplete category $\mathcal{A}$, we define $\Gamma\mathcal{B} \subset \mathcal{A}$ to be the full subcategory of all small colimits of $\mathcal{B}$-objects. Then $\Gamma\mathcal{B}$ is always closed under coproducts, but not necessarily under coequalizers. So we may have to iterate Γ in order to obtain a colimit-closed subcategory. We define $\Gamma^0\mathcal{B} := \mathcal{B}$, $\Gamma^{\alpha+1}\mathcal{B} := \Gamma\Gamma^\alpha\mathcal{B}$ for every ordinal α, $\Gamma^\lambda\mathcal{B} := \bigcup_{\xi<\lambda}\Gamma^\xi\mathcal{B}$ for λ a limit ordinal or a fixed symbol ∞ with $\alpha < \infty$ for all ordinals α. Then $\Gamma^{\alpha+1}\mathcal{B} = \Gamma\Gamma^\alpha\mathcal{B}$ is always closed under coproducts, while $\Gamma^\lambda\mathcal{B}$ is closed under finite colimits — in particular coequalizers — for each limit ordinal λ. If ρ is a regular cardinal, then $\Gamma^\rho\mathcal{B}$ is even closed under colimits of diagrams of size $< \rho$. More generally, $\Gamma^\alpha\mathcal{B}$ is always closed under colimits of diagrams of size less than the cofinality type of α (see [10] for the definition). Finally, $\Gamma^\infty\mathcal{B}$ is closed under all colimits. Conversely, by transfinite induction on α we see that $\Gamma^\alpha\mathcal{B} \subset \mathcal{C}$ for any colimit-closed $\mathcal{C}$ with $\mathcal{B} \subset \mathcal{C} \subset \mathcal{A}$. This gives $\Gamma^\infty\mathcal{B} \subset \mathcal{C}$, hence $\Gamma^\infty\mathcal{B}$ is the colimit closure of $\mathcal{B}$, i.e. the smallest colimit-closed subcategory containing $\mathcal{B}$.

Now let $\mathcal{R}$ be the class of regular epimorphisms of $\mathcal{B}$ (cf. [8]), and define $\mathcal{R}^\alpha$ for any ordinal α in the following way: $\mathcal{R}^0$ is the class of isomorphisms, for $\alpha > 0$ a morphism e belongs to $\mathcal{R}^\alpha$ if there exist families of morphisms $(e_\rho)_{\rho\leq\alpha}$ and $(q_\rho)_{\rho<\alpha}$ such that the following conditions hold:

(i) $e_0 = 1,\ e_\alpha = e$.
(ii) $q_\xi \in \mathcal{R}$ for all $\xi < \alpha$.
(iii) $e_{\xi+1} = q_\xi e_\xi$ for all $\xi < \alpha$.
(iv) For any limit ordinal $\lambda \leq \alpha,\ e_\lambda$ is the cointersection (= generalized pushout) of all e_ξ with $\xi < \lambda$.

This coincides with the definitions in 2.1 of [2].

By $\Sigma\mathcal{B}$ we denote the full subcategory of $\mathcal{A}$ whose objects are small coproducts of $\mathcal{B}$-objects. We obtain the following:

Proposition 2.2. *If $Y \in |\Gamma^\alpha\mathcal{B}|$, then there exist $X \in |\Sigma\mathcal{B}|,\ e : X \longrightarrow Y$ with $e \in \mathcal{R}^\alpha$.*

Proof: For $\alpha = 0$, the statement is trivial. Now assume that the statement is true for some α and let $Y \in |\Gamma^{\alpha+1}\mathcal{B}|$. Then Y is a small colimit of certain $Z_i \in |\Gamma^\alpha\mathcal{B}|$, where i ranges over some small set I. Now the usual construction of colimits from coproducts and coequalizers renders a regular epimorphism $q : \coprod_{i\in I} Z_i \longrightarrow Y$. Since $Z_i \in |\Gamma^\alpha \mathcal{B}|$, our induction hypothesis gives an $X_i \in |\Sigma\mathcal{B}|$ and an $\mathcal{R}^\alpha$-morphism $e_i : X_i \longrightarrow Z_i$. But then $e := \coprod_{i\in I} e_i : \coprod_{i\in I} X_i \longrightarrow \coprod_{i\in I} Z_i$ is also in $\mathcal{R}^\alpha$, and $\coprod_{i\in I} X_i \in |\Sigma\mathcal{B}|$. Now $e \in \mathcal{R}^\alpha$ and $q \in \mathcal{R}$ give $qe \in \mathcal{R}^{\alpha+1}$, proving that the statement is true for $\alpha + 1$.

Now let λ be a limit ordinal such that the statement is true for all $\xi < \lambda$. If $Y \in |\Gamma^\lambda \mathcal{B}|$, then $Y \in |\Gamma^\xi \mathcal{B}|$ for some $\xi < \lambda$, and by induction hypothesis there exist $X \in |\Sigma\mathcal{B}|$ and $e : X \longrightarrow Y$ with $e \in \mathcal{R}^\xi \subset \mathcal{R}^\lambda$. □

There is some kind of converse of the above result:

Proposition 2.3. *Let $|\mathcal{B}|$ be a generator of $\mathcal{A}$, let $X \in |\Sigma\mathcal{B}|$, $e : X \longrightarrow Y$, with $e \in \mathcal{R}^\alpha$. Then $Y \in |\Gamma^{\alpha+1}\mathcal{B}|$.*

PROOF: Since $\Sigma\mathcal{B} \subset \Gamma\mathcal{B}$, the statement is trivial for $\alpha = 0$. Now assume that it holds for some α. If $X \in |\Sigma\mathcal{B}|$, $e : X \longrightarrow Y$, $e \in \mathcal{R}^{\alpha+1}$, let e_ξ (for $\xi \leq \alpha + 1$) and q_ξ (for $\xi \leq \alpha$) be as in the definition of $\mathcal{R}^{\alpha+1}$. Then we have $e = e_{\alpha+1} = q_\alpha e_\alpha$ and $q := q_\alpha \in \mathcal{R}$, $e_\alpha \in \mathcal{R}^\alpha$. For the codomain Y' of e_α, our induction hypothesis yields $Y' \in |\Gamma^{\alpha+1}\mathcal{B}|$.

Since q is regularly epic and since $|\mathcal{B}|$ is a generator, q is the joint coequalizer of the small family of all pairs (u, v), where $u, v : B \longrightarrow Y', B \in |\mathcal{B}| \subset |\Gamma^{\alpha+1}\mathcal{B}|$, $qu = qv$. This yields $Y \in |\Gamma\Gamma^{\alpha+1}\mathcal{B}| = |\Gamma^{\alpha+2}\mathcal{B}|$, proving that the statement holds for $\alpha + 1$.

Now let λ be a limit ordinal and assume the statement to be true for all $\xi < \lambda$. For $X \in |\Sigma\mathcal{B}|$ and $e : X \longrightarrow Y$ with $e \in \mathcal{R}^\lambda$, let $e_\xi : X \longrightarrow Y_\xi$ $(\xi \leq \lambda)$ be as in the definition of $\mathcal{R}^\lambda$. Then we have $X \in |\Sigma\mathcal{B}| \subset |\Gamma^\lambda\mathcal{B}|$. For $\xi < \alpha$ we have $e_\xi \in \mathcal{R}^\xi$, hence our induction hypothesis yields $Y_\xi \in |\Gamma^{\xi+1}\mathcal{B}| \subset |\Gamma^\lambda\mathcal{B}|$. Since $e_\lambda = e : X \longrightarrow Y$ is the small cointersection of all $e_\xi : X \longrightarrow Y_\xi$ with $\xi < \alpha$, it follows that $Y \in |\Gamma\Gamma^\lambda\mathcal{B}| = |\Gamma^{\lambda+1}\mathcal{B}|$. This proves the statement for λ. By transfinite induction, it holds for all α. □

Corollary 2.4. *If $|\mathcal{B}|$ is an almost-$\mathcal{R}^\alpha$-generator, then $\Gamma^{\alpha+1}\mathcal{B} = \mathcal{A}$.*

PROOF: Since all morphisms in $\mathcal{R}^\alpha$ are epic, $|\mathcal{B}|$ is a generator. Then for any $Y \in |\mathcal{A}|$, we can apply 2.3 for suitable X, e. □

Corollary 2.5. *Assume that $\mathcal{B}$ is a strong generator and all extremal epimorphisms belong to $\mathcal{R}^\alpha$. Then $\Gamma^{\alpha+1}\mathcal{B} = \mathcal{A}$.*

□

A closer look at the initial step of the inductions of 2.2 and 2.3 leads to the following:

Corollary 2.6. *Let $\mathcal{B}$ be a small subcategory of a cocomplete category $\mathcal{A}$. Then $|\mathcal{B}|$ is an almost-regular generator if and only if $\Gamma\Sigma\mathcal{B} = \mathcal{A}$.*

□

2.7 Now we study the question of when a category $\mathcal{A}$ is the colimit closure $\Gamma^\infty\mathcal{B}$ of a small subcategory $\mathcal{B}$. For $e \in \mathcal{A}(X, Y)$, $A \in |\mathcal{A}|$ we write $e \perp A$ if

$\mathcal{A}(A,e) : \mathcal{A}(A,X) \longrightarrow \mathcal{A}(A,Y)$ is bijective. We define the *co-orthogonal closure* $\overset{\wedge}{\mathcal{B}}$ of a subcategory $\mathcal{B}$ to be the subcategory of all $A \in |\mathcal{A}|$ satisfying

$$(\forall\, B \in |\mathcal{B}| : e \perp B) \Longrightarrow e \perp A$$

for all $\mathcal{A}$-morphisms e. Note that $e\perp A$ holds for all $A \in |\mathcal{A}|$ if and only if e is invertible. Thus a category $\mathcal{A}$ is a co-orthogonal closure of some subcategory $\mathcal{B}$, if and only if every morphism e with $e\perp B$ for all $B \in |\mathcal{B}|$ is an isomorphism.

Our next result is essentially known (cf. [9, Prop. 3.4.0] or [11, Cor. 2.1 d]), but we include it in order to clarify the relationship between the above notions.

Theorem 2.8. *For* (i), (ii), (iii) *below the implication* (i)$\Longrightarrow$(ii) *holds. If every $\mathcal{A}$-morphism admits an (extremal-epi, mono) factorization then also* (ii)$\Longrightarrow$(iii) *holds. If, moreover, there exists an α, such that $\mathcal{R}^\alpha$ contains all strong epimorphisms, then all three statements are equivalent:*

(i) *$\mathcal{A}$ is the colimit closure of $\mathcal{B}$.*
(ii) *$\mathcal{A}$ is the co-orthogonal closure of $\mathcal{B}$.*
(iii) *$|\mathcal{B}|$ is a strong generator in $\mathcal{A}$.*

Proof: (i)$\Longrightarrow$(ii) follows immediately from the well-known (and easily established) fact that the co-orthogonal closure $\hat{\mathcal{B}}$ is always closed under all existing (even large) colimits.

(ii)$\Longrightarrow$(iii) Consider an (extremal-epi, mono)-factorization $\varepsilon_A = me$ for any $A \in |\mathcal{A}|$. We easily see $m\perp B$ for all $B \in |\mathcal{B}|$, hence m is an isomorphism by (ii), and therefore ε_A is extremally epic.

(iii)$\Longrightarrow$(i) If $\mathcal{R}^\alpha$ contains all extremal epimorphisms, then the result follows from 2.5. □

2.9 In 2.8, the condition about α cannot be omitted. Indeed, $\{A_0\}$ is a strong generator in the category $\mathcal{C}_\infty$ mentioned in 3.5 (3) of [2], but the colimit closure $\mathcal{A}$ of $\{A_0\}$ consist of all $(A, (\varphi_\nu)_{\nu\leq\infty})$ with φ_∞ nowhere defined. One easily sees that $\mathcal{A}$ has no terminal object. In particular, a terminal object of $\mathcal{C}_\infty$ (e.g. $(\{0\}, (\varphi_0))$ with $\varphi_\nu(0) = 0$ for all $\nu \leq \infty$) does not belong to $\mathcal{A}$, hence $\mathcal{A} \neq \mathcal{C}_\infty$. A similar example is given in [5, Remark 3.6.3].

3. Totality.

In this section, we shall apply the above results to obtain sufficient criteria for totality of a category $\mathcal{A}$. Here $\mathcal{A}$ is called *total* [13] if the Yoneda embedding $\mathcal{A} \longrightarrow [\mathcal{A}, \mathbf{Set}]$ has a left adjoint (disregarding the size of $[\mathcal{A}, \mathbf{Set}]$). This can be equivalently expressed by saying that a functor $H : \mathcal{D} \longrightarrow \mathcal{A}$ admits a colimit whenever $\mathcal{A}(\mathcal{A}, H-)$ has a colimit in **Set**. Every total category is *cocomplete* but also *complete* ([9]) and existence of pullbacks yields that all extremal epimorphisms are *strong*. Moreover, a total category always has all (possibly large) *cointersections of regular epimorphisms* ([4, 4.1]), *but not neccessarily of strong epimorphisms* ([4, 4.3]). It need not have a generator([4, 4.2]). Our investigations are based on the following criterion ([4, Thm. 5.2]) which generalizes a result of [6]:

Theorem 3.2. *Assume that $\mathcal{A}$ is complete admits large cointersections of $\mathcal{E}$-morphisms and has an almost-$\mathcal{E}$-generator for some $\mathcal{E}$ which contains all split-epimorphisms. Then $\mathcal{A}$ is total.*

□

Our first result strengthens [1, Thm. 2] and [4, Thm. 5.5]:

Theorem 3.3. *If $\mathcal{A} = \Gamma^\alpha \mathcal{B}$ is a cocomplete category for some ordinal α and some small $\mathcal{B} \subset \mathcal{A}$, then $\mathcal{A}$ is total, and $|\mathcal{B}|$ is a strong generator of $\mathcal{A}$.*

Proof: By 2.2, $|\mathcal{B}|$ is an almost-$\mathcal{R}^\alpha$-generator, in particular a generator. Therefore, $\mathcal{A}$ is cowellpowered with respect to $\mathcal{R}$ [12]. Consequently, the cocomplete category $\mathcal{A}$ admits all cointersections of $\mathcal{R}$-morphisms, hence of $\mathcal{R}^\alpha$-morphisms by 2.2 of [2]. If $\alpha \geq 1$, then $\mathcal{R}^\alpha$ contains all split-epimorphisms, and we can apply 3.2. If $\alpha = 0$, then $\mathcal{A} = \mathcal{B}$ is small and cocomplete, hence essentially a small complete lattice, and the result is trivial. □

Theorem 3.4. *Let $\mathcal{A}$ be a cocomplete category admitting all colimits of all chains of regular epimorphisms. Let $\mathcal{B} \subset \mathcal{A}$ be small and assume that $\mathcal{A}$ is the smallest subcategory that contains $\mathcal{B}$ and is closed under small colimits and under colimits of chains of regular epimorphisms. Then $\mathcal{A}$ is total and admits cointersections of strong epimorphisms. Moreover, $|\mathcal{B}|$ is a strong generator of $\mathcal{A}$.*

Proof: By 1.1 (2) and 3.4 of [2], $\mathcal{A}$ admits cointersections of extremal epimorphisms. The co-orthogonal closure $\overset{\wedge}{\mathcal{B}} \subset \mathcal{A}$ is even closed under all existing (possibly large) colimits; particularly under colimits of chains. Therefore, we get $\overset{\wedge}{\mathcal{B}} = \mathcal{A}$, and $|\mathcal{B}|$ is a strong generator by 2.8. Thus $\mathcal{A}$ is total by 3.2. □

Our next result partly strengthens Theorem 4 of [1]:

Theorem 3.5. *Let $\mathcal{A}$ be a cocomplete category, which is the colimit closure of a small subcategory. Assume that $\mathcal{R}^\alpha$ contains all extremal epimorphisms for some α. Then $\mathcal{A}$ is total and admits cointersections of strong epimorphisms, and $|\mathcal{B}|$ is a strong generator of $\mathcal{A}$.*

Proof: Since $\mathcal{A}$ is cocomplete, it follows from 1.1.2 and 2.5 of [2] that $\mathcal{A}$ admits all cointersections of $\mathcal{R}^\alpha$-morphisms, i.e. of extremal epimorphisms. By 2.2, $|\mathcal{B}|$ is a strong generator, and from 2.5 we conclude $\mathcal{A} = \Gamma^{\alpha+1}\mathcal{B}$. Now we can apply 5.3.

□

3.6 In 3.2. the condition $\Gamma^\alpha \mathcal{B} = \mathcal{A}$ cannot be weakened to $\Gamma^\infty \mathcal{B} = \mathcal{B}$, i.e. in 3.4 the existence of colimits of chains of regular epimorphisms is essential, and in 3.5 we cannot weaken the hypothesis about strong epimorphisms to the statement that the class of strong epimorphisms is the union of all $\mathcal{R}^\alpha$. Indeed, the colimit closure $\mathcal{A}$ of $\{A_o\}$ in $\mathcal{C}_\infty$ from 3.5 of [2] is cocomplete and every extremal epimorphism belongs to some $\mathcal{R}^\alpha$. Since $\mathcal{A}$ has no terminal object, it is not complete and therefore not total. Moreover, $\{A_o\}$ is a strong generator in $\mathcal{A}$, but $\mathcal{A}$ does not have an almost-regular one. Indeed, existence of a generator implies that $\mathcal{A}$ is cowellpowered with respect to regular epimorphisms. Thus $\mathcal{A}$ has cointersections of regular epimorphisms by cocompleteness. Now the existence of an almost-regular generator would imply totality by 5.2. A similar example is furnished by [4, Remark 4.4 (3)].

Note that the hypothesis of 3.5 implies that one of 3.4 since the existence of colimits of chains of regular epimorphisms follows from 3.4 of [2]. But we do not know whether the hypothesis of 3.3 always implies the existence of cointersections of strong epimorphisms.

We conclude with another problem: Does every total category with a regular generator admit cointersections of strong epimorphisms?

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