On generalized homogeneity of locally connected plane continua

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Abstract. The well-known result of S. Mazurkiewicz that the simple closed curve is the only nondegenerate locally connected plane homogeneous continuum is extended to generalized homogeneity with respect to some other classes of mappings. Several open problems in the area are posed.

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All spaces considered in this paper are assumed to be metric, and all mappings are continuous. By a *continuum* we mean a compact connected space.

A surjective mapping $f : X \to Y$ between spaces X and Y is said to be:

- a *local homeomorphism* provided that for every point $x \in X$ there is an open neighbourhood U of x such that $f(U)$ is an open subset of Y and the partial mapping $f \mid U : U \to f(U)$ is a homeomorphism;
- *light* provided for each point $y \in Y$ the components of the set $f^{-1}(y)$ are singletons (equivalently, if X is compact: $f^{-1}(y)$ is 0-dimensional);
- *monotone* provided for each point $y \in Y$ the set $f^{-1}(y)$ is connected;
- *open* if images of open sets under f are open;
- *confluent* provided for each subcontinuum Q in Y each component of $f^{-1}(Q)$ is mapped onto Q under f.

Locally connected plane curves and their mappings are studied in the paper, with a special emphasis put on the homogeneity. A topological space X is said to be *homogeneous* provided, for every two points $p, q \in X$, there is a homeomorphism f of X onto itself such that $f(p) = q$. See survey articles [16] and [17] on homogeneous continua. In 1924, S. Mazurkiewicz ([13, p. 137]) showed the following result.

Theorem 1 (Mazurkiewicz)**.** *Each nondegenerate locally connected homogeneous continuum is a simple closed curve.*

Later, in 1958, R.D. Anderson [1] (see also [2]) proved that each locally connected 1-dimensional homogeneous continuum is either the Menger universal curve or the simple closed curve. D.C. Wilson [18] proved in 1972 that there exists (i) a monotone and open, (ii) a light and open, mapping of the Menger universal curve onto any locally connected continuum such that each point-inverse set is homeomorphic to (i) the Menger universal curve, (ii) the Cantor set. This Wilson result leads to a variety of spaces that are homogeneous in a more general sense.

A class $\mathcal{M}$ of mappings is said to be *admissible* if it contains all homeomorphisms and if the composition of any two mappings in $\mathcal{M}$ is also in $\mathcal{M}$. A space X is said to be *homogeneous with respect to a class* $\mathcal{M}$ *of mappings* of X onto itself provided, for every two points $p, q \in X$, there is a mapping $f \in \mathcal{M}$ such that $f(p) = q$. Two spaces X and Y are said to be $\mathcal{M}$*-equivalent* if there are in $\mathcal{M}$ surjections of X onto Y and of Y onto X. Note that if the class $\mathcal{M}$ is admissible and if X and Y are $\mathcal{M}$-equivalent, then X is homogeneous with respect to $\mathcal{M}$ if and only if Y is so. Since the Menger universal curve is homogeneous, it is homogeneous with respect to any admissible class of mappings. Therefore the following observation is an immediate consequence of the Wilson result.

Observation 2. *Let $\mathcal{M}$ be one of the following five classes of mappings: open, monotone, light, open and monotone, open and light. If, for a locally connected continuum X, there exists a mapping in $\mathcal{M}$ from X onto the Menger universal curve, then X is homogeneous with respect to $\mathcal{M}$.*

For example, it is evident that the one-point union of two copies of the Menger universal curve is homogeneous with respect to the classes of open, of monotone and of light and open mappings. This, and many other examples, each containing a copy of the Menger universal curve, are not embeddable in the plane. Hence it is quite natural to study a more interesting case, namely the case of plane continua. We do not have any full answer to this problem for a given class $\mathcal{M}$ of mappings nor a characterization of these continua. Some partial results are presented below.

In the light of the above mentioned extension of the concept of homogeneity, it is natural to ask the following question:

Question 3. To what wider (admissible) classes of mappings can the Mazurkiewicz result be extended?

This question was already asked for open mappings (compare [5, Problem 6, p. 5] and [6, Problem 4, p. 132]), and it has a negative answer (see below).

Only some partial answers to Question 3 are known. We recall them for the reader's convenience.

Remark 4. The Mazurkiewicz result cannot be extended to monotone mappings, because the following plane locally connected continua are homogeneous with respect to this class of mappings:
(a) the Sierpiński universal plane curve ([6, Theorem 5, p. 131]);
(b) each standard universal dendrite D_n of order $n \in \{3, 4, \ldots, \omega\}$ (for $n = 3$ see [10, Example 2.4, p. 59] and [11, Proposition 2.4, p. 223]; for arbitrary integer $n \geq 3$ and $n = \omega$ see [8, Theorem 7.1]).
On the other hand, for any $n \in \{3, 4, \ldots, \omega\}$, each confluent, hence each monotone mapping from a simple n-od (i.e. a locally connected continuum homeomorphic to the union of n arcs, every two of which have their end point as the only point in common) onto itself preserves the center of the n-od (see e.g. [4, Theorem 12, p. 32]), and so each n-od is an example of a dendrite which is not homogeneous with

respect to confluent, and therefore to monotone, mappings. The following problems are then of some interest.

Problems 5. Characterize (a) locally connected plane continua, (b) locally connected plane curves, (c) dendrites which are homogeneous with respect to monotone mappings.

Remark 6. The Mazurkiewicz result cannot be extended to open mappings, because — as was recently shown by J.R. Prajs ([14, Corollary 5]) — the disc $B_2 = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1\}$ is homogeneous with respect to open mappings. This is a negative solution to the above mentioned problems ([5, Problem 6, p. 5] and [6, Problem 4, p. 132]). More generally, it is shown in [14] that (a) the n-dimensional ball B_n is homogeneous with respect to open mappings if and only if $n \geq 2$, and that (b) if a plane continuum X is a finite union of discs $X_1, \dots, X_k$ such that each X_i is a cyclic element of X and $X_i \cap X_j \subset \partial X_i \cap \partial X_j$ for $i \neq j$, then X is homogeneous with respect to open mappings. The following problems (which are taken from [14]) show how little is known to us about locally connected continua which are homogeneous with respect to open mappings.

Problems 7 (Prajs)**.** (a) Construct a one-dimensional locally connected plane continuum which is nonhomogeneous and homogeneous with respect to open mappings.

(b) Characterize 2-manifolds with boundary which are homogeneous with respect to open mappings. Find an example different from a disc.

In connection with Problem 7 (a), it is known that no continuum having the properties listed there can contain a point having a dendrite as its neighbourhood, because no dendroid (i.e. an arcwise connected and hereditarily unicoherent continuum), and hence no dendrite, is homogeneous with respect to open mappings, [7]. Thus we have the following corollary.

Corollary 8. *If a nondegenerate locally connected continuum distinct from a simple closed curve is homogeneous with respect to open mappings, then each of its open subsets contains a simple closed curve.*

It follows from Remarks 4 and 6 above that the classes of all monotone and of all open mappings are too large for a possible extension of the Mazurkiewicz result. So, it is natural to consider their intersection, i.e. the mappings which are both open and monotone, or other subclasses of the class of open mappings, e.g. mappings which are both open and light, as candidates for such an extension, and to check the examples previously discussed.

For mappings which are open and monotone simultaneously, no dendrite can be a counterexample because dendrites are not homogeneous with respect to open mappings, as was noted above. We do not know if the Sierpiński universal plane curve is a counterexample, since the following questions are open.

Question 9. Is the Sierpiński universal plane curve homogeneous with respect to the class of open mappings?

If so, one can ask the next question.

Question 10. Is the Sierpiński universal plane curve homogeneous with respect to the class of open and monotone mappings?

Now we pass to open light mappings. Recall that Theorem 1 of Mazurkiewicz has been generalized by R.H. Bing ([3]) as follows.

Theorem 11 (Bing)**.** *Each plane homogeneous continuum that contains an arc is a simple closed curve.*

Recently, Bing's generalization has been in turn extended by J.R. Prajs [15].

Theorem 12 (Prajs)**.** *Each plane continuum that is homogeneous with respect to the class of open and light mappings and that contains an arc is a simple closed curve.*

Since each locally connected continuum is arcwise connected, we get

Corollary 13 (Prajs)**.** *Each nondegenerate plane locally connected continuum that is homogeneous with respect to the class of open and light mappings is a simple closed curve.*

Remark 14. Let a light surjective mapping f be defined on a compact space. If either the domain or the range of f is locally connected, then f is confluent if and only if f is open (compare [9, Proposition 1]).

As a consequence of Theorem 1, Corollary 13 and Remark 14, we have the following result.

Theorem 15. *If a plane continuum X is locally connected, then the following conditions are equivalent:*

(1) *X is homogeneous;*
(2) *X is homogeneous with respect to local homeomorphism;*
(3) *X is homogeneous with respect to open light mappings;*
(4) *X is homogeneous with respect to confluent light mappings;*
(5) *X is a simple closed curve.*

As another consequence of Corollary 13, we have a result that concerns open light mappings of the Sierpiński universal plane curve S onto itself.

Recall that the union of all boundaries of complementary domains of S in the plane is called the *rational part* of S and is denoted by R. The remaining part, $S \setminus R$, of the curve is called its *irrational part* (cf. [12, p. 255]). A mapping $f : X \to Y$ is said to be *simple* provided $\operatorname{card} f^{-1}(y) \leq 2$ for each $y \in Y$. Thus each simple mapping is obviously light.

Proposition 16. *Let a pair of points x and y of the Sierpiński universal plane curve S be given.*

(a) *If x and y are either both in R or both in $S \setminus R$, there is a homeomorphism $f : S \to S$ such that $f(x) = y$ and $f(R) = R$.*
(b) *If $x \in S \setminus R$ and $y \in R$, there exists a simple open mapping $f : S \to S$ such that $f(x) = y$ and $f(R) \subset R$.*
(c) *If $x \in R$ and $y \in S \setminus R$, there is no open light mapping f of S onto itself such that $f(x) = y$.*

Proof: Statement (a) is just the Krasinkiewicz result in [12, p. 255]. Statement (b) is my Proposition 5 of [6, p. 130]. To prove statement (c), suppose on the contrary that there is an open light surjection $f : S \to S$ and a point $p \in R$ such that $f(p) \in S \setminus R$. We are going to show that then S would be homogeneous with respect to the class of all open light mappings of S onto itself, contrary to Corollary 13. Let x and y be any points of S. By Statements (a) and (b), only the case when $x \in R$ and $y \in S \setminus R$ needs a proof. But then, according to (a), there are homeomorphisms h_1 and h_2 of S onto itself such that $h_1(x) = p$ and $h_2(f(p)) = y$. Thus the composition $h_2 f h_1 : S \to S$ maps x to y, and the proof is complete. □

Remark 17. It should be stressed that the implication (c) of Proposition 16 is an essential part of Prajs' proof in [15] of Theorem 12.

Finally recall the following concept. A space X is said to be *homogeneous with respect to a class $\mathcal{M}$ of mappings of X onto itself from a point $p \in X$ to a point $q \in X$* provided there is a mapping $f \in \mathcal{M}$ such that $f(p) = q$ ([6, p. 125]). This is a generalization of the concept of a space being homogeneous between points p and q due to Krasinkiewicz [12]. Therefore as a direct consequence of Proposition 16, we have the following result which supplements Corollary 2 of [6] giving an affirmative answer to Problem 1 of [6, p. 130].

Theorem 19. *Let a pair of points x and y of the Sierpiński universal plane curve S be given. The following conditions are equivalent:*

(1) *S is homogeneous with respect to open simple mappings from x to y;*
(2) *S is homogeneous with respect to open light mappings from x to y;*
(3) *either $x \in S \setminus R$ and y is arbitrary, or both x and y are in R.*

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