Radicals which define factorization systems

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Abstract. A method due to Fay and Walls for associating a factorization system with a radical is examined for associative rings. It is shown that a factorization system results if and only if the radical is strict and supernilpotent. For groups and non-associative rings, no radical defines a factorization system.

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Introduction.

A *factorization system* in a category is an ordered pair $(\mathbf{E}, \mathbf{M})$ of classes of morphisms satisfying the following conditions.

(F1) All isomorphisms are in $\mathbf{E} \cap \mathbf{M}$.
(F2) Both $\mathbf{E}$ and $\mathbf{M}$ are closed under composition.
(F3) Every morphism f has a factorization $f = me$ (first e, then m) $m \in \mathbf{M}, e \in \mathbf{E}$.
(F4) For every commutative square

$$\begin{array}{ccc} A & \xrightarrow{e} & B \\ {\scriptstyle r}\downarrow & & \downarrow{\scriptstyle s} \\ C & \xrightarrow{m} & D \end{array}$$

with $e \in \mathbf{E}$ and $m \in \mathbf{M}$, there is a unique d with $de = r$ and $md = s$.

This is equivalent to the usual definition by 2.2 of [1] and is the most convenient characterization for our purposes.

Fay [2] has shown that in the category R-**mod** of left unital modules over a ring R with identity, every radical class (= not necessarily hereditary torsion class) $\mathcal{R}$ can be used to define a factorization system $(\mathbf{E}_{\mathcal{R}}, \mathbf{M}_{\mathcal{R}})$ as follows:

$$A \xrightarrow{f} B \in \mathbf{E}_{\mathcal{R}} \Leftrightarrow B/f(A) \in \mathcal{R};$$
$$C \xrightarrow{g} D \in \mathbf{M}_{\mathcal{R}} \Leftrightarrow g \text{ is injective and } \mathcal{R}(D/g(C)) = 0.$$

If one seeks analogous factorization systems in other categories such as the category of rings, in which subobjects need not be normal, the defining conditions for $\mathbf{E}_{\mathcal{R}}$ and $\mathbf{M}_{\mathcal{R}}$ have to be modified. The natural way to do this was demonstrated by

Fay and Walls [3] who considered specific radicals in some categories of groups, not varieties. (For an account of what happens in the category of all groups, see §2.) As it happens, radicals define factorizations comparatively rarely in non-abelian situations.

Following [3], but without making any assumptions about whether $(\mathbf{E}_{\mathcal{R}}, \mathbf{M}_{\mathcal{R}})$ is a factorization system, we define, for every radical class $\mathcal{R}$ of (associative) rings, the following classes of ring homomorphisms:

$$\mathbf{E}_{\mathcal{R}} = \{A \xrightarrow{f} B : B/[f(A)]_B \in \mathcal{R}\};$$
$$\mathbf{M}_{\mathcal{R}} = \{C \xrightarrow{g} D : g \text{ is injective, } g(C) \triangleleft D \text{ and } \mathcal{R}(D/g(C)) = 0\}.$$

Here $[S]_R$ denotes the ideal of a ring R generated by a subset S, and the symbol $\triangleleft$ indicates an ideal.

In the next section we shall characterize the radical classes $\mathcal{R}$ for which $(\mathbf{E}_{\mathcal{R}}, \mathbf{M}_{\mathcal{R}})$ is a factorization system. The corresponding question for groups and non-associative rings will be discussed in §2.

Mrówka [9] proved that a topological space A is compact if and only if for every space B, the projection $\pi_2 : A \times B \to B$ is closed. In the category of topological spaces,

$$(\{\text{maps with dense image}\}, \{\text{closed embeddings}\})$$

is a factorization system. The concatenation of these ideas led Manes [8] and Herrlich, Salicrup and Strecker [7] to study analogues of compactness in other categories. The papers of Fay and Walls [2], [3], [4] are primarily concerned with the investigation of such "compactnesses" in categories of modules and groups. In the category of rings it turns out that "everything is compact"; this is proved in §3.

All rings treated are associative unless we indicate otherwise, but need not, of course, have identities. For unexplained terminology pertaining to radical theory, we refer to [6].

1. Associative rings.

In this section we shall be exclusively concerned with radical classes of associative rings. A radical class $\mathcal{R}$ is *strict* if its semi-simple class is closed under subrings or, equivalently, for every ring A, $\mathcal{R}(A)$ contains all subrings of A belonging to $\mathcal{R}$ ([6, p. 153]); $\mathcal{R}$ is *supernilpotent* if it contains all nilpotent rings.

Clearly $(\mathbf{E}_{\mathcal{R}}, \mathbf{M}_{\mathcal{R}})$ satisfies (F1) for all $\mathcal{R}$. We shall now consider (F2).

Proposition 1.1. *The following conditions are equivalent for a radical class $\mathcal{R}$ of associative rings.*

(i) *If $A \triangleleft B \triangleleft C$, $\mathcal{R}(B/A) = 0$ and $\mathcal{R}(C/B) = 0$, then $A \triangleleft C$.*
(ii) *$\mathcal{R}$ is supernilpotent.*
(iii) *If $A \triangleleft B \triangleleft C$ and $\mathcal{R}(B/A) = 0$, then $A \triangleleft C$.*

Proof: $\neg$(ii) $\Rightarrow$ $\neg$(i): If $\mathcal{R}$ is not supernilpotent then $\mathbb{Z}^0$, the zeroring on the integers, is not in $\mathcal{R}$ and so (as $\mathbb{Z}^0$ is isomorphic to each of its non-zero ideals)

$\mathcal{R}(\mathbb{Z}^0) = 0$. It then follows (e.g. as in [10, Proof of Lemma 2]) that $\mathcal{R}(Y\mathbb{Z}[Y]) = 0$ ($Y\mathbb{Z}[Y]$ being the free ring on the generator Y). Let

$$\begin{aligned} C &= (Y\mathbb{Z}[Y])[X] \quad \text{(polynomial ring)}, \\ B &= \{a_1X + a_2X^2 + \cdots : a_i \in Y\mathbb{Z}[Y]\}, \\ A &= \{kYX + a_2X^2 + \cdots : k \in \mathbb{Z},\ a_i \in Y\mathbb{Z}[Y]\}. \end{aligned}$$

Then $A \triangleleft B \triangleleft C$. Since $B^2 \subseteq A$ and additively $B/A \cong Y\mathbb{Z}[Y]/\langle Y\rangle$, B/A is a zeroring on a free group, so $\mathcal{R}(B/A) = 0$. Also $C/B \cong Y\mathbb{Z}[Y]$, so $\mathcal{R}(C/B) = 0$. However, A is not an ideal of C.

(ii) $\Rightarrow$ (iii): If $A \triangleleft B \triangleleft C$, then $[A]_C^3 \subseteq A$ (by Andrunakievich's lemma) so $[A]_C/A$ is a nilpotent ideal of B/A. Since $\mathcal{R}(B/A) = 0$, B/A is semiprime, so $A = [A]_C \triangleleft C$. (This implication is well known.) □

Note that the implication (iii) $\Rightarrow$ (ii) can also be obtained from Theorem 2 of [11].

Thus since semi-simple classes are closed under extensions, $\mathbf{M}_{\mathcal{R}}$ is closed under composition if and only if $\mathcal{R}$ is supernilpotent.

Proposition 1.2. *If $\mathcal{R}$ is strict, then $\mathbf{E}_{\mathcal{R}}$ is closed under composition.*

Proof: Let $f : A \to B$ and $g : B \to C$ be in $\mathbf{E}_{\mathcal{R}}$. Then $B/[f(A)]_B \in \mathcal{R}$, so $g(B)/g([f(A)]_B) \in \mathcal{R}$. Since $g([f(A)]_B) \subseteq [g([f(A)]_B)]_C$, $\mathcal{R}$ also contains

$$(g(B) + [g([f(A)]_B)]_C)/[g([f(A)]_B)]_C.$$

Since $\mathcal{R}$ is strict, we then have

$$\begin{aligned} &[g(B)]_C/[g([f(A)]_B)]_C \\ &\qquad =[g(B) + [g([f(A)]_B)]_C]_C/[g([f(A)]_B)]_C \\ &\qquad = [[g(B) + [g([f(A)]_B)]_C]/[g([f(A)]_B)]_C]_{\overline{C}} \in \mathcal{R}, \end{aligned}$$

where $\overline{C} = C/[g([f(A)]_B)]_C$. (This follows, for example, from the proof of Theorem 2.1 of [12].) From the exact sequence

$$\begin{aligned} 0 \to [g(B)]_C/[g([f(A)]_B)]_C \to & C/[g([f(A)]_B)]_C \\ \to & C/[g(B)]_C \to 0 \end{aligned}$$

and the fact that $C/[g(B)]_C \in \mathcal{R}$, we deduce that $C/[g([f(A)]_B)]_C \in \mathcal{R}$. But

$$\begin{aligned} g([f(A)]_B) =& gf(A) + g(B)gf(A) + gf(A)g(B) + g(B)gf(A)g(B) \\ \subseteq& gf(A) + Cgf(A) + gf(A)C + Cgf(A)C \\ =& [gf(A)]_C, \end{aligned}$$

so $[g([f(A)]_B)]_C \subseteq [gf(A)]_C$, while the reverse inclusion is clear, so we conclude that

$$C/[gf(A)]_C = C/[g([f(A)]_B)]_C \in \mathcal{R}.$$

Thus $gf \in \mathbf{E}_{\mathcal{R}}$. □

Proposition 1.3. *If $\mathcal{R}$ is supernilpotent and strict, then $(\mathbf{E}_{\mathcal{R}}, \mathbf{M}_{\mathcal{R}})$ is a factorization system.*

PROOF: By 1.1 and 1.2, $(\mathbf{E}_{\mathcal{R}}, \mathbf{M}_{\mathcal{R}})$ satisfies (F2). Let $f : A \to B$ be a ring homomorphism. Let $I/[f(A)]_B = \mathcal{R}(B/[f(A)]_B)$. Then we have a commutative diagram

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "(f)"'] & B \arrow[r] & B/[f(A)]_B \\ f(A) \arrow[r] & I \arrow[r] \arrow[u, "\triangleleft"] & \mathcal{R}(B/[f(A)]_B). \end{tikzcd}$$

Let $m : I \to B$ be inclusion. Then

$$\begin{aligned} \mathcal{R}(B/m(I)) = \mathcal{R}(B/I) &\cong \mathcal{R}((B/[f(A)]_B)/(I/[f(A)]_B)) \\ &= \mathcal{R}((B/[f(A)]_B)/\mathcal{R}(B/[f(A)]_B)) = 0, \end{aligned}$$

so m is in $\mathbf{M}_{\mathcal{R}}$.

Now let $e : A \to I$ be induced by f. Then $[e(A)]_B = [f(A)]_B$, so $I/[e(A)]_B = I/[f(A)]_B \in \mathcal{R}$. Also $[e(A)]_I \triangleleft I \triangleleft B$, and $[[e(A)]_I]_B = [e(A)]_B$, so by Andruakievich's lemma, $[e(A)]_B/[e(A)]_I$ is nilpotent, hence in $\mathcal{R}$. Since

$$(I/[e(A)]_I)/([e(A)]_B/[e(A)]_I) \cong I/[e(A)]_B \in \mathcal{R},$$

we conclude that $I/[e(A)]_I \in \mathcal{R}$. Thus e is in $\mathbf{E}_{\mathcal{R}}$. Since $me = f$, we have established (F3).

Finally we examine (F4). If

$$\begin{tikzcd} A \arrow[r, "e"] \arrow[d, "f"'] & B \arrow[d, "g"] \\ C \arrow[r, "m"] & D \end{tikzcd}$$

commutes, with $e \in \mathbf{E}_{\mathcal{R}}$ and $m \in \mathbf{M}_{\mathcal{R}}$, then for the natural maps $\alpha : B \to B/[e(A)]_B$, $\beta : D \to D/m(C)$, we have $\beta ge = \nu mf = 0$, so $\beta g([e(A)]_B) = 0$ and there is a unique map $\gamma = B/[e(A)]_B \to D/m(C)$ such that $\gamma\alpha = \beta g$. But $B/[e(A)]_B \in \mathcal{R}$ while $\mathcal{R}(D/m(C)) = 0$, so as $\mathcal{R}$ is strict, $\gamma = 0$ and thus $\beta g = 0$. This means that there is a unique $d : B \to C$ $(= \mathrm{Ker}(\beta m))$ such that $md = g$. Then also $mde = ge = mf$, so as m is injective, $de = f$. (Cf. the proof on p. 2257 of [3], and note that strictness (or something like it) is required there for the conclusion $u = 0$. All subsequent results in that paper are obtained for the torsion radical in categories of groups for which it *is* strict.) $\square$

All we need for the converse of 1.3 is

Proposition 1.4. *If $\mathcal{R}$ is not strict, then* $(\mathbf{E}_{\mathcal{R}}, \mathbf{M}_{\mathcal{R}})$ *does not satisfy* (F4).

Proof: If $\mathcal{R}$ is not strict, there is a ring $A \in \mathcal{R}$ and a ring $B \supseteq A$ for which $A \nsubseteq \mathcal{R}(B)$. Then

$$0 \neq A/A \cap \mathcal{R}(B) \cong (A + \mathcal{R}(B))/\mathcal{R}(B) \subseteq B/\mathcal{R}(B)$$

where $A/A \cap \mathcal{R}(B) \in \mathcal{R}$ and $\mathcal{R}(B/\mathcal{R}(B)) = 0$. Thus we may as well assume that $\mathcal{R}(B) = 0$.

Recall that the *split null extension* $A * A$ of A is the additive direct sum $A \oplus A$ with multiplication given by

$$(x, y)(z, w) = (xw + yz, yw).$$

Let $A_0 = \{(x, 0) : x \in A\}$. Then $A_0 \lhd A * A$ and $(A * A)/A_0 \cong A$. We have a commutative diagram

$$\begin{array}{ccc} A_0 & \xrightarrow{\ e\ } & A * A \\ {\scriptstyle f}\big\downarrow & & \big\downarrow{\scriptstyle g} \\ B_0 & \xrightarrow{\ m\ } & B * B \end{array}$$

where all maps are inclusions. Since $e(A_0) \lhd A * A$ and $(A * A)/A_0 \cong A \in \mathcal{R}$, e is in $\mathbf{E}_{\mathcal{R}}$, while $B * B/m(B_0) \cong B$ and $\mathcal{R}(B) = 0$, so $m \in \mathbf{M}_{\mathcal{R}}$. But if there were a homomorphism $d : A * A \to B_0$ with $md = g$ we would have, for every $a \in A$,

$$(0, a) = g(0, a) = md(0, a) \in B_0$$

— a contradiction, as clearly $A \neq 0$. □

We now have all the ingredients for our principal result.

Theorem 1.5. *Let $\mathcal{R}$ be a radical class of associative rings. Then* $(\mathbf{E}_{\mathcal{R}}, \mathbf{M}_{\mathcal{R}})$ *is a factorization system if and only if $\mathcal{R}$ is supernilpotent and strict.*

Proof: “If”: 1.3. “Only if”: 1.1 and 1.4. □

A rather natural “smaller” category which has connections with radical theory has all associative rings as objects but as morphisms just those homomorphisms whose images are *accessible* (= subideals). For a radical class $\mathcal{R}$, let

$$\mathbf{E}^*_{\mathcal{R}} = \{f : f \in \mathbf{M}_{\mathcal{R}} \text{ and } f \text{ has accessible image}\}.$$

Then by adapting some of the preceding arguments we can prove

Theorem 1.6. *Let $\mathcal{R}$ be a radical class of associative rings. Then* $(\mathbf{E}^*_{\mathcal{R}}, \mathbf{M}_{\mathcal{R}})$ *is a factorization system for the category of associative rings and homomorphisms with accessible images if and only if $\mathcal{R}$ is supernilpotent.*

2. Non-associative rings; groups.

We quickly examine two other non-abelian categories for radical-based factorization systems. The notation of the previous section (with obvious changes in meaning) will be retained.

Proposition 2.1. *Let $\mathcal{R}$ be a radical class of non-associative rings. Then $(\mathbf{E}_{\mathcal{R}}, \mathbf{M}_{\mathcal{R}})$ is a factorization system if and only if $\mathcal{R}$ is the class of all rings.*

PROOF: Suppose $(\mathbf{E}_{\mathcal{R}}, \mathbf{M}_{\mathcal{R}})$ is a factorization system. Then $\mathbf{M}_{\mathcal{R}}$ is closed under composition so by the argument used in the proof of 1.1, $\mathcal{R}$ contains all zerorings. Let A be any ring with $\mathcal{R}(A) = 0$. A construction in [5] yields a ring R with the following properties (where A^0 is the zeroring on the additive group of A, and so on):

$$A \oplus A_0 \triangleleft R,\ R/(A \oplus A_0) \cong A,\ [A_0]_R = A^2 \oplus A_0.$$

Thus we have $A_0 \triangleleft A \oplus A_0 \triangleleft R$ with $\mathcal{R}((A \oplus A_0)/A_0) \cong \mathcal{R}(A) = 0$ and $\mathcal{R}(R/A \oplus A_0) \cong \mathcal{R}(A) = 0$. Again using the fact that $\mathbf{M}_{\mathcal{R}}$ is composition-closed, we see that $A_0 \triangleleft R$. But then $A_0 = A^2 \oplus A_0$, i.e. $A^2 = 0$. This means that $\mathcal{R}(A) = A$, so $A = 0$ and $\mathcal{R}$ contains all rings.

Conversely, if $\mathcal{R}$ is the class of all rings, then

$$(\mathbf{E}_{\mathcal{R}}, \mathbf{M}_{\mathcal{R}}) = \quad (\{\text{all homomorphisms}\}, \{\text{all isomorphisms}\})$$

and this is a factorization system. □

There is a lot of degeneracy associated with radical theory for non-associative rings. The category of groups, however, is in many ways well-behaved. The following result may therefore be a little surprising.

Proposition 2.2. *Let $\mathcal{R}$ be a radical class of groups. Then $(\mathbf{E}_{\mathcal{R}}, \mathbf{M}_{\mathcal{R}})$ is a factorization system for the category of groups if and only if $\mathcal{R}$ is the class of all groups.*

PROOF: Suppose there is a group G with $|G| \neq 1$ and $|\mathcal{R}(G)| = 1$. As in the previous proof, we just have to show that $\mathbf{M}_{\mathcal{R}}$ is not closed under composition. Consider the wreath product $G \wr G$. Let x be any element of G. Then we have (up to isomorphism)

$$G_x \triangleleft \prod_{g \in G} G_g \triangleleft G \wr G$$

where each $G_g \cong G, \mathcal{R}(\prod G_g/G_x) \cong \mathcal{R}(\prod_{g \neq x} G_g) = 0$ and $\mathcal{R}((G \wr G)/\prod_{g \in G}) \cong \mathcal{R}(G) = 0$. But G_x is not normal in $G \wr G$ as $|G| > 1$. □

3. Compactness.

As we have noted, there is an analogue of compactness associated with a factorization system. We conclude by describing the compact rings relative to $(\mathbf{E}_{\mathcal{R}}, \mathbf{M}_{\mathcal{R}})$ where $\mathcal{R}$ is a supernilpotent strict radical class. A ring A is said to be *compact relative to* $(\mathbf{E}_{\mathcal{R}}, \mathbf{M}_{\mathcal{R}})$ if for every ring B the projection $\pi_2 : A \times B \to B$ satisfies the condition

$$(C \xrightarrow{g} A \times B) \in \mathbf{M}_{\mathcal{R}} \Longrightarrow (C \xrightarrow{\pi_2 g} B) \in \mathbf{M}_{\mathcal{R}}.$$

Proposition 3.1. *Let $\mathcal{R}$ be a supernilpotent strict radical class of associative rings. Then every associative ring is compact relative to $(\mathbf{E}_{\mathcal{R}}, \mathbf{M}_{\mathcal{R}})$.*

Proof: Let A, B be any (associative) rings. Let $I \triangleleft A \oplus B$ and let

$$J = \{a \in A : \exists\, (a,b) \in I\},$$
$$K = \{b \in B : \exists\, (a,b) \in I\}.$$

Then $J \triangleleft A$ and $K \triangleleft B$ so $I \subseteq J \oplus K \triangleleft A \oplus B$. We first prove that

$$(A \oplus B)(J \oplus K) \subseteq I.$$

Let $a \in J, b \in K, (r,s) \in A \oplus B$. Then there exist $b' \in B, a' \in A$ such that $(a,b'), (a',b) \in I$. Now we have

$$\begin{aligned}(r,s)(a,b) = (ra, sb) &= (r,0)(a,b') + (0,s)(a',b)\\ &\in (A \oplus B)I \subseteq I.\end{aligned}$$

Thus $(J \oplus K)/I$ is contained in the right annihilator of $(A \oplus B)/I$.

Now suppose the embedding of I into $A \oplus B$ is in $\mathbf{M}_{\mathcal{R}}$, i.e. that $\mathcal{R}((A \oplus B)/I) = 0$. Since $\mathcal{R}$ is supernilpotent, $(A \oplus B)/I$ is semiprime, so it has zero annihilator and thus $(J \oplus K)/I = 0/$. We therefore have $J \oplus K = I$. Then

$$\begin{aligned}0 = \mathcal{R}((A \oplus B)/I) &= \mathcal{R}((A \oplus B)/(J \oplus K))\\ &\cong \mathcal{R}(A/J) \oplus \mathcal{R}(B/K).\end{aligned}$$

In particular, $\mathcal{R}(B/K) = 0$, so the embedding $\pi_2(I) \to B$, i.e. $K \to B$, is in $\mathbf{M}_{\mathcal{R}}$. □

References

[1] Bousfield A.K., *Construction of factorization systems in categories*, J. Pure Appl. Algebra **9** (1977), 207–220.

[2] Fay T.H., *Compact modules*, Comm. Algebra **16** (1988), 1209–1219.

[3] Fay T.H., Walls G.L., *Compact nilpotent groups*, Comm. Algebra **17** (1989), 2255–2268.

[4] ———, *Categorically compact locally nilpotent groups*, Comm. Algebra **18** (1990), 3423–3435.

[5] Gardner B.J., *Some degeneracy and pathology in non-associative radical theory*, Annales Univ. Sci. Budapest Sect. Math. 22–23 (1979–80), 65–74.

[6] ———, *Radical Theory*, Longman, Harlow, 1989.

[7] Herrlich H., Salicrup G., Strecker G.E., *Factorizations, denseness, separation, and relatively compact objects*, Topology Appl. **27** (1987), 157–169.

[8] Manes E.G., *Compact Hausdorff objects*, General Topology Appl. **4** (1974), 341–360.

[9] Mrówka S., *Compactness and product spaces*, Colloq. Math. **7** (1959), 19–22.

[10] Puczylowski E.R., *On unequivocal rings*, Acta Math. Acad. Sci Hungar. **36** (1980), 57–62.

[11] Sands A.A., *On ideals in over-rings*, Publ. Math. Debrecen **35** (1988), 273–279.

[12] Stewart P.N., *Strict radical classes of associative rings*, Proc. Amer. Math. Soc. **39** (1973), 273–278.

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