Hercules versus Hidden Hydra Helper

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Abstract. L. Kirby and J. Paris introduced the Hercules and Hydra game on rooted trees as a natural example of an undecidable statement in Peano Arithmetic. One can show that Hercules has a "short" strategy (he wins in a primitively recursive number of moves) and also a "long" strategy (the finiteness of the game cannot be proved in Peano Arithmetic). We investigate the conflict of the "short" and "long" intentions (a problem suggested by J. Nešetřil).

After each move of Hercules (trying to kill Hydra fast) there follow k moves of Hidden Hydra Helper (making the same type of moves as Hercules but trying to keep Hydra alive as long as possible). We prove that for $k = 1$ Hercules can make the game short, while for $k \geq 2$ Hidden Hydra Helper has a strategy for making the game long.

Keywords: rooted tree, unprovability, Kirby–Paris Theorem

Classification: 05C05, 90D99, 03B25

1. Introduction and statement of results.

L. Kirby and J. Paris introduced the Hercules and Hydra game (abbreviated H+H) on rooted trees (see [1]) as a natural example of an undecidable statement in Peano Arithmetic. One can show that Hercules has a "short" strategy (he wins in a primitively recursive number of moves) and also a "long" strategy (the finiteness of the game cannot be proved in Peano Arithmetic). We investigate the conflict of the "short" and "long" intentions (a problem suggested by J. Nešetřil).

Let us start by a definition of the original H+H game.

By the word "tree" we shall mean a finite rooted tree. The *height* of a tree T, denoted by $ht(T)$, is the maximum distance of a vertex of T from the root. The symbol P_n denotes a path of n edges, rooted in one endpoint. We use some more or less standard terminology concerning rooted trees; for completeness, we give the definitions in Section 2. We need two more special notions:

A *head* of a tree T is any vertex of T (different from the root) with no successors. The 2-*predecessor* of a head v in a tree T is the (unique) vertex w such that v is a successor of a successor of w. The *throat* of v is the subtree adjacent to the 2-predecessor containing v, minus the vertex v.

The H+H game is the following battle between Hercules and Hydra: Hydra is a tree, its initial (1-st) stage is given. In the t-th move (t is a positive integer) Hercules chops off a head v of Hydra. Hydra then grows t replicas of the throat of v from the 2-predecessor of v. If no 2-predecessor of v exists, nothing is grown. Hercules wins if Hydra is reduced to the root after some finite number of moves (the length of the battle).

If the t-th stage of Hydra is the tree T, we shall speak about the stage (T,t). Sometimes we shall use the obvious generalization of the game beginning at some stage (T,t) for a $t>1$. Fig. 1 shows an example of the Hercules and Hydra game.

Note that the height of Hydra never increases and if the initial stage of the game is (T_0,t_0), then for any stage (T,t), $t>t_0$ of this game $|T|<|T_0|.(t_0{+}1).(t_0{+}2)\dots(t)$.

Hercules must win in every battle, but this is unprovable in Peano Arithmetic ([1]).

The fact that Hercules never makes infinitely many moves cannot be fully formalized in Peano Arithmetic. We must restrict ourselves to effectively generated sequences of moves. A *strategy* of Hercules is a general recursive function assigning to every initial segment of the H+H game of length n (i.e. the full information about the first n stages of the game and the heads chopped off at first $n-1$ stages) the head to be chopped off at the n-th stage. The initial stage together with the strategy completely determines the game. All concrete strategies we shall use are primitively recursive.

A special strategy MAX of Hercules, introduced in [3], essentially appears already in [1]. Informally, this strategy can be described as follows: Imagine that we draw the stages of Hydra into the plane, modifying the drawing between successive stages by deleting the chopped head and by drawing the newly grown replicas of a throat on the right in the same manner as the original throat. Then MAX says: always chop off the rightmost head.

The strategy MAX always gives the longest possible game and the statement

"For every n, the Hercules and Hydra game with initial stage $(P_n,1)$
and strategy MAX is finite"

cannot be proved in Peano Arithmetic (cf. [2], [3]). On the other hand, certain strategy MIN gives a primitively recursive bound on the length of the game (see Proposition 4.1).

J. Nešetřil suggested the following problem. Suppose that there are two players chopping off the heads of Hydra: Hercules, who honestly tries to kill the Hydra as fast as possible, and a strange person called Hidden Hydra Helper (abbreviated HHH), who wants to keep Hydra alive infinitely long. Hercules wins if Hydra is reduced to the root at some stage, HHH wins otherwise (in contrast to the original H+H game, where only Hercules may choose his moves, this is a true 2-player game where both players have a choice of moves). Hercules must always win, but the question we want to investigate is the provability of this by finite means, i.e. in Peano Arithmetic. We denote **H+k.HHH** $(k\ge 1)$ the game where HHH has the first k moves, then there follows one move of Hercules, k moves of HHH etc. (each move of both Hercules and HHH creates a new stage). Hercules uses one strategy and HHH another one (the input for the strategy contains the information who has played which move). The provability results are the following:

Theorem 1.1.

(i) *There exists a strategy S_0 of Hercules in the* H+1HHH *game such that the statement*

"For every strategy L of HHH *and for every initial stage*
(T,t) the H+1HHH *game with the strategies S_0, L is finite"*

is a theorem of Peano Arithmetic.

(ii) *There exists a strategy L_0 of* HHH *in the* H+2HHH *game such that for every recursive sequence of trees $\{T_1, T_2, \dots\}$ with $ht(T_n) \geq n$ and for every strategy S of Hercules the statement*

"For all n, the H+2HHH *game with the initial stage $(T_n, 1)$ and the strategies S, L_0 is finite"*

cannot be proved in Peano Arithmetic.

Part (i) of this theorem will be proved in Section 3: we exhibit a specific strategy of Hercules and prove that the game is finished in a number of moves bounded by a provably recursive function. Part (ii) will be proved in Section 5, essentially by showing that HHH can simulate the strategy MAX (from the H+H game) on a suitable subtree of Hydra.

2. Preliminaries.

Let us define several notions concerning rooted trees. By a rooted subtree of a tree T we shall mean a subtree with the same root as T. If T is a tree, then $|T|$ is the number of vertices of T. The k-th *level* of T consists of all vertices of T having distance exactly k from the root. We say that w *lies above* v if the (unique) path from the root to w contains v. The *successors* of a vertex v are all vertices adjacent to v lying above v. If w is a successor of v, then all vertices lying above w together with v form a tree rooted in v, which we call a *subtree adjacent to* v. The tree $T_1 + T_2$ arises by identifying the roots of T_1 and T_2.

We shall deal with functions whose order of growth is unusually large in most parts of mathematics. We use the so-called *Fast Growing Hierarchy* ([4]) for comparing the order of growth of functions. For every ordinal number $\alpha < \varepsilon_0 = \lim\{\omega, \omega^\omega, \omega^{\omega^\omega}, \dots\}$ we define a function f_α:

$$f_0(n) = n + 1$$
$$f_{\alpha+1}(n) = f_\alpha^n(n)$$

(the exponent n means n-fold application of f_α), and

$$f_\alpha(n) = f_{\alpha_n}(n)$$

if α is a limit. Here $\{\alpha_n\}_{n<\omega}$ is a *fundamental sequence*, a certain increasing sequence of ordinals cofinal in α (for example, $\omega_n = n$, $(\omega^2)_n = \omega \cdot n$; for the exact definition see [4]). The functions f_k ($k < \omega$) are primitively recursive and f_ω is essentially the Ackermann function.

3. The H+1HHH game.

Proof of Theorem 1.1, the part (i): Let v be a vertex of a tree T. Then $Deg(T, v)$ is the number of successors of v which are not heads of T and $deg(T, v)$ is the number of successors of v which are heads of T. We put

$$Deg(T, n) = \max\{Deg(T, v);\ v \text{ is on the } n\text{-th level of } T\},$$
$$deg(T, n) = \max\{deg(T, v);\ v \text{ is on the } n\text{-th level of } T\}.$$

Then we let

$$D(T) = (deg(T,n), Deg(T,n-1), deg(T,n-1), Deg(T,n-2), deg(T,n-2), \dots, \\ Deg(T,0), deg(T,0)), \quad \text{where } n = ht(T) - 1.$$

We order the finite dimensional integer vectors by their length and the vectors of equal length lexicographically: $(x_1, \dots, x_n) < (y_1, \dots, y_m)$ iff $n < m$ or $n = m$ and $x_i < y_i$ for the first index i such that $x_i \neq y_i$.

The *skeleton* $C(T)$ is the subtree T' of T satisfying

(1) All heads of T' are also heads of T.
(2) $Deg(T', v) = Deg(T', w)$ and $deg(T', v) = deg(T', w)$ for every two vertices v, w on the same level of T'.
(3) The vector $D(C(T))$ is maximal (in the above defined ordering) among all $D(T'')$, T'' a subtree satisfying (1).

The *type* of a tree T is the vector $D(C(T))$. Fig. 2 shows a tree with its skeleton (thick line); the type of this tree is $(4, 2, 3, 1, 2, 3, 0, 1, 5)$.

The skeleton is uniquely determined (proceed from top to bottom).

The heads of a skeleton are of two types:

- *maxheads*, which lie on the top level, and
- *lowheads*, which lie on lower levels.

The strategy S_0 of Hercules is defined as follows: Let $(T_{-1}, t-1)$ be the stage of Hydra before the last move of HHH, (T_0, t) the stage after the last move of HHH and $(T_1, t+1)$ will be the stage after the next move of Hercules. We distinguish two possibilities:

(i) $D(C(T_{-1})) \geq D(C(T_0))$. Then Hercules chops off an arbitrary maxhead of $C(T_0)$, hence $D(C(T_1)) < D(C(T_{-1}))$.

(ii) $D(C(T_{-1})) < D(C(T_0))$. The only way the type of Hydra may increase is producing a head which becomes a lowhead of the new skeleton (for example, chopping off the head A or B on Fig. 2). Furthermore, all such lowheads which caused the increasing of the skeleton are the successors of a single vertex. This vertex is called a *critical vertex* and it belongs to the new skeleton. The strategy S_0 of Hercules says: chop off any maxhead lying above the critical vertex. One can easily verify that $D(C(T_1)) \leq D(C(T_{-1}))$.

The last inequality need not be strict, but we show that the case (ii) can be repeated only a limited number of times (namely $|T|.t!$ times if the repetition starts at stage (T, t)) and then the case (i) must occur.

Suppose that we have a series of moves where only the case (ii) occurs. The key observation is that no vertex can be the critical vertex twice in this series of moves and that only heads grow as new vertices. Therefore the length of the series is limited by the number of non-heads at the stage where the case (i) occurred the last time, and this gives the above mentioned bound.

We have shown that the type of Hydra strictly decreases during the game and hence the game must be finite. If some component of the vector $C(T)$ decreases at time t, then the components with higher indices increase to at most $|T_0|.t!$ (T_0 is the initial stage of the game). This proof can be formalized inside Peano Arithmetic.

Our considerations give a recursive relation for the length of the battle. From this relation, one can compute an explicit bound, namely that starting from an initial stage (T, t), Hercules always wins in less than $f_{\omega^2+1}(\max(t, |T|))$ moves. This bound will be used in the proof of the second part of Theorem 1.1. □

4. Short H+H game.

In this section we estimate the minimal length of the H+H game:

Proposition 4.1. *Denote by $\mu(T,t)$ the minimal length of the* H+H *game with the initial stage (T,t). If $ht(T) = h$ and T has s heads, then*

$$\mu(T,t) \leq \tau(2h-2, 2.\max(s,t)),$$

where the "tower function" τ is defined by $\tau(0,x) = x$, $\tau(i+1,x) = 2^{\tau(i,x)}$.

Proof: We shall use the strategy MIN of Hercules ([2],[3]): At each stage (T,t), pick a vertex v that lies on level $ht(T)-1$ and has a maximal number of successors among such vertices, and choose the head to chop off among its successors.

Let m_i denote the number of vertices of T on level $ht(T)-1$ with exactly i successors ($i = 1, 2, \dots$) and let r be the number of heads of T which are on lower levels than $ht(T)$ (hence $1.m_1+2.m_2+\ldots+s.m_s+r = s$). Let k be the maximal index such that $m_k \neq 0$. We define $\psi(h,t,k,(m_1,\dots,m_k),r)$ as the maximal length of the game (with the initial stage (T,t) having parameters $h, k, m_1, \dots, m_k, r$) when Hercules uses the strategy MIN.

It is easy to see that ψ is nondecreasing in each of its arguments. It may happen that the function ψ is not defined for some combination of arguments because the parameters correspond to no tree (for example, for an actual tree, if $h = 1$, then necessarily $r = 0$ and only $m_k = 1$ is nonzero among the m_i's). To avoid such cases, we define ψ for any given combination of nonnegative arguments (with $h > 0$, $t > 0$, $k > 0$ and $m_k > 0$) as the maximum of values of ψ for all admissible combinations of arguments with all arguments majorised by the given combination.

From the definition of MIN and monotonicity considerations, we get the following recurrence relations:

$$\psi(1,t,k,(0,0,\dots,1),0) = t+k \tag{1}$$

$$\psi(h,t,1,(m_1),r) \leq \psi(h-1,t+m_1,s',(s',\dots,s'),r) \quad \text{where } s' = r + m_1.(2t+m_1+1)/2,\ h>1 \tag{2}$$

$$\psi(h,t,k,(m_1,\dots,m_k),r) \leq \psi(h,t+m_k,k-1,(m_1,\dots,m_{k-2}, m_{k-1}+m_k.(2.t+m_k+1)/2),r),\ k>1. \tag{3}$$

Using these recurrences, we estimate $(s,t \geq 1)$

$$
\begin{array}{ll}
\psi(h,t,s,(s,s,\dots,s),s) & \\
\leq \psi(h,t+s,s-1,(s,\dots,s,s+s(2t+s+1)/2)),s) & \text{by (3)} \\
\leq \psi(h,t+s,s-1,(s,\dots,s,(s+t)^2),s) & \text{by monotonicity} \\
 & \text{(induction basis)} \\
\vdots & \\
\leq \psi(h,(t+s)^{2^j-1},s-j,(s,\dots,s,(s+t)^{2^j},s) & \text{inductive hypothesis} \\
\leq \psi(h,(t+s)^{2^j-1}+(t+s)^{2^j},s-j-1,(s,\dots,s, & \\
\quad s+(s+t)^{2^j}(2(s+t)^{2^j-1}+(s+t)^{2^j}+1)/2),s) & \text{by (3)} \\
\leq \psi(h,(t+s)^{2^{j+1}-1},s-j-1,(s,\dots,s,(s+t)^{2^{j+1}}),s) & \text{induction step} \\
 & j=1,2,\dots,s-1 \\
\vdots & \\
\leq \psi(h-1,s_1,s_1,s_1,(s_1,s_1,\dots,s_1),s_1), \text{ where } s_1=(s+t)^{2^s} & \\
\quad - \text{ by (2) and monotonicity.} &
\end{array}
$$

Putting $s_0 = \max(s,t)$ and $s_{i+1} = (2s_i)^{2^{s_i}}$, we see that $\psi(h,t,s,(s,\dots,s),s) \leq s_{h-1}$. After some calculation one can verify that this implies the statement of the proposition. □

Remark. Although the estimates used in the preceding proof are quite rough, a more detailed analysis shows that the height of the tower of 2's can be shrunk only by a constant.

Corollary 4.2. *There exist a primitively recursive function φ and a strategy* MIN *of Hercules such that for every tree T and for every t the* H+H *game with an initial stage (T,t) and the strategy* MIN *finishes before the stage number $\varphi(\max(t,|T|))$.*

Remark. For a technical reason, we shall need this corollary for the modified Hercules and Hydra game, where $C \cdot t$ copies are grown on the t-th stage for a fixed C (instead of t). By inspection of the proof we see that the corollary also holds for this case.

5. The H+2HHH game.

In this section we prove the second part of Theorem 1.1. We start by stating one (almost trivial) lemma about monotonicity of the H+H game:

Lemma 5.1. *Let S be a rooted subtree of a tree T, $s \leq t$ and suppose that Hercules chops off some head at the stage (S,s) of the* H+H *game, passing to the stage (S',s+1). Then there exists a move of Hercules at the stage (T,t) passing to the stage $(T',t+1)$ such that either S or S' is a rooted subtree of T'.*

We shall build the tools for the strategy of HHH in the part (ii) of Theorem 1.1.

Let B_k denote the tree which has a root, one vertex at level 1 and k heads on level 2 (i.e. a k-fan with a stick) and let $C_k = B_1 + B_2 + \ldots + B_k$.

Lemma 5.2. *Consider the* H+H *game where Hercules is allowed to make dummy moves (i.e. chop off no head if he wishes). Let (C_k, t_0), $t_0 \geq k$ be the initial stage of Hydra. Then Hercules has a strategy S_k such that for every $t \leq f_k(t_0) - t_0$ Hydra has at least t heads adjacent to the root at the stage $t_0 + t$.*

Proof: We prove the following stronger statement by induction on k:

There exists a strategy S_k which applied from the initial stage (C_k, t_0) on (where $t_0 \geq k$) guarantees the following:

(i) the moves $t_0 + k, \ldots, 2t_0$ are dummy moves
(ii) for all $t \leq f_k(t_0) - t_0$ Hydra has at least t heads adjacent to the root at the stage $t_0 + t$
(iii) Hydra includes C_{k-1} at the stage $f_k(t_0)$.

For $k = 1$ the statement holds (Hercules chops off the head of $B_1 = P_2$ and makes t_0 dummy moves; $f_1(t_0) = 2t_0$). Having C_{k+1} at the stage t_0, Hercules starts the strategy S_k on $C_k \subseteq C_{k+1}$ and he chops off one head of the B_{k+1} instead of the dummy move at the stage $t_0 + k$. This gives him $t_0 + k$ replicas of B_k, so he may repeat the strategy S_k t_0 times with C_k left at the stage $f_{k+1}(t_0) = f_k \circ f_k \circ \cdots \circ f_k(t_0)$ (t_0-fold composition). □

Lemma 5.3. *There exists a primitively recursive function ψ satisfying the following: Let (T_0, t_0) be a stage in the* H+H *game, v a vertex of T_0 on level at least* 3. *Then Hercules can play so that at some stage $t_1 < \psi(\max(|T_0|, t_0))$, Hydra contains $B_1, B_2, \ldots, B_{t_0-1}$ adjacent to the root and $T_0 \setminus \{v\}$ as a rooted subtree (disjoint from the B_i's).*

Proof: Consider the following generalization of the above situation: Let T be Hydra at some stage $t \geq t_0$ and M its subtree adjacent to some vertex w lying on level k (such that $k + ht(M) \geq 3$); initially $T = T_0$ and $M = \{w, v\}$. We prove the following statement by induction on k: There exists a primitively recursive fuction ρ of two variables and a strategy of Hercules, so that when this strategy is applied in the above described situation, then at some stage $t_1 < \rho(k, \max(|T|, t))$, Hydra will contain $B_1, \ldots, B_{t_0-1}$ adjacent to the root and $(T \setminus M) \cup \{w\}$ as a rooted subtree.

For $k = 0$ the task is to extract C_{t_0-1} from a tree M of height at least 3, which is possible in a primitively recursive number $\rho(0, \max(|T_0|, t_0))$ of moves (use Corollary 4.2 and the fact that before being reduced to the root, Hydra will contain C_m with m large enough). If $k > 0$, Hercules reduces M to heads adjacent to w and then he chops off one of them. If we denote by T' the Hydra at this moment, by w' the predecessor of w and by M' one of the newly grown copies, we have the situation as before with k reduced by one (the case $k = 1$ requires some additional discussion in order to preserve a sufficient height of M), so for every s, $\rho(k, s) \leq \rho(k-1, (\varphi(s) + 1)!)$ (again by Corollary 4.2), thus $\psi(s) \leq \rho(s, s)$ is primitively recursive as required. □

The next result is the heart of the proof of the part (ii) of Theorem 1.1:

Lemma 5.4. *There exists a constant k such that the following holds: Let (U, t_0), (U', t_0+1) be two successive stages of the* H+H *game, $t > k$, $t > t_0$, $t! \geq |U|$ and suppose that (T, t) (see Fig.* 3a) *is a stage of the* H+2HHH *game, with* HHH *on the first of his two moves. Then* HHH *can pass to the stage (T', t') depicted on Fig.* 3b, *where he is again on the first of his two moves (regardless of the strategy of Hercules).*

Proof: Divide the moves of HHH as follows:

stage $t+0$	S-move of HHH
stage $t+1$	C-move of HHH
stage $t+2$	move of Hercules
stage $t+3$	U-move of HHH
stage $t+4$	C-move of HHH
stage $t+5$	move of Hercules

⋮

(continue modulo 6)

Initially, call the whole T the *healthy subtree* of Hydra. In the S-moves and U-moves HHH plays something in the healthy subtree and in every C-move he chops off a head adjacent to the vertex v of the healthy subtree (see Fig. 3), obtaining many copies of the healthy subtree. Only one of them may be affected by the successive move of Hercules, so some unaffected copy is defined as the healthy subtree for the next stage.

In his S-moves, HHH plays the strategy S_k described in Lemma 5.2 on the part of the healthy subtree arisen from C_k (v serves as the root). This supports enough heads adjacent to v for C-moves for time at least $f_k(t/3)$ (there are two C-moves for every S-move, but there grow three times more heads at S-moves compared to Lemma 5.2).

At his first U-move, HHH executes the move converting U to U', but because $t > t_0$, there have grown more new heads than necessary. HHH chooses one of the superfluous heads and he produces in further U-moves a new C_k adjacent to v. According to Lemma 5.3, this takes a primitively recursive number of moves, so choosing k large enough, the new C_k is prepared before the old one is destroyed by S-moves. □

Proof of Theorem 1 the part (ii): By induction and by the monotonicity we see from Lemma 5.4 that for trees of a special form (Fig. 3a) HHH can make the game at least as long as the strategy MAX in the H+H game (started on P_n if $ht(U) \geq n$). We indicate how HHH may convert any (sufficiently high) tree into this special form (omitting some technical details).

Let the H+2HHH game start on a sufficiently high tree. There necessarily appears a vertex u of degree at least $t-1 > k$ (k is as in Lemma 5.4) on level $ht(T)-1$ at some stage (T, t). HHH can obtain the subtree C_k rooted at a vertex v_1 on level $ht(T)-2$ (Hercules cannot prevent this). His goal is to get C_k rooted at a predecessor v_2 of v_1 and a successor w of v_1, having a high degree $m \geq t/2$ (this is expressed by Fig. 4). Such a successor w of v_1 exists and it can be preserved by the method of Lemma 5.2 for a long time. During this time, HHH simultaneously produces a subtree of height 2 rooted at v_2 and then he converts it to a new C_k, so the goal is reached. This happens at some stage (T', t'), where t' and hence also $|U|$ is bounded by a fixed primitively recursive function of m.

Note the difference: while Hercules cannot destroy all successors of v_2 on level $ht(T)$, it is easy for him and also for HHH to produce some subtree of height 2 adjacent to v_2.

Now (by a slight modification of Lemma 5.4) HHH can simulate the strategy MAX on the subtree denoted by V on Fig. 4. He chooses a small (primitively recursive in m) subtree U adjacent to v_3 (the predecessor of v_2) and he makes from it a new C_k adjacent to v_3. This can be done in at most $f_{\omega^2+1}-(\max(|U|, t'))$ moves — it follows from the part (i) of Theorem 1.1, where Hercules and HHH exchange their roles. Moreover, HHH must do this simultaneously with the simulation of MAX on V. He must suitably divide his moves between these two tasks according to the last move of Hercules (only the affected part of Hydra needs immediate action), but it is not difficult to see that it works. We need one more lemma:

Lemma 5.5. *The strategy* MAX *with the initial stage* (V, m) (*Fig.* 4) *has at least* $f_{\omega^\omega}(m)$ *moves.*

PROOF: This is an easy corollary of more general theorems in [3]. □

The class of functions majorised by f_{ω^ω} is closed under primitive recursion, so for sufficiently large m $f_{\omega^\omega}(m) > f_{\omega^2+1}(\max(|U|, t'))$ and MAX on V is longer than the destroying of U.

The above described "descend of C_k" can be repeated any desired number of times, so finally HHH can simulate the strategy MAX on an arbitrarily high tree.

Any proof that the H+2HHH game on every T_n with the above described strategy is finite would yield a proof of finiteness of MAX on P_n. This proves Theorem 1.1, the part (ii).

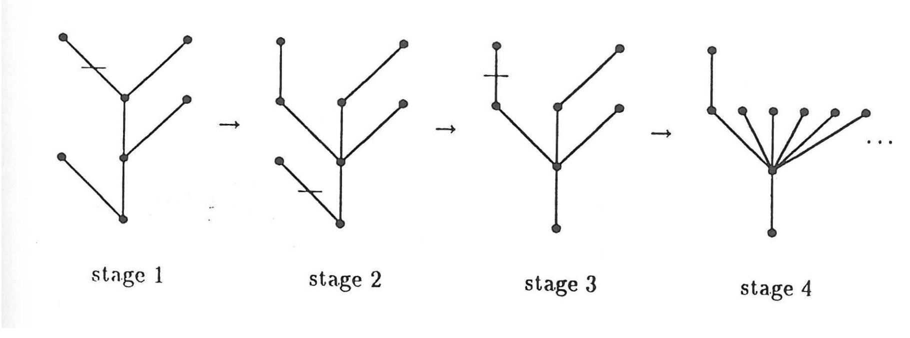

Figure 1: Example of the Hercules and Hydra game

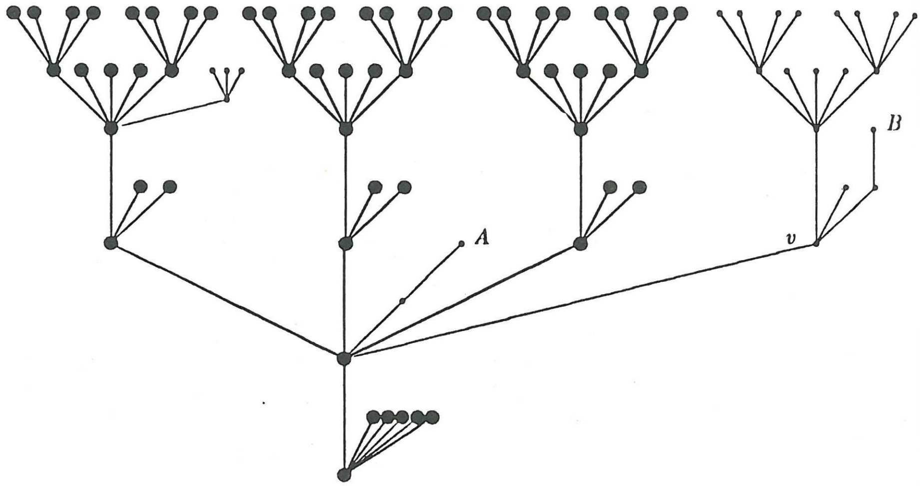

Figure 2: Tree with its skeleton

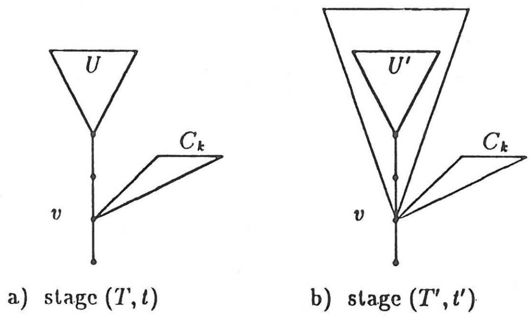

Figure 3: Illustration to the strategy of HHH

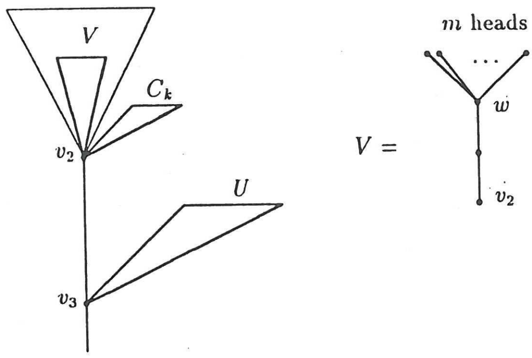

Figure 4: Initial phase of the H+2HHH game

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