Some results on the product of distributions and the change of variable

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Abstract. Let F and G be distributions in $\mathcal{D}'$ and let f be an infinitely differentiable function with $f'(x) > 0$, (or < 0). It is proved that if the neutrix product $F \circ G$ exists and equals H, then the neutrix product $F(f) \circ G(f)$ exists and equals $H(f)$.

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In the following, we let N be the neutrix, see van der Corput [1], having domain $N' = \{1, 2, \dots, n, \dots\}$ and range the real numbers, with negligible functions finite linear sums of the functions

$$n^\lambda \ln^{r-1} n, \quad \ln^r n : \lambda > 0, \quad r = 1, 2, \dots$$

and all functions which converge to zero in the normal sense as n tends to infinity.

We will use n or m to denote a general term in N' so that if $\{a_n\}$ is a sequence of real numbers, then $\mathrm{N}-\lim_{n\to\infty} a_n$ means exactly the same thing as $\mathrm{N}-\lim_{m\to\infty} a_m$.

Note that if $\{a_n\}$ is a sequence of real numbers which converges to a in the normal sense as n tends to infinity, then the sequence $\{a_n\}$ converges to a in the neutrix sense as n tends to infinity and

$$\lim_{n\to\infty} a_n = \mathrm{N}-\lim_{n\to\infty} a_n$$

We now let $\rho(x)$ be a fixed infinitely differentiable function having the following properties:

(i) $\rho(x) = 0$ for $|x| \geq 1$,
(ii) $\rho(x) \geq 0$,
(iii) $\rho(x) = \rho(-x)$,
(iv) $\int_{-1}^{1} \rho(x)\, dx = 1$.

Putting $\delta_n(x) = n\rho(nx)$ for $n = 1, 2, \dots$, it follows that $\{\delta_n(x)\}$ is a regular sequence of infinitely differentiable functions converging to the Dirac delta-function $\delta(x)$.

Now let $\mathcal{D}$ be the space of infinitely differentiable functions with compact support and let $\mathcal{D}'$ be the space of distributions defined on $\mathcal{D}$. Then, if F is an arbitrary distribution in $\mathcal{D}'$, we define

$$F_n(x) = (F * \delta_n)(x) = \langle F(t), \delta_n(x-t)\rangle$$

for $n = 1, 2, \ldots$. It follows that $\{F_n(x)\}$ is a regular sequence of infinitely differentiable functions converging to the distribution $F(x)$.

The following definition for the product of two distributions was given in [2].

Definition 1. Let F and G be distributions in $\mathcal{D}'$ and let $G_n = G * \delta_n$. We say that the neutrix product $F \circ G$ of F and G exists and is equal to the distribution H on the interval (a, b) if

$$\operatorname*{N-lim}_{n\to\infty} \langle F G_n, \phi \rangle = \langle H, \phi \rangle \tag{1}$$

for all functions ϕ in $\mathcal{D}$ with support contained in the interval (a, b). If

$$\lim_{n\to\infty} \langle F G_n, \phi \rangle = \langle H, \phi \rangle,$$

we simply say that the product $F.G$ exists and equals H.

Note that if we put $F_m = F * \delta_m$, we have

$$\langle F G_n, \phi \rangle = \operatorname*{N-lim}_{m\to\infty} \langle F_m G_n, \phi \rangle$$

and so the equation (1) could be replaced by the equation

$$\operatorname*{N-lim}_{n\to\infty} \big[\operatorname*{N-lim}_{m\to\infty} \langle F_m G_n, \phi \rangle\big] = \langle H, \phi \rangle. \tag{2}$$

The next definition for the change of variable in distributions was given in [3].

Definition 2. Let F be a distribution in $\mathcal{D}'$ and let f be a locally summable function. We say that the distribution $F(f(x))$ exists and is equal to the distribution H on the interval (a, b) if

$$\operatorname*{N-lim}_{n\to\infty} \int_{-\infty}^{\infty} F_n(f(x))\phi(x)\, dx = \langle H, \phi \rangle$$

for all test functions ϕ in $\mathcal{D}$ with support contained in the interval (a, b), where

$$F_n(x) = (F * \delta_n)(x).$$

The following theorem was proved in [5].

Theorem 1. *Let F be a distribution in $\mathcal{D}'$ and let f be an infinitely differentiable function with $f'(x) > 0$, (or < 0), for all x in the interval (a, b). Then the distribution $F(f(x))$ exists on the interval (a, b).*

Further, if F is the p-th derivative of a locally summable function $F^{(-p)}$ on the interval $(f(a), f(b))$, (or $f(b), f(a)$), (g inverse of f), then

$$\langle F(f(x)), \phi(x) \rangle = (-1)^p \int_{f(a)}^{f(b)} F^{(-p)}(x)[g'(x)\phi(g(x))]^{(p)}\, dx = \tag{3}$$

$$= (-1)^p \int_{-\infty}^{\infty} F^{(-p)}(f(x)) f'(x) \left[\frac{1}{f'(x)} \frac{d}{dx}\right]^p \left[\frac{\phi(x)}{f'(x)}\right] dx \tag{4}$$

for all ϕ in $\mathcal{D}$ with support contained in the interval (a, b).

Using the equation (3), it was proved that if f had a single simple zero at the point $x = x_1$ in the interval (a, b), then

$$\delta^{(s)}(f(x)) = \frac{1}{|f'(x_1)|}\left[\frac{1}{f'(x)}\frac{d}{dx}\right]^s \delta(x - x_1) \tag{5}$$

on the interval (a, b) for $s = 0, 1, 2, \dots$, showing that the Definition 2 is in agreement with the definition of $\delta^{(s)}(f(x))$ given by Gel'fand and Shilov [6].

The problem of defining the product $F(f) \circ G(g)$ was considered in [4]. Putting $F(f) = F_1$ and $G(g) = G_1$, the product $F_1 \circ G_1 = H_1$ is of course defined by the equation

$$\mathrm{N}\text{-}\lim_{n\to\infty}\big[\mathrm{N}\text{-}\lim_{m\to\infty}\langle F_{1m}G_{1n}, \phi\rangle\big] = \langle H_1, \phi\rangle,$$

for all ϕ in $\mathcal{D}$, where $F_{1m} = F_1 * \delta_m$ and $G_{1n} = G_1 * \delta_n$.

However, it was pointed out that since the distributions $F(f)$ and $G(g)$ were defined by the sequences $\{F_m\}$ and $\{G_n\}$, the product $F(f) \circ G(g)$ should be defined by these sequences, leading to the following definition.

Definition 3. Let F and G be distributions in $\mathcal{D}'$, let f and g be locally summable functions and let $F_m = F * \delta_m$ and $G_n = G * \delta_n$. We say that the neutrix product $F(f) \circ G(g)$ of $F(f)$ and $G(g)$ exists and is equal to the distribution H on the interval (a, b) if $F_m(f)\, G_n(g)$ is a locally summable function on the interval (a, b) and

$$\mathrm{N}\text{-}\lim_{n\to\infty}\big[\mathrm{N}\text{-}\lim_{m\to\infty}\langle F_m(f)G_n(g), \phi\rangle\big] = \langle H_1, \phi\rangle,$$

for all ϕ in $\mathcal{D}$ with support contained in the interval (a, b).

The following two examples were given in [4] and show that the neutrix product $F(f) \circ G(g)$ can be equal to, but is not necessarily equal to the neutrix product $F_1 \circ G_1$.

Example 1. Let $F = x_+^{1/2}$, $G = \delta'(x)$, $f = x_+^2$ and $g = x_+$. Then

$$F(f) = F_1 = x_+, \quad G(g) = G_1 = \frac{1}{2}\delta'(x)$$

and

$$F(f) \circ G(g) = -\frac{1}{2}\delta(x) = F_1 \circ G_1.$$

Example 2. Let $F = x_+^{-1/2}$, $G = \delta(x)$, $f = x$ and $g = x_+^{1/2}$. Then

$$F(f) = F_1 = x_+^{-1/2}, \quad G(g) = G_1 = 0$$

and

$$F(f) \circ G(g) = \delta(x) \neq 0 = F_1 \circ G_1.$$

The following theorem was, however, proved in [4].

Theorem 2. *Let F and G be distributions in $\mathcal{D}'$, let f be a locally summable function and let g be an infinitely differentiable function. If the distributions $F(f) = F_1$ and $G(g) = G_1$ exist and the neutrix product $F(f) \circ G(g)$ exists on the interval (a, b), then*

$$F(f) \circ G(g) = F_1 \circ G(g)$$

on the interval (a, b). In particular, if $g(x) = x$, then

$$F(f) \circ G(g) = F_1 \circ G_1$$

on the interval (a, b).

In this theorem, $F_1 \circ G(g)$ was used to denote the distribution defined by

$$\mathrm{N}-\lim_{n\to\infty}\langle F_1 G_n, (g), \phi\rangle.$$

We now prove the following theorem.

Theorem 3. *Let F and G be distributions in $\mathcal{D}'$ and let f be an infinitely differentiable function with $f'(x) > 0$, (or < 0), for all x in the interval (a, b). If the neutrix product $F \circ G$ exists and is equal to H on the interval $(f(a), f(b))$, (or $(f(b), f(a))$), then*

$$F(f) \circ G(f) = H(f)$$

on the interval (a, b).

Proof: Note first of all that the distributions $F(f)$ and $G(f)$ exist on the interval $(f(a), f(b))$, (or $(f(b), f(a))$), by Theorem 1.

We will suppose that $f'(x) > 0$ and that g is the inverse of f on the interval (a, b). Letting ϕ be an arbitrary function in $\mathcal{D}$ with support contained in the interval (a, b) and making the substitution $t = f(x)$, we have

$$\begin{aligned}\int_{-\infty}^{\infty} F_m(f(x))G_n(f(x))\phi(x)\,dx &= \int_{-\infty}^{\infty} F_m(t)G_n(t)\phi(g(t))g'(t)\,dt = \\ &= \int_{-\infty}^{\infty} F_m(t)G_n(t)\psi(t)\,dt,\end{aligned}$$

where $\psi(t) = \phi(g(t))g'(t)$ is a function in $\mathcal{D}$ with support contained in the interval $(f(a), f(b))$. It follows that

$$\mathrm{N}-\lim_{n\to\infty}\Big[\mathrm{N}-\lim_{m\to\infty}\langle F_m(f)G_n(f), \phi\rangle\Big] = \langle H, \psi\rangle$$

for all ϕ or ψ.

Further, on making the substitution $t = f(x)$, we have

$$\begin{aligned}\int_{-\infty}^{\infty} H_n(t)\psi(t)\,dt &= \int_{-\infty}^{\infty} H_n(t)\phi(g(t))g'(t)\,dt = \\ &= \int_{-\infty}^{\infty} H_n(f(x))\phi(x)\,dx\end{aligned}$$

and so

$$\mathrm{N}-\lim_{n\to\infty}\langle H_n, \psi\rangle = \langle H(f), \phi\rangle.$$

The result of the theorem follows. □

Theorem 4. *Let F and G be distributions in $\mathcal{D}'$ and let f be an infinitely differentiable function with $f'(x) > 0$, (or < 0), for all x in the interval (a,b). If the neutrix products $F \circ G$ and $F \circ G'$, (or $F' \circ G$), exist on the interval $(f(a), f(b))$, (or $(f(b), f(a))$), then*

$$[F(f) \circ G(f)]' = [F(f)]' \circ G(f) + F(f) \circ [G(f)]'$$

on the interval (a,b).

PROOF: The usual law

$$(F \circ G)' = F' \circ G + F \circ G'$$

for the differentiation of a product holds, see [2], and so the result of the theorem follows immediately from Theorem 3. □

Theorem 5. *Let f be an infinitely differentiable function with $f'(x) > 0$, (or < 0), for all x in the interval (a,b) and having a simple zero at the point $x = x_1$ in the interval (a,b). Then the neutrix products $(f(x))_+^r \circ \delta^{(s)}(f(x))$ and $\delta^{(s)}(f(x)) \circ (f(x))_+^r$ exist and*

$$(f(x))_+^r \cdot \delta^{(s)}(f(x)) = \delta^{(s)}(f(x)) \cdot (f(x))_+^r = 0 \tag{6}$$

for $s = 0, 1, \dots, r-1$ and $r = 1, 2, \dots$ and

$$\begin{aligned}(f(x))_+^r \circ \delta^{(s)}(f(x)) =& \delta^{(s)}(f(x)) \circ (f(x))_+^r = \\ =& \frac{(-1)^r s\,!}{2(s-r)\,!} \frac{1}{|f'(x_1)|} \left[\frac{1}{f'(x)} \frac{d}{dx}\right]^{s-r} \delta(x - x_1),\end{aligned} \tag{7}$$

for $r = 0, 1, \dots, s$ and $s = r, r+1, r+2, \dots$ on the interval (a,b).

PROOF: If g is an s times continuously differentiable function at the origin, then the product $g \cdot \delta^{(s)} = \delta^{(s)} \cdot g$ is given by

$$g(x) \cdot \delta^{(s)}(x) = \delta^{(s)}(x) \cdot g(x) = \sum_{i=0}^{s} (-1)^{s+i} \binom{s}{i} g^{s-i}(0) \delta^{(i)}(x).$$

It follows that

$$x_+^r \cdot \delta^{(s)}(x) = \delta^{(s)}(x) \cdot x_+^r = 0$$

for $s = 1, 2, \dots, r-1$ and $r = 1, 2, \dots$ and the equation (6) follows immediately on using Theorem 3.

It was proved in [2] that

$$x_+^r \circ \delta^{(s)}(x) = \delta^{(s)}(x) \circ x_+^r = \frac{(-1)^r s\,!}{2(s-r)\,!} \delta^{(s-r)}(x),$$

for $r, s = 0, 1, 2, \dots, s \geq r$, and it follows on using Theorem 3 that

$$(f(x))_+^r \circ \delta^{(s)}(f(x)) = \delta^{(s)}(f(x)) \circ (f(x))_+^r = \frac{(-1)^r s\,!}{2(s-r)\,!} \delta^{(s-r)}(f(x)),$$

for $r, s = 0, 1, 2, \dots$. The equation (7) follows immediately on using equation (5). □

Example 3.

$$(x+x^2)_+^r \circ \delta^{(r)}(x+x^2) = \delta^{(r)}(x+x^2) \circ (x+x^2)_+^r = \\ = \frac{1}{2}(-1)^r r\,![\delta(x)+\delta(x+1)], \tag{8}$$

$$(x+x^2)_+^r \circ \delta^{(r+1)}(x+x^2) = \delta^{(r+1)}(x+x^2) \circ (x+x^2)_+^r = \\ = \frac{1}{2}(-1)^r (r+1)\,!\,[\delta'(x)+2\delta(x)+\delta'(x+1)+2\delta(x+1)] \tag{9}$$

for $r = 0, 1, 2, \dots$ on the real line.

Proof: The function $f(x) = x + x^2$ has simple zeros at the points $x = 0, -1$. It follows from the equations (5) and (7) that

$$\begin{aligned}(x+x^2)_+^r \circ \delta^{(r)}(x+x^2) =&\delta^{(r)}(x+x^2) \circ (x+x^2)_+^r = \\ =&\frac{1}{2}(-1)^r r\,!\,\delta(x+x^2) = \\ =&\frac{1}{2}(-1)^r r\,!\,[\delta(x)+\delta(x+1)],\end{aligned}$$

proving the equation (8) for $r = 0, 1, 2, \dots$.

It again follows from the equations (5) and (7) that

$$\begin{aligned}(x+x^2)_+^r \circ \delta^{(r+1)}(x+x^2) =&\delta^{(r+1)}(x+x^2) \circ (x+x^2)_+^r = \\ =&\frac{1}{2}(-1)^r (r+1)\,!\,\frac{1}{1+2x}\,[\delta'(x)+\delta'(x+1)] = \\ =&\frac{1}{2}(-1)^r (r+1)\,!\,[\delta'(x)+2\delta(x)+\delta'(x+1)+2\delta(x+1)],\end{aligned}$$

proving the equation (9) for $r = 0, 1, 2, \dots$. □

Theorem 6. *Let f be an infinitely differentiable function with $f'(x) > 0$, (or < 0), for all x in the interval (a, b) and having a simple zero at the point $x = x_1$ in the interval (a, b). Then the neutrix products $(f(x))^{-r} \circ \delta^{(s)}(f(x))$ and $\delta^{(s)}(f(x)) \circ (f(x))^{-r}$ exist and*

$$(f(x))^{-r} \circ \delta^{(s)}(f(x)) = \frac{(-1)^r s\,!}{(r+s)\,!}\,\frac{1}{|f'(x_1)|}\left[\frac{1}{f'(x)}\,\frac{d}{dx}\right]^{r+s}\delta(x-x_1), \tag{10}$$

$$\delta^{(s)}(f(x)) \circ (f(x))^{-r} = 0, \tag{11}$$

for $r = 1, 2, \dots$ and $s = 0, 1, 2, \dots$ on the interval (a, b).

Proof: It was proved in [2] that

$$x^{-r} \circ \delta^{(s)}(x) = \frac{(-1)^r s\,!}{(r+s)\,!}\,\delta^{(r+s)}(x),$$

$$\delta^{(s)}(x) \circ x^{-r} = 0$$

for $r = 1, 2, \dots$ and $s = 0, 1, 2, \dots$. Equations (10) and (11) follow immediately as in the proof of Theorem 6. □

Example 4.

$$(12)\ (x^2-1)^{-1}\circ\delta(x^2-1) = -\frac{1}{4}[\delta'(x-1)+\delta(x-1)-\delta'(x+1)+\delta(x+1)],$$

$$(13)\qquad \delta^{(s)}(x^2-1)\circ(x^2-1)^{-r}=0,$$

for $r=1,2,\dots$ and $s=0,1,2,\dots$ on the real line.

PROOF: The function $f(x)=x^2-1$ has simple zeros at the points $x=\pm 1$. It follows from the equations (5) and (10) that

$$\begin{aligned}(x^2-1)^{-1}\circ\delta(x^2-1) &= -\frac{1}{4x}[\delta'(x-1)+\delta'(x+1)] = \\ &= -\frac{1}{4}[\delta'(x-1)+\delta(x-1)-\delta'(x+1)+\delta(x+1)]\end{aligned}$$

proving equation (12). □

The equation (13) follows immediately from the equations (5) and (11) for $r=1,2,\dots$ and $s=0,1,2,\dots$.

Theorem 7. *Let f be an infinitely differentiable function with $f'(x)>0$, (or <0), for all x in the interval (a,b) and having a simple zero at the point $x=x_1$ in the interval (a,b). Then the neutrix products $(f(x))_+^{\lambda}\circ(f(x))_-^{-\lambda-r}$ and $(f(x))_-^{-\lambda-r}\circ(f(x))_+^{\lambda}$ exist and*

$$(14)\quad \begin{aligned}(f(x))_+^{\lambda}\circ(f(x))_-^{-\lambda-r} =& (f(x))_-^{-\lambda-r}\circ(f(x))_+^{\lambda} = \\ =& -\frac{\pi\operatorname{cosec}(\pi\lambda)}{2(r-1)!}\,\frac{1}{|f'(x_1)|}\left[\frac{1}{f'(x_1)}\,\frac{d}{dx}\right]^{r-1}\delta(x-x_1),\end{aligned}$$

for $\lambda\neq 0,\pm1,\pm2,\dots$ and $r=1,2,\dots$ on the interval (a,b)

PROOF: It was proved in [2] that

$$x_+^{\lambda}\circ x_-^{-\lambda-r} = x_-^{-\lambda-r}\circ x_+^{\lambda} = -\frac{\pi\operatorname{cosec}(\pi\lambda)}{2(r-1)!}\delta^{(r-1)}(x),$$

for $\lambda\neq 0,\pm1,\pm2,\dots$ and $r=1,2,\dots$. Equation (14) follows immediately as in the proof of Theorem 6. □

Example 5. Let $f(x)=t$ be the inverse of the function $g(t)=t+t^3=x$. Then

$$(15)\quad \begin{aligned}(f(x))_+^{\lambda}\circ(f(x))_-^{-\lambda-1} =& (f(x))_-^{-\lambda-1}\circ(f(x))_+^{\lambda} = \\ =& -\frac{1}{2}\pi\operatorname{cosec}(\pi\lambda)\delta(x),\end{aligned}$$

$$(16)\quad \begin{aligned}(f(x))_+^{\lambda}\circ(f(x))_-^{-\lambda-2} =& (f(x))_-^{-\lambda-2}\circ(f(x))_+^{\lambda} = \\ =& -\frac{1}{2}\pi\operatorname{cosec}(\pi\lambda)[\delta'(x)+\delta(x)],\end{aligned}$$

for $\lambda \neq 0, \pm 1, \pm 2, \ldots$ on the real line.

PROOF:

$$g'(t) = 1 + 3t^2 > 0$$

for all t, it follows that $f'(x) > 0$ for all x and so on using the equation (3) with $p = 1$, we have for all ϕ in $\mathcal{D}$

$$\begin{aligned}\langle \delta(f(x)), \phi(x)\rangle &= -\int_{-\infty}^{\infty} H(x) d[(1+3x^2)\phi(x+x^3)] = \\ &= -\int_{-\infty}^{\infty} d[(1+3x^2)\phi(x+x^3)] = \phi(0).\end{aligned}$$

It follows that

$$\delta(f(x)) = \delta(x). \tag{17}$$

Using the equation (3) again with $p = 2$, we have for all x in $\mathcal{D}$

$$\begin{aligned}\langle \delta'(f(x)), \phi(x)\rangle &= \int_{0}^{\infty} d[(1+3x^2)\phi(x+x^3)]' = \\ &= -\phi'(0) - \int_{0}^{\infty} d[(1+3x^2)\phi(x+x^3)] = \\ &= -\phi'(0) + \phi(0).\end{aligned}$$

It follows that

$$\delta'(f(x)) = \delta'(x) + \delta(x). \tag{18}$$

It now follows from the equations (15) and (17) that

$$\begin{aligned}(f(x))_+^{\lambda} \circ (f(x))_-^{-\lambda-1} =&(f(x))_-^{-\lambda-1} \circ (f(x))_+^{\lambda} = \\ &= -\frac{1}{2}\pi \operatorname{cosec}(\pi\lambda)\delta(f(x)) = \\ &= -\frac{1}{2}\pi \operatorname{cosec}(\pi\lambda)\delta(x),\end{aligned}$$

proving the equation (15) for $\lambda \neq 0, \pm 1, \pm 2, \ldots$.

It again follows from the equations (14) and (18) that

$$\begin{aligned}(f(x))_+^{\lambda} \circ (f(x))_-^{-\lambda-2} =&(f(x))_-^{-\lambda-2} \circ (f(x))_+^{\lambda} = \\ &= -\frac{1}{2}\pi \operatorname{cosec}(\pi\lambda)\delta'(f(x)) = \\ &= -\frac{1}{2}\pi \operatorname{cosec}(\pi\lambda)[\delta'(x) + \delta(x)],\end{aligned}$$

proving the equation (16) for $\lambda \neq 0, \pm 1, \pm 2, \ldots$. □

References

[1] van der Corput J.G., *Introduction to the neutrix calculus*, J. Analyse Math. **7** (1959–60), 291–398.

[2] Fisher B., *A non-commutative neutrix product of distributions*, Math. Nachr. **108** (1982), 117–127.

[3] ———, *On defining the distribution* $\delta^{(r)}(f(x))$ *for summable* f, Publ. Math. Debrecen **32** (1985), 233–241.

[4] ———, *On the product of distributions and the change of variable*, Publ. Math. Debrecen **35** (1988), 37–42.

[5] Fisher B., Özcag̃ E., *A result on distributions and the change of variable*, submitted for publication.

[6] Gel'fand I.M., Shilov G.E., *Generalized Functions*, vol. I., Academic Press, 1964.

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