On a certain class of subspectra

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Abstract. The aim of this paper is to characterize a class of subspectra for which the geometric spectral radius is the same and depends only upon a commuting n-tuple of elements of a complex Banach algebra. We prove also that all these subspectra have the same capacity.

Keywords: Banach algebra, joint spectrum, subspectrum, spectroid, geometrical spectral radius, (joint) capacity

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1. Preliminaries.

For the convenience of the reader, we shall recall briefly some definitions and results on Żelazko's axiomatic theory of joint spectra (cf. [7]). Let A be a complex unital Banach algebra. Denote by A^n_{com} the set of all n-tuples of mutually commuting elements in A and let $A_{\text{com}} = \bigcup_{n=1}^{\infty} A^n_{\text{com}}$, in particular, A identified with A^1_{com} is a subset of A_{com}. Suppose that to each n-tuple $(a_1, \dots, a_n)$ in A_{com}, there corresponds a non-void compact subset $\tilde{\sigma}(a_1, \dots, a_n)$ in $\mathbb{C}^n$.

$$(a_1, \dots, a_n) \mapsto \tilde{\sigma}(a_1, \dots, a_n). \tag{1}$$

We shall formulate several axioms for such a map.

$$\tilde{\sigma}(a_1, \dots, a_n) \subset \prod_{j=1}^{n} \sigma(a_j), \tag{I}$$

where $\sigma(a)$ denotes the usual spectrum of an element $a \in A$ and $(a_1, \dots, a_n) \in A_{\text{com}}$. In particular, for a single element a, we have

$$\tilde{\sigma}(a) \subset \sigma(a). \tag{2}$$

The next axiom is the equality in the above formula

$$\tilde{\sigma}(a) = \sigma(a) \tag{II}$$

for an arbitrary a in A.

Let $p : \mathbb{C}^n \to \mathbb{C}^k$ be a polynomial map, i.e. a map given by the formula

$$p(z_1, \dots, z_n) = (p_1(z_1, \dots, z_n), \dots, p_k(z_1, \dots, z_n)),$$

where p_j are polynomials in n variables with complex coefficients. Such a polynomial map induces a map (denoted also by p) from A^n_{com} to A^k_{com} given by

$$(a_1,\dots,a_n) \mapsto (p_1(a_1,\dots,a_n),\dots,p_k(a_1,\dots,a_n)).$$

The third axiom is the so-called spectral mapping property.

$$\text{(III)} \qquad \tilde\sigma(p(a_1,\dots,a_n)) = p(\tilde\sigma(a_1,\dots,a_n))$$

for all polynomial maps on $\mathbb{C}^n$ and an arbitrary $(a_1,\dots,a_n) \in A^n_{\text{com}}$, $n = 1,2,\dots$.

The last axiom gives the translation property of $\tilde\sigma$, it is a particular case of (III) (to keep the numeration of axioms from [7], we shall give it the number (V)).

$$\text{(V)} \qquad \tilde\sigma(a_1+\lambda_1,\dots,a_n+\lambda_n) = \tilde\sigma(a_1,\dots,a_n) + (\lambda_1,\dots,\lambda_n)$$

for all n-tuples $(a_1,\dots,a_n) \in A^n_{\text{com}}$, $(\lambda_1,\dots,\lambda_n) \in \mathbb{C}^n$, $n = 1,2,\dots$. (Here, we write $a_j + \lambda_j$ instead of $a_j + \lambda_j 1$, 1 — the unit of A.)

Definition 1. A (joint) *spectrum* on A is a map (1) such that the axioms (I), (II), and (III) (and consequently also (V)) are satisfied. If a map (1) satisfies only (I) and (III), then it is called a *subspectrum*. A *spectroid* is a map satisfying (I) and (V).

Hence every spectrum is a subspectrum and every subspectrum is a spectroid.

Definition 2. For a spectroid $\tilde\sigma$ and an n-tuple $(a_1,\dots,a_n)$ in A_{com}, the *geometric spectral radius* of $(a_1,\dots,a_n)$ relative to $\tilde\sigma$ is defined by the formula (see [1] or [2])

$$r_{\tilde\sigma}(a_1,\dots,a_n) = \max\{|z| : z \in \tilde\sigma(a_1,\dots,a_n)\},$$

where $|z| = |(z_1,\dots,z_n)| = \left(\sum_{j=1}^n |z_j|^2\right)^{1/2}$.

The main result of [2] says that the geometrical spectral radius relative to a spectrum is in fact independent of this spectrum and is equal to

$$r(a_1,\dots,a_n) = \max\{|z| : \sigma^{[a_1,\dots,a_n]}(a_1,\dots,a_n)\},$$

where $\sigma^{[a_1,\dots,a_n]}(a_1,\dots,a_n)$ is the joint (Harte) spectrum of $(a_1,\dots,a_n)$ in the Banach algebra $[a_1,\dots,a_n]$ generated by $a_1,\dots,a_n$ and the unit. The same result appeared to be true for many subspectra (such as the joint approximate point spectrum σ_π, the left and right joint spectra, σ_l and σ_r) and for some spectroids (the commutant and bicommutant spectra, σ' and σ''). The class of all spectroids $\tilde\sigma$ for which the formula

$$\text{(3)} \qquad r_{\tilde\sigma}(a_1,\dots,a_n) = r(a_1,\dots,a_n)$$

($(a_1,\dots,a_n) \in A^n_{\text{com}}$, $n = 1,2,\dots$) is satisfied, is denoted by Σ_0 in [2]. It is also proved in the same paper that for a given n-tuple of elements in A convex hulls of all spectroids in Σ_0 coincide.

Observe that the proofs in [2] were done for $A = \mathcal{B}(\mathcal{X})$ (the Banach algebra of all bounded endomorphisms of a complex Banach space $\mathcal{X}$), but they can be repeated without any changes in a general case.

We shall need the following two results.

Theorem A [2, Theorem 5.1]. *Let $\tilde{\sigma}$ be a subspectrum on a complex Banach algebra A with unit. Then*

$$\tilde{\sigma}(a_1, \ldots, a_n) \subset \sigma^B(a_1, \ldots, a_n)$$

for every $(a_1, \ldots, a_n) \in A^n_{\mathrm{com}}$, $n = 1, 2, \ldots$, and each closed commutative subalgebra B of A containing the unit of A and the elements $a_1, \ldots, a_n$.

(Here $\sigma^B(a_1, \ldots, a_n)$ denotes the joint spectrum of $(a_1, \ldots, a_n)$ in the commutative Banach algebra B.)

Theorem B [7, Theorem 5.3 and the remarks after Def. 5.5]. *Let $\tilde{\sigma}$ be a subspectrum on a complex unital Banach algebra A. Then for each closed commutative subalgebra B of A containing the unit of A there is a compact subset $\Delta(\tilde{\sigma}; B) \subset \mathfrak{M}(B)$ — the maximal ideal space of B, such that for an arbitrary n-tuple of elements $(a_1, \ldots, a_n)$ in B^n we have*

$$\tilde{\sigma}(a_1, \ldots, a_n) = \{(\phi(a_1), \ldots, \phi(a_n)) \in \mathbb{C}^n : \phi \in \Delta(\tilde{\sigma}; B)\}.$$

If, moreover, $\tilde{\sigma}$ is a spectrum, then the Shilov boundary

$$\Gamma(B) \subset \Delta(\tilde{\sigma}; B). \tag{4}$$

Note that the proofs of Theorems 5.1 and 5.3 in [7] were done for B being a maximal abelian subalgebra of A, but they work as well for any closed commutative subalgebra containing the unit of A.

The main result of this paper says that a subspectrum $\tilde{\sigma}$ on a Banach algebra A belongs to the class Σ_0 if and only if the equality

$$\max\{|\lambda| : \lambda \in \tilde{\sigma}(a)\} = r(a) (= \text{spectral radius of } a)$$

holds for every $a \in A$, and this is equivalent to one of the conditions (3) or (4).

Let us also recall the notions of joint capacities of elements of a Banach algebra A. An arbitrary polynomial p of degree k in n variables may be written in the form

$$p(z) = \sum_{|j| \le k} a_j z^j,$$

where we use the notation:

$$j = (j_1, \ldots, j_n) \quad \text{an } n\text{-tuple of non-negative integers},$$
$$|j| = j_1 + \cdots + j_n, \ z = (z_1, \ldots, z_n), \ z^j = z_1^{j_1} z_2^{j_2} \ldots z_n^{j_n}.$$

The coefficients a_j are complex numbers. Let $v(p)$ and $u(p)$ be defined as

$$v(p) = \sum_{|j| \le k} a_j, \quad u(p) = \sum_{|j| \le k} |a_j|.$$

Denote by $P_k^1(n)$ ($\tilde{P}_k^1(n)$ respectively) the set of all polynomials p of degree k in n variables such that $v(p) = 1$ ($u(p) = 1$ respectively). For an n-tuple $(a_1, \ldots, a_n)$ of mutually commuting elements in A put

$$\operatorname{cap}_k(a_1, \ldots, a_n) = \inf\{\|p(a_1, \ldots, a_n)\| : p \in P_k^1(n)\},$$
$$\text{and} \quad \operatorname{cap}(a_1, \ldots, a_n) = \lim_k (\operatorname{cap}_k(a_1, \ldots, a_n))^{1/k}$$
$$(\tilde{\operatorname{cap}}_k(a_1, \ldots, a_n) = \inf\{\|p(a_1, \ldots, a_n)\| : p \in \tilde{P}_k^1(n)\},$$
$$\text{and} \quad \tilde{\operatorname{cap}}(a_1, \ldots, a_n) = \liminf_k (\tilde{\operatorname{cap}}_k(a_1, \ldots, a_n))^{1/k}, \text{ respectively}).$$

The quantities $\operatorname{cap}_k(a_1, \ldots, a_n)$ and $\tilde{\operatorname{cap}}_k(a_1, \ldots, a_n)$ are the joint capacities of $(a_1, \ldots, a_n)$ in the sense of [4] and [6], respectively. In particular, let $A = \mathcal{C}(\Omega)$ be the Banach algebra of all continuous complex-valued functions defined on the compact subset $\Omega \subset \mathbb{C}^n$. Let $\pi_j(z) = z_j$ be the j-th projection, $j = 1, 2, \ldots, n$. Then the quantities $\operatorname{cap}(\pi_1, \ldots, \pi_n) = \operatorname{Cap} \Omega$ and $\tilde{\operatorname{cap}}(\pi_1, \ldots, \pi_n) = \tilde{\operatorname{Cap}}\, \Omega$ are called the joint capacities of the set Ω (see [4], [5], and [6]).

In [5], it is proved that, for n-tuples of mutually commuting operators on a complex Banach space, many known spectroids have the same capacity (in any of the sense defined above). In this paper, we show that the same is true for all subspectra of class Σ_0 on a complex Banach algebra with unit.

2. Results.

Theorem. *Let A be a complex Banach algebra with unit. For an arbitrary subspectrum $\tilde{\sigma}$ on A, the following conditions are equivalent:*

(i) *$\tilde{\sigma}$ is of class Σ_0, i.e. $r_{\tilde{\sigma}}(a_1, \ldots, a_n) = r(a_1, \ldots, a_n)$ for all $(a_1, \ldots, a_n) \in A^n_{\mathrm{com}}$ and $n = 1, 2, \ldots$;*

(ii) *$\max\{|\lambda| \in \tilde{\sigma}(a)\} = r(a)$ for every $a \in A$;*

(iii) *$\Gamma(B) \subset \Delta(\tilde{\sigma}; B)$ for every closed commutative subalgebra B of A containing the unit of A;*

(iii′) *$\{(\phi(a_1), \ldots, \phi(a_n)) : \phi \in \Gamma(B)\} \subset \tilde{\sigma}(a_1, \ldots, a_n)$ for every $(a_1, \ldots, a_n) \in A^n_{\mathrm{com}}$, $n = 1, 2, \ldots$, and each closed commutative subalgebra B of A containing the unit of A and the elements $a_1, \ldots, a_n$;*

(iii″) *$\{\phi(a) : \phi \in \Gamma(B)\} \subset \tilde{\sigma}(a)$ for every $a \in A$ and each closed commutative subalgebra B of A containing the unit of A and the element a;*

(iv) *$\Gamma(\mathcal{A}) \subset \Delta(\tilde{\sigma}; \mathcal{A})$ for every maximal abelian subalgebra $\mathcal{A}$ of A;*

(iv′) *$\{(\phi(a_1), \ldots, \phi(a_n)) : \phi \in \Gamma(\mathcal{A})\} \subset \tilde{\sigma}(a_1, \ldots, a_n)$ for every $(a_1, \ldots, a_n) \in A^n_{\mathrm{com}}$, $n = 1, 2, \ldots$, and each maximal abelian subalgebra $\mathcal{A}$ of A containing the elements $a_1, \ldots, a_n$;*

(iv″) *$\{\phi(a) : \phi \in \Gamma(\mathcal{A})\} \subset \tilde{\sigma}(a)$ for every $a \in A$ and each maximal abelian subalgebra $\mathcal{A}$ of A containing a;*

(v) *$\Gamma([a_1, \ldots, a_n]) \subset \Delta(\tilde{\sigma}; [a_1, \ldots, a_n])$ for every $(a_1, \ldots, a_n) \in A^n_{\mathrm{com}}$ and $n = 1, 2, \ldots$;*

(v′) *$\{(\phi(a_1), \ldots, \phi(a_n)) : \phi \in \Gamma([a_1, \ldots, a_n])\} \subset \tilde{\sigma}(a_1, \ldots, a_n)$ for every $(a_1, \ldots, a_n) \in A^n_{\mathrm{com}}$ and $n = 1, 2, \ldots$;*

(v″) *$\{\phi(a) : \phi \in \Gamma([a])\} \subset \tilde{\sigma}(a)$ for every $a \in A$.*

Proof: The implications (i) $\Rightarrow$ (ii), (iii) $\Rightarrow$ (iii$'$) $\Rightarrow$ (iii$''$), (iv) $\Rightarrow$ (iv$'$) $\Rightarrow$ (iv$''$), (v) $\Rightarrow$ (v$'$) $\Rightarrow$ (v$''$), (iii) $\Rightarrow$ (iv), (iii$'$) $\Rightarrow$ (iv$'$), (iii$''$) $\Rightarrow$ (iv$''$), (iii) $\Rightarrow$ (v), (iii$'$) $\Rightarrow$ (v$'$), and (iii$''$) $\Rightarrow$ (v$''$) are obvious. To see that (ii) $\Rightarrow$ (iii), notice that by Theorem B

$$\tilde{\sigma}(a) = \{\phi(a) : \phi \in \Delta(\tilde{\sigma}; B)\}$$

for every a in a given closed commutative subalgebra B of A containing the unit. This implies that

$$\begin{gathered}\max\{|\phi(a)| : \phi \in \Delta(\tilde{\sigma}; B)\} = \max\{|\lambda| : \lambda \in \tilde{\sigma}(a)\} = \\ = r(a) = \max\{|\phi(a)| : \phi \in \mathfrak{M}(B)\}.\end{gathered}$$

Hence by the definition of the Shilov boundary (cf. [8, p. 61]), we obtain $\Gamma(B) \subset \Delta(\tilde{\sigma}; B)$.

Now, we prove that (iv$''$) $\Rightarrow$ (ii). Take an arbitrary maximal abelian subalgebra $\mathcal{A}$ of A and $a \in \mathcal{A}$. Since by (2) and our assumption

$$\{\phi(a) : \phi \in \Gamma(\mathcal{A})\} \subset \tilde{\sigma}(a) \subset \sigma(a) = \sigma^{\mathcal{A}}(a),$$

we get

$$r(a) = \max\{|\phi(a)| : \phi \in \Gamma(\mathcal{A})\} \le \max\{|\lambda| : \lambda \in \tilde{\sigma}(a)\} \le r(a)$$

which gives the required equality.

The proof of the implication (v$''$) $\Rightarrow$ (ii) is similar.

To conclude the proof, we show that (v) $\Rightarrow$ (i). In view of Theorem A we have

$$\{(\phi(a_1), \dots, \phi(a_n)) : \phi \in \Gamma([a_1, \dots, a_n])\} \subset \tilde{\sigma}(a_1, \dots, a_n) \subset \sigma^{[a_1,\dots,a_n]}(a_1, \dots, a_n).$$

This implies

$$\begin{gathered}\gamma(a_1, \dots, a_n) = \max\left\{\left(\sum_{j=1}^{n} |\phi(a_j)|^2\right)^{1/2} : \phi \in \Gamma([a_1, \dots, a_n])\right\} \le \\ \le r_{\tilde{\sigma}}(a_1, \dots, a_n) \le r(a_1, \dots, a_n).\end{gathered}$$

Therefore it is enough to prove that $\gamma(a_1, \dots, a_n) = r(a_1, \dots, a_n)$. Suppose that it is not true. Hence

$$\gamma(a_1, \dots, a_n) < \left(\sum_{j=1}^{n} |\phi_0(a_j)|^2\right)^{1/2}$$

for some $\phi_0 \in \mathfrak{M}([a_1, \dots, a_n])$. Take $\vartheta_j \in \mathbb{R}$ such that $e^{i\vartheta_j}\phi_0(a_j) = |\phi_0(a_j)|$ for $j = 1, \dots, n$. Then

$$\sum_{j=1}^{n} |\phi_0(a_j)|^2 = \sum_{j=1}^{n} e^{2i\vartheta_j}\phi_0(a_j^2) = \phi_0\left(\sum_{j=1}^{n} e^{2i\vartheta_j} a_j^2\right)$$

and thus

$$\gamma(a_1,\dots,a_n)^2 < \phi_0\left(\sum_{j=1}^n e^{2i\vartheta_j}a_j^2\right) \le$$
$$\le \max\left\{|\phi\left(\sum_{j=1}^n e^{2i\vartheta_j}a_j^2\right)| : \phi \in \mathfrak{M}([a_1,\dots,a_n])\right\} =$$
$$= \max\left\{|\phi\left(\sum_{j=1}^n e^{2i\vartheta_j}a_j^2\right)| : \phi \in \Gamma([a_1,\dots,a_n])\right\} \le$$
$$\le \max\left\{\sum_{j=1}^n |\phi(a_j)|^2 : \phi \in \Gamma([a_1,\dots,a_n])\right\} = \gamma(a_1,\dots,a_n)^2.$$

This is impossible and so we are done. □

Remarks. 1. If for every maximal abelian subalgebra $\mathcal{A}$ of the Banach algebra A we have $\Gamma(\mathcal{A}) = \mathfrak{M}(\mathcal{A})$, then there is only one subspectrum on A satisfying the equivalent conditions of the above Theorem. It coincides with the uniquely determined spectrum in this case (cf. [7, Cor. 5.6]). Such a situation holds e.g. for the algebra $\mathcal{M}_n$ of all complex $n \times n$ matrices or for the group algebra $L^1(G)$ of a compact group G. By Theorem 5.7 of [7], we also see that all such subspectra coincide on n-tuples of commuting normal operators on a complex Hilbert space H.

2. The conditions of the Theorem are no longer equivalent in the class of all spectroids. E.g. if we consider the spectroid $\sigma_0(a_1,\dots,a_n) = \prod_{j=1}^n \sigma(a_j)$ on the algebra of all $n\times n$ matrices, then it is easy to see that it satisfies the conditions (ii), (iii′), (iii″), (iv′), (iv″), (v′), and (v″), but it is not of class Σ_0. (Notice that the conditions (iii), (iv), and (v) have no sense for spectroids.)

Now we proceed to capacities of subspectra of class Σ_0.

Proposition. *Let A be a complex unital Banach algebra. For an arbitrary n-tuple $(a_1,\dots,a_n) \in A^n_{\mathrm{com}}$, $n = 1,2,\dots$, and each subspectrum $\tilde\sigma$ on A belonging to the class Σ_0, the following relations are satisfied:*

(a) $\operatorname{cap}(a_1,\dots,a_n) = \operatorname{Cap}\tilde\sigma(a_1,\dots,a_n)$;
(a′) $\widetilde{\operatorname{cap}}(a_1,\dots,a_n) = 0$ *if and only if* $\widetilde{\operatorname{Cap}}\,\tilde\sigma(a_1,\dots,a_n) = 0$;
(b) $\operatorname{Cap}\tilde\sigma(a_1,\dots,a_n) = \operatorname{Cap}\sigma^{[a_1,\dots,a_n]}(a_1,\dots,a_n)$;
(b′) $\widetilde{\operatorname{Cap}}\,\tilde\sigma(a_1,\dots,a_n) = \widetilde{\operatorname{Cap}}\,\sigma^{[a_1,\dots,a_n]}(a_1,\dots,a_n)$.

Proof: In view of Theorem 2 of [4] and Theorem 1 of [6], it is enough to prove (b) and (b′). Let us start with the proof of (b). By Theorem A and the Theorem, we have

$$\{(\phi(a_1),\dots,\phi(a_n)) : \phi \in \Gamma([a_1,\dots,a_n])\} \subset \tilde\sigma(a_1,\dots,a_n) \subset \sigma^{[a_1,\dots,a_n]}(a_1,\dots,a_n).$$

Now we can repeat the argument from the proof of Theorem 5 in [5]. Taking an arbitrary polynomial p in n variables, we get

$$\begin{aligned}&\max\{|p(\phi(a_1),\ldots,\phi(a_n))| : \phi \in \Gamma([a_1,\ldots,a_n])\} =\\ &= \max\{|p(\phi(a_1),\ldots,\phi(a_n))| : \phi \in \mathfrak{M}([a_1,\ldots,a_n])\} =\\ &= \max\{|p(\lambda)| : \lambda \in \sigma^{[a_1,\ldots,a_n]}(a_1,\ldots,a_n)\}.\end{aligned}$$

Polynomial convexity of the joint spectrum $\sigma^{[a_1,\ldots,a_n]}(a_1,\ldots,a_n)$ (see [8, p. 78]) implies the following equality for the polynomially convex hull $P\big(\{(\phi(a_1),\ldots,\phi(a_n)) : \phi \in \Gamma([a_1,\ldots,a_n])\}\big)$:

$$P\big(\{(\phi(a_1),\ldots,\phi(a_n)) : \phi \in \Gamma([a_1,\ldots,a_n])\}\big) = \sigma^{[a_1,\ldots,a_n]}(a_1,\ldots,a_n).$$

Since obviously $\operatorname{Cap}\Omega = \operatorname{Cap}P(\Omega)$ for an arbitrary compact subset Ω of $\mathbb{C}^n$, we get

$$\begin{aligned}\operatorname{Cap}\{(\phi(a_1),\ldots,\phi(a_n)) : \phi \in \Gamma([a_1,\ldots,a_n])\} = \operatorname{Cap}\tilde{\sigma}(a_1,\ldots,a_n) =\\ = \operatorname{Cap}\sigma^{[a_1,\ldots,a_n]}(a_1,\ldots,a_n).\end{aligned}$$

The proof of (b′) is analogous. □

Remarks. 1. The converse of the Proposition is not true. Since on the algebra $\mathcal{M}_2$ of all 2×2 matrices we have a single point subspectrum which evidently satisfies the conditions (a), (a′), (b), and (b′), but is not a subspectrum of class Σ_0 (cf. [3]).

2. From the above proof, it follows that for a given n-tuple of elements in a complex Banach algebra A, all subspectra of class Σ_0 have the same polynomially convex hull.

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