Comment.Math.Univ.Carolin. 32,4 (1991)609–614

On centralizers of semiprime rings

Borut Zalar

Abstract. Let $\mathcal{K}$ be a semiprime ring and $T : \mathcal{K} \to \mathcal{K}$ an additive mapping such that $T(x^2) = T(x)x$ holds for all $x \in \mathcal{K}$. Then T is a left centralizer of $\mathcal{K}$. It is also proved that Jordan centralizers and centralizers of $\mathcal{K}$ coincide.

Keywords: semiprime ring, left centralizer, centralizer, Jordan centralizer

Classification: 16N60, 16W10, 16W25

Throughout this paper, $\mathcal{K}$ will represent an associative ring with the center $\mathcal{Z}$. $\mathcal{K}$ is called prime if $a\mathcal{K}b = (0)$ implies $a = 0$ or $b = 0$ and semiprime if $a\mathcal{K}a = (0)$ implies $a = 0$. A mapping $D : \mathcal{K} \to \mathcal{K}$ is called derivation if $D(xy) = D(x)y + xD(y)$ holds for all $x, y \in \mathcal{K}$. A left (right) centralizer of $\mathcal{K}$ is an additive mapping $T : \mathcal{K} \to \mathcal{K}$ which satisfies $T(xy) = T(x)y\,(T(xy) = xT(y))$ for all $x, y \in \mathcal{K}$. If $a \in \mathcal{K}$, then $L_a(x) = ax$ is a left centralizer and $R_a(x) = xa$ is a right centralizer.

If $\mathcal{K}$ is a ring with involution $*$, then every additive mapping $E : \mathcal{K} \to \mathcal{K}$ which satisfies $E(x^2) = E(x)x^* + xE(x)$ for all $x \in \mathcal{K}$ is called Jordan $*$-derivation. These mappings are closely connected with a question of representability of quadratic forms by bilinear forms. Some algebraic properties of Jordan $*$-derivations are considered in [1], where further references can be found. For quadratic forms see [6].

In [2] M. Brešar and the author obtained a representation of Jordan $*$-derivations in terms of left and right centralizers on the algebra of compact operators on a Hilbert space. We arrived at a problem whether an additive mapping T which satisfies a weaker condition $T(x^2) = T(x)x$ is automatically a left centralizer. We proved in [2] that this is in fact so if $\mathcal{K}$ is a prime ring (generally without involution). In the present paper, we generalize this result on semiprime rings.

Our second result is motivated by the study of Jordan mappings in associative rings. If we introduce a new product in $\mathcal{K}$ given by $x \circ y = xy + yx$, then Jordan derivation is an additive mapping D which satisfies $D(x \circ y) = D(x) \circ y + x \circ D(y)$ for all $x, y \in \mathcal{K}$ and Jordan homomorphism is an additive mapping A which satisfies $A(x \circ y) = A(x) \circ A(y)$ for all $x, y \in \mathcal{K}$. Therefore we can define a Jordan centralizer to be an additive mapping T which satisfies $T(x \circ y) = T(x) \circ y = x \circ T(y)$. Since the product $\circ$ is commutative, there is no difference between the left and right Jordan centralizers.

A centralizer of $\mathcal{K}$ is an additive mapping which is both left and right centralizer. An easy computation shows that every centralizer is also a Jordan centralizer. We prove as our second result that every Jordan centralizer of a semiprime ring is a centralizer.

*This research was partially supported by the Research Council of Slovenija.

1. The first result.

To prove our first result, we need three simple lemmas which we now state.

Lemma 1.1. *Let $\mathcal{K}$ be a semiprime ring. If $a, b \in \mathcal{K}$ are such that $axb = 0$ for all $x \in \mathcal{K}$, then $ab = ba = 0$.*

PROOF: Take any $x \in \mathcal{K}$.

$$(ab)x(ab) = a(bxa)b = 0,$$
$$(ba)x(ba) = b(axb)b = 0.$$

By the semiprimeness of $\mathcal{K}$, it follows $ab = ba = 0$. □

Lemma 1.2. *Let $\mathcal{K}$ be a semiprime ring and $A : \mathcal{K} \times \mathcal{K} \to \mathcal{K}$ biadditive mappings. If $A(x,y)wB(x,y) = 0$ for all $x, y, w \in \mathcal{K}$, then $A(x,y)wB(u,v) = 0$ for all $x, y, u, v, w \in \mathcal{K}$.*

PROOF: First we shall replace x with $x + u$.

$$A(x+u,y)wB(x+u,y) = 0,$$
$$A(x,y)wB(u,y) = -A(u,y)wB(x,y).$$

We used the biadditivity of A and B.

$$\begin{aligned}(A(x,y)wB(u,y))z(A(x,y)wB(u,y)) &= \\ &= -A(u,y)wB(u,y)zA(x,y)wB(x,y) = 0.\end{aligned}$$

Hence $A(x,y)wB(u,y) = 0$ by semiprimeness of $\mathcal{K}$. Now we replace y by $y + v$ and obtain the assertion of the lemma with a similar approach as above. □

Lemma 1.3. *Let $\mathcal{K}$ be a semiprime ring and $a \in \mathcal{K}$ some fixed element. If $a[x,y] = 0$ for all $x, y \in \mathcal{K}$, then there exists an ideal $\mathcal{U}$ of $\mathcal{K}$ such that $a \in \mathcal{U} \subset \mathcal{Z}$ holds.*

PROOF:

$$\begin{aligned}[z,a]x[z,a] = zax[z,a] - azx[z,a] &= \\ &= za[z,xa] - za[z,x]a - a[z,zxa] + a[z,zx]a = 0.\end{aligned}$$

Hence $a \in \mathcal{Z}$. Since $zaw[x,y] = 0$ for all $z, w, x, y \in \mathcal{K}$ we can repeat the above argument with zaw instead of a to obtain $\mathcal{K}a\mathcal{K} \subset \mathcal{Z}$ and now it is obvious that the ideal generated by a is central. □

Proposition 1.4. *Let $\mathcal{K}$ be a semiprime ring of characteristic not two and $T : \mathcal{K} \to \mathcal{K}$ an additive mapping which satisfies $T(x^2) = T(x)x$ for all $x \in \mathcal{K}$. Then T is a left centralizer.*

PROOF:

$$T(x^2) = T(x)x. \tag{1}$$

If we replace x by $x+y$, we get

$$T(xy+yx) = T(x)y + T(y)x. \tag{2}$$

By replacing y with $xy+yx$ and using (2), we arrive at

$$T(x(xy+yx)+(xy+yx)x) = T(x)xy + T(x)yx + T(x)yx + T(y)x^2. \tag{3}$$

But this can also be calculated in a different way.

$$T(x^2y+yx^2) + 2T(xyx) = T(x)xy + T(y)x^2 + 2T(xyx). \tag{4}$$

Comparing (3) and (4) we obtain

$$T(xyx) = T(x)yx. \tag{5}$$

If we linearize (5), we get

$$T(xyz+zyx) = T(x)yz + T(z)yx. \tag{6}$$

Now we shall compute $j = T(xyzyx + yxzxy)$ in two different ways. Using (5) we have

$$j = T(x)yzyx + T(y)xzxy. \tag{7}$$

Using (6) we have

$$j = T(xy)zyx + T(yx)zxy. \tag{8}$$

Comparing (7) and (8) and introducing a biadditive mapping $B(x,y) = T(xy) - T(x)y$ we arrive at

$$B(x,y)zyx + B(y,x)zxy = 0. \tag{9}$$

Equality (2) can be rewritten in this notation as $B(x,y) = -B(y,x)$. Using this fact and equality (9) we obtain

$$B(x,y)z[x,y] = 0. \tag{10}$$

Using first Lemma 1.2 and then Lemma 1.1 we have

$$B(x,y)z[u,v] = 0. \tag{11}$$

Now fix some $x,y \in \mathcal{K}$ and write B instead of $B(x,y)$ to simplify further writing. Using Lemma 1.3 we get the existence of an ideal $\mathcal{U}$ such that $B \in \mathcal{U} \subset \mathcal{Z}$ holds. In particular, $bB, Bb \in \mathcal{Z}$ for all $b \in \mathcal{K}$. This gives us

$$x \cdot B^2y = B^2y \cdot x = yB^2 \cdot x = y \cdot B^2x.$$

This gives us $4T(x\cdot B^2y) = 4T(y\cdot B^2x)$. Both sides of this equality will be computed in few steps using (2) and the above remarks.

$$
\begin{gathered}
2T(xB^2y + B^2yx) = 2T(yB^2x + B^2xy),\\
2T(x)B^2y + 2T(B^2y)x = 2T(y)B^2x + 2T(B^2x)y,\\
2T(x)B^2y + T(B^2y + yB^2)x = 2T(y)B^2x + T(B^2x + xB^2)y,\\
2T(x)B^2y + T(B)Byx + T(y)B^2x = 2T(y)B^2x + T(B)Bxy + T(x)B^2y,\\
T(x)B^2y + T(B)Byx = T(y)B^2x + T(B)Bxy.
\end{gathered}
$$

Since

$$Byx = By\cdot x = x\cdot By = xBy = Bxy,$$

we obtain

$$T(x)B^2y = T(y)B^2x. \tag{12}$$

On the other hand, we also have

$$
\begin{gathered}
4T(xyB^2) = 4T(xB\cdot yB),\\
2T(xyB^2 + B^2xy) = 2T(xByB + yBxB),\\
2T(xy)B^2 + 2T(B)Bxy = 2T(Bx)By + 2T(By)Bx,\\
2T(xy)B^2 + 2T(B)Bxy = T(xB + Bx)By + T(yB + By)Bx,\\
2T(xy)B^2 + 2T(B)Bxy = T(x)B^2y + T(B)Bxy + T(y)B^2x + T(B)Bxy,\\
2T(xy)B^2 = T(x)yB^2 + T(y)xB^2.
\end{gathered}
$$

Using (12) we finally arrive at $T(xy)B^2 = T(x)yB^2$. But this means that $B^3 = 0$ so that

$$
\begin{aligned}
B^2\mathcal{K}B^2 &= B^4\mathcal{K} = (0),\\
B\mathcal{K}B &= B^2\mathcal{K} = (0),
\end{aligned}
$$

which implies $B = 0$ and the proof is complete. □

It was proved in [5] that left centralizers of semisimple Banach algebras are automatically continuous.

Corollary 1.5. *Let $\mathcal{A}$ be a semisimple Banach algebra and $T : \mathcal{A} \to \mathcal{A}$ an additive mapping such that $T(x^2) = T(x)x$ holds for all $x \in \mathcal{A}$. Then T is a continuous linear operator.*

Proof: Every semisimple Banach algebra is a semiprime ring. Linearity follows from

$$(T(\lambda x) - \lambda T(x))y = T(\lambda x)y - T(x)\lambda y = T(\lambda xy) - T(x\lambda y) = 0.$$

□

The concept of a left and right centralizer are symmetric, therefore it is obvious that every additive mapping T which satisfies $T(x^2) = xT(x)$ is a right centralizer if $\mathcal{K}$ is semiprime ring of characteristic not two.

2. The second result.

We again divide the proof in few lemmas.

Lemma 2.1. *Let $\mathcal{K}$ be a semiprime ring, D a derivation of $\mathcal{K}$ and $a \in \mathcal{K}$ some fixed element.*

(i) *$D(x)D(y) = 0$ for all $x, y \in \mathcal{K}$ implies $D = 0$.*
(ii) *$ax - xa \in \mathcal{Z}$ for all $x \in \mathcal{K}$ implies $a \in \mathcal{Z}$.*

Proof: (i)

$$D(x)yD(x) = D(x)D(yx) - D(x)D(y)x = 0.$$

(ii) Define $D(x) = ax - xa$. It is easy to see that D is a derivation. Since $D(x) \in \mathcal{Z}$ for all $x \in \mathcal{K}$, we have $D(y)x = xD(y)$ and also $D(yz)x = xD(yz)$. Hence

$$D(y)zx + yD(z)x = xD(y)z + xyD(z),$$
$$D(y)[z,x] = D(z)[x,y].$$

Now take $z = a$. Obviously $D(a) = 0$, so we obtain

$$0 = D(y)[a,x] = D(y)D(x).$$

From (i) we get $D = 0$ and hence $a \in \mathcal{Z}$. □

Lemma 2.2. *Let $\mathcal{K}$ be a semiprime ring and $a \in \mathcal{K}$ some fixed element. If $T(x) = ax + xa$ is a Jordan centralizer, then $a \in \mathcal{Z}$.*

Proof:

$$T(xy + yx) = T(x)y + yT(x)$$

gives us

$$axy + ayx + xya + yxa = (ax + xa)y + y(ax + xa),$$
$$ayx + xya - xay - yax = 0 = (ay - ya)x - x(ay - ya).$$

The second part of Lemma 2.1 now gives us $a \in \mathcal{Z}$. □

Lemma 2.3. *Let $\mathcal{K}$ be a semiprime ring. Then every Jordan centralizer of $\mathcal{K}$ maps $\mathcal{Z}$ into $\mathcal{Z}$.*

Proof: Take any $c \in \mathcal{Z}$ and denote $a = T(c)$.

$$2T(cx) = T(cx + xc) = T(c)x + xT(c) = ax + xa.$$

A straightforward verification shows that $S(x) = 2T(cx)$ is also a Jordan centralizer. By Lemma 2.2, we have $T(c) \in \mathcal{Z}$. □

Lemma 2.4. *Let $\mathcal{K}$ be a semiprime ring and $a, b \in \mathcal{K}$ two fixed elements. If $ax = xb$ for all $x \in \mathcal{K}$, then $a = b \in \mathcal{Z}$.*

Proof: $xyb = axy = xby$ implies $x[b,y] = 0$ for all $x, y \in \mathcal{K}$. Hence $[b,y]x[b,y] = 0$ and by semiprimeness of $\mathcal{K}$, we have $b \in \mathcal{Z}$. Therefore $ax = bx$ and this clearly implies $a = b$. □

Proposition 2.5. *Every Jordan centralizer of semiprime ring $\mathcal{K}$ of characteristic not two is a centralizer.*

Proof: Let T be a Jordan centralizer, i.e.

$$T(xy + yx) = T(x)y + yT(x) = xT(y) + T(y)x.$$

If we replace y by $xy + yx$, we get

$$\begin{aligned} T(x)(xy + yx) + (xy + yx)T(x) &= T(xy + yx)x + xT(xy + yx) = \\ &= (T(x)y + yT(x))x + x(T(x)y + yT(x)). \end{aligned}$$

Now it follows that $[T(x), x]y = y[T(x), x]$ holds for all $x, y \in \mathcal{K}$ and so $[T(x), x] \in \mathcal{Z}$. The next goal is to show that $[T(x), x] = 0$ holds. Take any $c \in \mathcal{Z}$.

$$2T(cx) = T(cx + xc) = T(c)x + xT(c) = 2T(x)c.$$

Using Lemma 2.3 we get

$$\begin{gathered} T(cx) = T(x)c = T(c)x, \\ [T(x), x]c = T(x)xc - xT(x)c = T(c)x^2 - xT(c)x = 0. \end{gathered}$$

Since $[T(x), x]$ itself is central element, our goal is achieved.

$$2T(x^2) = T(xx + xx) = T(x)x + xT(x) = 2T(x)x = 2xT(x).$$

Proposition 1.4 now concludes the proof. □

References

[1] Brešar M., Vukman J., *On some additive mapping in rings with involution*, Aequationes Math. **38** (1989), 178–185.

[2] Brešar M., Zalar B., *On the structure of Jordan ∗-derivations*, Colloquium Math., to appear.

[3] Herstein I.N., *Topics in ring theory*, University of Chicago Press, 1969.

[4] ———, *Theory of rings*, University of Chicago Press, 1961.

[5] Johnson B.E., Sinclair A.M., *Continuity of derivations and a problem of Kaplansky*, Amer. J. Math. **90** (1968), 1067–1073.

[6] Šemrl P., *Quadratic functionals and Jordan ∗-derivations*, Studia Math. **97** (1991), 157–165.

Institute of Mathematics, University of Ljubljana, Jadranska 21, 61000 Ljubljana, Slovenija

(*Received* May 21, 1991)