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Distributivity law for the normal triples in the category of compacta and lifting of functors to the categories of algebras

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Abstract. We investigate the triples in the category of compacta whose functorial parts are normal functors in the sense of E.V. Shchepin (normal triples). The problem of lifting of functors to the categories of algebras of the normal triples is considered. The distributive law for normal triples is completely described.

Keywords: triple, normal functor, category of algebras, distributive law, compact Hausdorff space, power functor, projective triple

Classification: 54B30, 54D30, 18C15, 18C20

Let us recall some definitions from the category theory; see [1], [5] for details. A triple $\mathbb{T} = (T, \eta, \mu)$ in the category $\mathcal{E}$ consists of an endofunctor $T : \mathcal{E} \to \mathcal{E}$ and natural transformations $\eta : 1_{\mathcal{E}} \to T$ (unity), $\mu : T^2 \to T$ (multiplication) satisfying the relations $\mu \circ T\eta = \mu \circ \eta T = 1_T$ and $\mu \circ \mu T = \mu \circ T\mu$.

A couple (X, ξ), for which $\xi : TX \to X$ is a morphism such that $\xi \circ \eta X = 1_X, \xi \circ T\xi = \xi \circ \mu X$ is called $\mathbb{T}$-algebra. A morphism $f : X \to X'$ is said to be the morphism of $\mathbb{T}$-algebra (X, ξ) to a $\mathbb{T}$-algebra (X', ξ') if $f \circ \xi = \xi' \circ Tf$. The category of $\mathbb{T}$-algebras is denoted by $\mathcal{E}^{\mathbb{T}}$, and by $U = U^{\mathbb{T}} : \mathcal{E}^{\mathbb{T}} \to \mathcal{E}$, we denote the forgetful functor: $U(X, \xi) = X$, $Uf = f$.

A natural transformation $\psi : T \to T'$ is called a morphism of the triple $\mathbb{T}$ to the triple $\mathbb{T}' = (T', \eta', \mu')$ if $\psi \circ \eta = \eta'$ and $\psi \circ \mu = \mu' \circ \psi T' \circ T\psi$.

A triple $\mathbb{T} = (T, \eta, \mu)$ is called projective [8] if there exists a natural transformation $\pi : T \to 1_{\mathcal{E}}$ (projection), for which $\pi \circ \eta = 1, \pi \circ \mu = \pi \circ \pi T = \pi \circ T\pi$. Note that $\mathbb{T}$ is projective iff there exists a morphism of triples

$$\pi : \mathbb{T} \to \mathbb{I} = (1_{\mathcal{E}}, 1, 1).$$

A distributive law of a triple $\mathbb{T}_1 = (T_1, \eta_1, \mu_1)$ over a triple $\mathbb{T}_2 = (T_2, \eta_2, \mu_2)$ is a natural transformation $\lambda : T_1T_2 \to T_2T_1$ satisfying the relations:

(D1) $\lambda \circ T_1\eta_2 = \eta_2T_1$;
(D2) $\lambda \circ \eta_1T_2 = T_2\eta_1$;
(D3) $\lambda \circ T_1\mu_2 = \mu_2T_1 \circ T_2\lambda \circ \lambda T_2$;
(D4) $\lambda \circ \mu_1T_2 = T_2\mu_1 \circ \lambda T_1 \circ T_1\lambda$

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(see [1], [2]).

In this paper, we consider triples in the category Comp of compacta (= compact Hausdorff spaces) and continuous mappings whose functorial parts are normal functors in the sense of E.V. Shchepin [7] (normal triples). The problem of lifting of functors to the category of algebras of normal triple can be completely solved in the case of functors of finite degree (§2). In §3, the situation of the existence of a distributive law for normal triples is completely described.

1. Normal functors and triples.

All spaces and mappings under discussion are taken from the category Comp.

Definition 1 [7]**.** A functor $F : \mathrm{Comp} \to \mathrm{Comp}$ is called normal if it is continuous, monomorphic, epimorphic, preserves weight, intersections, preimages, singleton and empty space.

Remark that in [7], the notion of continuity is used in the sense of preserving limits of inverse spectra. All the definitions and results concerning normal functors which we shall use in the sequel can be found in [3], [7]. See [3], [6], [9] for the examples of triples whose functorial parts are normal.

Recall that the support of a point $a \in FX$, where F is a normal functor, is the set $\mathrm{supp}_F(a) = \bigcap\{A \mid A \text{ is closed in } X \text{ and } a \in FA \subset FX\}$ [7]. The cardinality of $\mathrm{supp}_F(a)$ is called degree of $a \in FX$ and is denoted by $\deg(a)$. A functor F is said to be functor of degree $\leq n$ if $\deg(a) \leq n$ for each $a \in FX$.

For a couple (F_1, F_2) of normal functors and $a \in F_1F_2X$, define $\mathrm{Supp}_{F_1F_2}(a) = \{\mathrm{supp}_{F_2}(b) \mid b \in \mathrm{supp}_{F_1}(a)\}$.

If F is a normal functor and $a \in FX, \deg(a) < \omega$, then the subfunctor F_a of F, defined by $F_aY = \bigcap\{GY \mid G \text{ is a normal subfunctor of } F \text{ with } a \in GX\}$, is normal (see [10]).

Definition 2. A normal functor F is called profinitely power functor if for every $a \in FX, \deg(a) = n < \omega$, the functor F_a is isomorphic to the power functor $(-)^n$.

Definition 3. A normal functor F is called weakly bicommutative if for every $a \in FX, b \in FY$, there exists $c \in F(X \times Y)$ with $Fpr_1(c) = a, Fpr_2(c) = b$ (pr_i denotes the i-th projection).

Theorem 1. *A weakly bicommutative profinitely power normal functor is isomorphic to a power functor $(-)^\alpha$, $1 \leq \alpha \leq \omega$.*

PROOF: See [11]. □

Let (F_1, F_2) be a couple of normal functors, and $a \in F_2X_1$, $b \in F_1X_2$. For each $y \in X_2$, let $i_y : X_1 \to X_1 \times X_2$ be a map sending $x \in X_1$ to $(x, y) \in X_1 \times X_2$. Let $f_a : X_2 \to F_2(X_1 \times X_2)$ be a map acting by the formula $f_a(y) = F_2i_y(a)$, $y \in X_2$. Remark that in [3], the continuity of f_a is established. Put $a \oplus b = F_1f_a(b) \in F_1F_2(X_1 \times X_2)$.

Similarly, define the mapping $j_x : X_2 \to X_1 \times X_2$ by $j_x(y) = (x, y)$, $y \in X_2$, and the mapping $g_b : X_1 \to F_1(X_1 \times X_2)$ by $g_b(x) = F_1j_x(b)$, $x \in X_1$. Put $a\widetilde{\oplus}b = F_2g_b(a) \in F_2F_1(X_1 \times X_2)$.

In the sequel, we shall specialize the choice of a couple (F_1, F_2).

We have immediately

Lemma 1. *The operations* $\oplus$ *and* $\widetilde{\oplus}$ *are natural by both arguments.*

Let $\deg(F) = n < \omega$ and $a \in Fn, \deg(a) = n$. Let $(F/a)X = \{b \in F(n \times X) \mid Fpr_1(b) = a\}$. It is easy to see that F/a is a subfunctor of $F(n \times (-))$ and is isomorphic to the power functor $(-)^n$. For each $m \in n$, define the natural transformation $\pi_m : F/a \to 1_{\mathrm{Comp}}$ by the property: $\pi_m X(b) = y$, where $(m, y) \in \mathrm{supp}_F(b)$.

2. Lifting of functors of finite degree to the category of $\mathbb{T}$-algebras.

Let $\mathbb{T} = (T, \eta, \mu)$ be a triple on a category $\mathcal{E}$. By definition, a lifting of a functor $F : \mathcal{E} \to \mathcal{E}$ to the category $\mathcal{E}^{\mathbb{T}}$ is a functor $\overline{F} : \mathcal{E}^{\mathbb{T}} \to \mathcal{E}^{\mathbb{T}}$ such that $FU = U\overline{F}$.

The following theorem can be considered in some sense as dual to the theorem of J. Vinárek [8] concerning extensions of functors to the Kleisli category.

Theorem 2. *There exists a bijective correspondence between liftings of functor* F *to the category* $\mathcal{E}^{\mathbb{T}}$ *and natural transformations* $\delta : TF \to FT$*, such that*

(a) $\delta \circ \eta F = F\eta$ *and*
(b) $F\mu \circ \delta T \circ T\delta = \delta \circ \mu F$.

Sketch of the proof: Given a natural transformation $\delta : TF \to FT$ satisfying (a) and (b), define the lifting $\overline{F}$ using the formula $\overline{F}(X, \xi) = (FX, F\xi \circ \delta X)$.

Conversely, given a lifting $\tilde{F}$ of F, let $(FTX, \tilde{\mu}X) = \tilde{F}(TX, \mu X)$ and define the natural transformation $\tilde{\delta} : TF \to FT$ by $\tilde{\delta}X = \tilde{\mu}X \circ TF\eta X$. $\square$

Proposition 1. *There exists a lifting of the functor* T *to the category* $\mathcal{E}^{\mathbb{T}}$.

Proof: Put $\delta = T\eta \circ \mu$ and use Theorem 2. $\square$

The following proposition is dual to the corresponding result of J. Vinárek [8].

Proposition 2. *Let* $\mathbb{T}$ *be a projective triple. Then every endofunctor* $F : \mathcal{E} \to \mathcal{E}$ *has a lifting to the category* $\mathcal{E}^{\mathbb{T}}$.

Proof: Let $\pi : T \to 1_{\mathcal{E}}$ be the projection. It is easy to check that the natural transformation $\delta = F\eta \circ \pi F : TF \to FT$ satisfies the conditions (a) and (b) of Theorem 2. $\square$

Proposition 3. *Let* $\mathcal{E}$ *be a category with products. Then the power functor* $(-)^\alpha : \mathcal{E} \to \mathcal{E}$ *has a lifting to the category* $\mathcal{E}^{\mathbb{T}}$.

Proof: Define $\delta : T \circ (-)^\alpha \to (-)^\alpha \circ T$ by the conditions $pr_i \circ \delta X = Tpr_i, i \in \alpha$, and use Theorem 2. $\square$

The following theorem shows that in the case of normal triple $\mathbb{T}$ all the situations when there exists a lifting of a functor of finite degree to the category $\mathcal{E}^{\mathbb{T}}$ are described by Propositions 2 and 3.

Theorem 3. *Let a normal functor F of finite degree $n \geq 1$ has a lifting to the category* $\mathrm{Comp}^{\mathbb{T}}$ *of $\mathbb{T}$-algebras of a normal triple $\mathbb{T} = (T, \eta, \mu)$. Then either $F \cong (-)^n$ or $\mathbb{T}$ is a projective triple.*

PROOF: Here we use the denotation $\oplus$ for the case $F_1 = T, F_2 = F$. For each X, define the mapping $kX : n \times TX \to T(n \times X)$ by the formula $kX(m, b) = Tj_m(b), m \in n, b \in TX$ (see [3]).

Suppose F has a lifting to the category $\mathrm{Comp}^{\mathbb{T}}$ and let $\delta : TF \to FT$ be the natural transformation corresponding, by Theorem 2, to this lifting. Then for each $b \in TX$, we have $FTpr_1 \circ \delta(n \times X)(a \oplus b) = \delta X \circ TFpr_1(a \oplus b) = \delta X \circ \eta Fn(a) = F\eta n(a)$. Fix $a \in Fn$ with $\deg_F(a) = n$. Then $\delta(n \times X)(a \oplus b) \in FkX(n \times TX)$ and $FkX^{-1} \circ \delta(n \times X)(a \oplus b) \in (F/a)TX$, hence for each $m \in n$, we can correctly define the natural transformation $\psi_m : T \to T$ by the formula: $\psi_m X(b) = \pi_m TX \circ FkX^{-1} \circ \delta(n \times X)(a \oplus b), b \in TX$.

Lemma 2. *For each $d \in T(F/a)X \subset TF(n \times X)$, we have $\pi_m TX \circ FkX^{-1} \circ \delta(n \times X)(d) = \psi_m X \circ T\pi_m X(d)$.*

PROOF: There exists $b \in TX'$ for some X' and a mapping $\alpha : n \times X \to n \times X$ such that $TF\alpha(a \oplus b) = d$, and for each $l \in n$, there exists a mapping $\alpha_l : X' \to X$ such that $\alpha(l, x) = (l, \alpha_l(x)), x \in X$. Note that $T\alpha_l(b) = T\pi_l X(d)$.

Now, $\pi_m TX \circ FkX^{-1} \circ \delta(n \times X)(d) = \pi_m TX \circ FkX^{-1} \circ \delta(n \times X) \circ TF\alpha(a \oplus b) = \pi_m TX \circ FkX^{-1} \circ FT\alpha \circ \delta(n \times X')(a \oplus b) = T\alpha_m \circ \pi_m TX \circ FkX'^{-1} \circ \delta(n \times X')(a \oplus b) = T\alpha_m \circ \psi_m X'(b) = \psi_m X \circ T\alpha_m(b) = \psi_m X \circ T\pi_m X(d)$. The lemma is proved. □

Lemma 3. *$\psi_m = \{\psi_m X\}$ is a morphism of the triple $\mathbb{T}$ into itself.*

PROOF: Since the components of the natural transformation of normal functors do not enlarge supports, we have $\psi_m X \circ \eta X(x) = \eta X(x), x \in X$.

If $\mathcal{B} \in T^2 X$, then $\psi_m X \circ \mu X(\mathcal{B}) = \pi_m TX \circ FkX^{-1} \circ \delta(n \times X) \circ Tf_a \circ \mu X(\mathcal{B}) = \pi_m TX \circ FkX^{-1} \circ F\mu(n \times X) \circ \delta T(n \times X) \circ T\delta(n \times X) \circ T^2 f_a(\mathcal{B}) = \pi_m TX \circ F(1_n \times \mu X) \circ FkTX^{-1} \circ \delta T(n \times X) \circ T\delta(n \times X) \circ T^2 f_a(\mathcal{B}) = \mu X \circ \pi_m T^2 X \circ FkX^{-1} \circ \delta(n \times TX) \circ T(FkX^{-1} \circ \delta(n \times X) \circ Tf_a)(\mathcal{B}) = \mu X \circ \pi_m T^2 X \circ FkTX^{-1} \circ \delta(n \times TX)(a \oplus T\pi_m TX \circ T(FkX^{-1} \circ \delta(n \times X) \circ Tf_a)(\mathcal{B})) = \mu X \circ \pi_m T^2 X \circ FkTX^{-1} \circ \delta(n \times TX)(a \oplus T\psi_m X(\mathcal{B})) = \mu X \circ \psi_m TX \circ T\psi_m X(\mathcal{B})$, by Lemma 2. The lemma is proved. □

Return to the proof of Theorem 3. Each subfunctor $T_m = \psi_m(T)$ of T generates, by Lemma 3, the subtriple $\mathbb{T}_m = (T_m, \eta, \mu \mid T_m^2)$ of $\mathbb{T}$, $m \in n$. Suppose $\mathbb{T}$ is not projective, then $\mathbb{T}_m$ is not projective, too, and by the result of [12], $\deg(T_m) = \infty$.

From the elementary properties of normal functors, we can easily deduce that there exists a discrete two-point space $X = \{x_1, x_2\}$ and $b \in TX$ such that $\mathrm{Supp}_{FT}(\delta(n \times X)(a \oplus b)) = \{\{m\} \times X \mid m \in n\}$.

Show that for any mappings $f_1, f_2 : n \to n$ such that $f_1 \neq f_2$, we have $Ff_1(a) \neq Ff_2(a)$. Assuming the contrary, define the mappings $h_1, h_2 : n \times X \to n \times X$ by $h_1 = f_1 \times 1_X, h_2(m, x_i) = (f_i(m), x_i), m \in n, i = 1, 2$. Then obviously $TFh_1(a \oplus b) = TFh_2(a \oplus b)$, but $\mathrm{Supp}_{FT}(\delta(n \times X) \circ TFh_1(a \oplus b)) = \{h_1(\{m\} \times X) \mid m \in$

$n\} \neq \{h_2(\{m\} \times X) \mid m \in n\} = \mathrm{Supp}_{FT}(\delta(n \times X) \circ TFh_2(a \oplus b))$, and we get a contradiction.

In order to prove the isomorphism $F \cong (-)^n$, it is sufficient, by Theorem 1, to prove that for each $a' \in Fn$ there exists a mapping $f : n \to n$ such that $Ff(a) = a'$. Define the mapping $l : X \to F(n \times X)$ by $l(x_1) = a', l(x_2) = a$, and let $d = Tl(b)$.

For $i = 1, 2$, denote by $\mathcal{R}_i$ the partition of $n \times X$, whose unique non-trivial element is the set $n \times \{x_i\}$. Let $Y_i = (n \times X)/\mathcal{R}_i$ be the quotient space and $q_i : n \times X \to Y_i$ be the quotient mapping, $i = 1, 2$. Then obviously $TFq_1(d) = TFq_1(a \oplus b)$, and consequently for each $A \in \mathrm{Supp}_{FT}(FTq_1 \circ \delta(n \times X)(d))$, the set $A \cap (n \times \{x_2\})$ is a singleton. Similarly, for each $B \in \mathrm{Supp}_{FT}(FTq_2 \circ \delta(n \times X)(d))$, the set $B \cap (n \times \{x_1\})$ is a singleton.

Finally, we obtain that for each $C \in \mathrm{Supp}_{FT}(\delta(n \times X)(d))$ and for each $x \in X$, the set $C \cap (n \times \{x\})$ is a singleton. The intersections $C \cap (n \times \{x_2\})$, where $C \in \mathrm{Supp}_{FT}(\delta(n \times X)(d))$, form a disjoint cover of the set $n \times \{x_2\}$.

Now, we can construct the desired mapping f by the following manner. Let $m \in n$. There exists (unique as remarked above) the element $C_m \in \mathrm{Supp}_{FT}(\delta(n \times X)(d))$ such that $(m, x_2) \in C_m$. Take $f(m) \in n$ for which $(f(m), x_1) \in C_m$.

3. Distributive law for normal triples.

The main result of this paragraph is the following

Theorem 4. *Let $\mathbb{T}_i = (T_i, \eta_i, \mu_i)$, $i = 1, 2$, be normal triples in the category* Comp*, $\mathbb{T}_1 \neq \mathbb{I}$, and there exists a distributive law of the triple $\mathbb{T}_1$ over the triple $\mathbb{T}_2$. Then $\mathbb{T}_2$ is a power triple.*

Proof: Here we use the notations $\oplus$ and $\widetilde{\oplus}$ for the couple (T_1, T_2). For each spaces X and Y and each $a \in T_2X, b \in T_1Y$, we obtain $T_1T_2pr_1(a \oplus b) = \eta_1T_2X(a), T_1T_2pr_2(a \oplus b) = T_1\eta_2Y(b)$. Denoting the distributive law by λ and using the properties (D1), (D2), we obtain $T_2T_1pr_1 \circ \lambda(X \times Y)(a \oplus b) = T_2\eta_1X(a), T_2T_1pr_2 \circ \lambda(X \times Y)(a \oplus b) = \eta_2T_1Y(b)$. Hence $\lambda(X \times Y)(a \oplus b) = a \widetilde{\oplus} b$.

Show that T_2 is a profinitely power functor. Let X be finite, $\mathrm{supp}_{T_2}(a) = X$, and let $b \in T_1Y$ be such that $\deg(b) = 2$. Choose $y_1, y_2 \in \mathrm{supp}_{T_1}(b), y_1 \neq y_2$. It is sufficient to show that for all mappings $f_1, f_2 : X \to X$, we have $T_2f_1(a) \neq T_2f_2(a)$, whenever $f_1 \neq f_2$. Assuming the contrary, define the mappings $h_1, h_2 : X \times Y \to X \times Y$ by $h_1 = f_1 \times 1_y$,

$$h_2(x, y) = \begin{cases} (f_1(x), y) & \text{if } y \neq y_2; \\ (f_2(x), y) & \text{otherwise.} \end{cases}$$

Then, by the assumption, $T_1T_2h_1(a \oplus b) = T_1T_2h_2(a \oplus b)$. Analogously to the proof of Theorem 3 we can verify that $\lambda(X \times Y) \circ T_1T_2h_1(a \oplus b) \neq \lambda(X \times Y) \circ T_1T_2h_2(a \oplus b)$, thus obtaining a contradiction.

As a matter of fact, it is proved in [12] that every normal functor determining a triple, is weakly bicommutative. Now, it follows from Theorem 1 that T_2 is a power functor. □

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