Logarithmic capacity is not subadditive — a fine topology approach

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Abstract. In Landkof's monograph [8, p. 213] it is asserted that logarithmic capacity is strongly subadditive, and therefore that it is a Choquet capacity. An example demonstrating that logarithmic capacity is not even subadditive can be found e.g. in [6, Example 7.20], see also [3, p. 803]. In this paper we will show this fact with the help of the fine topology in potential theory.

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1. Introduction.

In potential theory we often use special set functions, called capacities. Their physical interpretation is relatively simple. If we denote Q the total charge on the conductor and V its equilibrium potential, the equality $Q = k \cdot V$ holds where the constant k (k independent on Q) is called the capacity of the conductor. From a physical point of view it is obvious that the capacity of the surface of the unit ball equals the capacity of the whole ball. The capacity therefore cannot be additive.

Around 1950, Choquet constructed a mathematical theory of capacities. A capacity was defined axiomatically as a non-negative, monotone, right continuous set function C, defined on the system of all compact sets fulfilling the **strong subadditivity axiom**

$$C(A \cup B) + C(A \cap B) \leq C(A) + C(B) \ .$$

Today every set function that fulfills the above axioms is called a **Choquet capacity**.

Similarly to measure theory, starting from a Choquet capacity both an inner and outer Choquet capacity, C_* and C^*, respectively, of an arbitrary set can be constructed. Choquet's main result in capacity theory is the following theorem : All Borel sets are capacitable (i.e. their inner and outer capacities coincide) [7]. Furthermore, the outer capacity is countably subadditive [1] :

$$C^*\Big(\bigcup_{n=1}^{\infty} A_n\Big) \leq \sum_{n=1}^{\infty} C^*(A_n) \ .$$

A Choquet capacity has found widespread use in potential theory in $\mathbb{R}^m$, $m \geq 3$.

Let us examine the case of the plane [8]: For any compact set K in the plane $\mathbb{R}^2$ denote

$$W = \inf_{\mu} \int \log \frac{1}{|x-y|} d\mu(x) d\mu(y)$$

where the infimum is taken over all positive probability Radon measures μ supported by K. Now the **logarithmic capacity** *cap* of compact K is defined by

$$cap(K) = e^{-W} .$$

The logarithmic capacity has a number of suitable properties (e.g. for a capacity of a closed disc $B(x,r)$ with a center x and a radius r the formula $cap\, B(x,r) = r$ holds), but it is not a Choquet capacity.

Theorem 1 shows that the outer logarithmic capacity is not countably subadditive and therefore cannot be an outer Choquet capacity. Furthermore, in Theorem 2 we show that there exist compact sets A, B such that

$$cap(A) + cap(B) < cap(A \cup B) ,$$

so that the logarithmic capacity does not fulfill the axiom of (strong) subadditivity for a Choquet capacity. Let us note that all Borel sets are capacitable for the logarithmic capacity despite of the fact that it is not a Choquet capacity (see [9, p. 170]). For Borel sets, we shall write *cap* instead of cap^*.

2. Fine topology.

In modern potential theory, **fine topology** introduced in the 40s by Cartan plays an important role. It is the coarsest topology that makes all superharmonic functions continuous. In connection with this topology we will use terms such as finely open, fine interior $\cdots$.

With the fine topology the notion "**a set thin at a point**" is closely related. If we denote for an arbitrary point $x \in \mathbb{R}^2$, a set $A \subset \mathbb{R}^2$ and $n \in \mathbb{N}$

$$A_n(x) = \{z \in A, \frac{1}{2^n} \leq |z-x| < \frac{2}{2^n}\} ,$$

we can characterize points x at which a set A is thin as those for which the series

$$\sum_{n=1}^{\infty} \frac{-n}{log\, cap^*\,(A_n(x))}$$

converges (**Wiener's test**). It can be shown that a set A is thin at a point x if and only if the point x is in the fine interior of the set $\{x\} \cup \complement A$ (see [2, Theorem IX,10]).

We will need the following properties of the fine topology in the plane :

Theorem A. ([7, Theorem 10.14]) *Let $E \subset \mathbb{R}^2$ and let $\complement E$ be thin at a point x. Then there exist arbitrarily small $r > 0$ such that*

$$\{z \in \mathbb{R}^2, |x-z| = r\} \subset E .$$

This statement holds in the plane. The fine topology does not have such a property in higher dimensions.

Theorem B. ([5, Lemma 7]) *Let $U \subset \mathbb{R}^2$ be a finely open set and let S be compact in $\mathbb{R}^2$. Define, for $\alpha \in \mathbb{R}$, the function*

$$h_\alpha(z) = \int_{S \setminus U} \frac{1}{|z-\zeta|^\alpha} d\zeta \quad , z \in U \ .$$

Then every point $z \in U$ has a fine neighbourhood $V \subset U$ such that, for every $\alpha \in \mathbb{R}$, h_α is bounded on V.

The proof of Theorem B is based on the inequality between the Lebesgue measure λ and the logarithmic capacity

$$\lambda(E) \le \pi (cap\, E)^2 \ ,$$

that holds for all Borel sets E in the plane, and a certain uniform version of Wiener's criterion obtained by Lyons (cf. [5, Lemma 6]). Using Theorem B, Fuglede derived in the 80s a theory of finely holomorphic functions having the property of unique continuation typical of holomorphic functions (cf. [4]).

Other properties of the fine topology and of other "fine" topologies (for example the density topology on the real axis) have been studied in the monograph [10].

3. Subadditivity.

To prove the following theorem we will use the fine topology property introduced in Theorem A.

Theorem 1. *Logarithmic capacity is not countably subadditive, in fact there exist nonempty Borel sets A_j such that*

$$cap \bigcup_{j=1}^{\infty} A_j > \sum_{j=1}^{\infty} cap\, A_j \ .$$

Proof: If $x \in \mathbb{R}$, set $\widehat{x} = (x, 0) \in \mathbb{R}^2$. Choose real numbers $a \ge b > 0$. If $0 \le x \le a$, and $T = (\frac{a}{2}, a + \frac{b}{100})$, we have

$$B(T, \frac{b}{100}) \subset \{z \in \mathbb{R}^2, a \le |z - \widehat{x}| < 2a\} \ .$$

For m, $k \in \mathbb{N}$, k odd, $1 \le k \le 2^{m+1}$ denote

$$T_{m,k} = (\frac{k}{2^{m+1}}, \frac{1}{2^m} + \frac{1}{100} \cdot \frac{1}{2^{2m}}) \ .$$

Set

$$F = \bigcup_{m=1}^{\infty} \bigcup_{\substack{k=1 \\ k \text{ odd}}}^{2^{m+1}} B(T_{m,k}, \frac{1}{100} \cdot \frac{1}{2^{2m}}) \ .$$

Fix now $x \in]0,1]$. For any $n \in \mathbb{N}$ there exists $k \in \mathbb{N}$, k odd, such that (for $A \subset \mathbb{R}^2$ is $A_n(x)$ defined in the part 2)

$$B(T_{n,k}, \frac{1}{100} \cdot \frac{1}{2^{2n}}) \subset (\mathbb{R}^2)_n(\widehat{x}) \ .$$

For any $n \in \mathbb{N}$ we have

$$cap\, F_n(\widehat{x}) \geq cap\, B(T_{n,k}, \frac{1}{100} \cdot \frac{1}{2^{2n}}) = \frac{1}{100} \cdot \frac{1}{2^{2n}} \ .$$

Since

$$\sum_{n=3}^{\infty} \frac{-n}{log\, cap\, F_n(\widehat{x})} \geq \sum_{n=3}^{\infty} \frac{-n}{log(\frac{1}{100} \cdot \frac{1}{2^{2n}})} = +\infty \ ,$$

F is not thin at $\widehat{x}$ according to Wiener's test. In the same way the set

$$G = \bigcup_{n=1}^{\infty} \bigcup_{m=n^3}^{\infty} \bigcup_{k \in D_{n,m}} B(T_{m,k}, \frac{1}{100} \cdot \frac{1}{2^{2m}}) \ .$$

(where $D_{n,m} = \{k \in \mathbb{N} : 1 \leq k \leq 2^{m+1}, k \text{ odd}, \frac{1}{2^n} \leq \frac{k}{2^{m+1}} \leq \frac{2}{2^n}\}$ for n, $m \in \mathbb{N}$) is not thin at $\widehat{x}$.

We shall prove that G is thin at the origin 0. We have

$$G_n(0) \subset \bigcup_{m=n^3}^{\infty} \bigcup_{\substack{k=1 \\ k \text{ odd}}}^{2^{m+1}} B(T_{m,k}, \frac{1}{100} \cdot \frac{1}{2^{2m}}) \ .$$

Assuming that the logarithmic capacity is countably subadditive, we get the estimate

$$cap\, G_n(0) \leq \sum_{m=n^3}^{\infty} \sum_{k=1}^{2^{m+1}} \frac{1}{100} \cdot \frac{1}{2^{2m}} = \sum_{m=n^3}^{\infty} \frac{2}{100 \cdot 2^m} = \frac{2}{50} \cdot \frac{1}{2^{n^3}} \ .$$

Since

$$\sum_{n=1}^{\infty} \frac{-n}{log\, cap\, G_n(0)} \leq \sum_{n=1}^{\infty} \frac{-n}{log\frac{2}{50} - n^3 \cdot log 2} < \infty \ ,$$

G is thin at 0. Now, let H denote the fine interior of the set $\complement G$. Since G is thin at the origin, $0 \in H$. According to Theorem A there exists $x \in]0,1]$ such that $\widehat{x} \in H$. This contradicts the fact that G is not thin at $\widehat{x}$. $\square$

The proof of the following theorem is not only based on Theorem A, but also on a deeper assertion of Theorem B.

Theorem 2. *There exist compact sets A, B such that*

$$cap(A) + cap(B) < cap(A \cup B) \ .$$

PROOF: Let $n \in \mathbb{N}$. Divide the interval $[\frac{1}{2^n}, \frac{2}{2^n}]$ into k_n intervals of the same length $d_n = 1/k_n \cdot 2^n$ with centers $x_{n,j}$ for $j = 1, \dots, k_n$. Set

$$M = \bigcup_{n=1}^{\infty} \bigcup_{j=1}^{k_n} B(\widehat{x}_{n,j}, r_n) \ ,$$

where $r_n = 1/k_n \cdot 2^{n^3}$. Fix $\alpha > 2$. Define a function $h : \mathbb{R}^2 \to [0, +\infty]$ by the formula

$$h : z \mapsto \int_M \frac{1}{|z - \zeta|^\alpha} d\zeta \ .$$

For $z \in [\frac{1}{2^n}, \frac{2}{2^n}]$, the inequalities

$$h(\widehat{z}) \geq \int_{B(\widehat{x}_{n,j}, r_n)} \frac{1}{|\widehat{z} - \zeta|^\alpha} d\zeta \geq \int_{B(\widehat{x}_{n,j}, r_n)} \frac{1}{d_n^\alpha} d\zeta$$

hold, where j $(1 \leq j \leq k_n)$ is the index of the interval such that $|z - x_{n,j}| \leq \frac{d_n}{2}$. Since

$$\int_{B(\widehat{x}_{n,j}, r_n)} \frac{1}{d_n^\alpha} d\zeta = \pi \cdot (r_n)^2 \cdot (2^n \cdot k_n)^\alpha = c_n \cdot k_n^{\alpha - 2} \ ,$$

where c_n is a constant independent of k_n, there exists a k_n such that for all $z \in [\frac{1}{2^n}, \frac{2}{2^n}]$

$$(*) \qquad h(\widehat{z}) \geq n \ .$$

Supposing, that the logarithmic capacity is subadditive, we get (the capacity of a disc equals its diameter)

$$cap\, M_n(0) \leq cap\, B(\widehat{x}_{n,1}, r_n) + \ldots + cap\, B(\widehat{x}_{n,k_n}, r_n) = k_n \cdot \frac{1}{k_n \cdot 2^{n^3}} = \frac{1}{2^{n^3}} \ .$$

We see that M is thin at the origin according to Wiener's test. The function h is bounded according to Theorem B in a certain fine neighbourhood V of the origin by a certain constant K. Using Theorem A there is a sequence $z_j \downarrow 0$ such that $\widehat{z}_j \in V$ and

$$h(\widehat{z}_j) \leq K \ .$$

This contradicts $(*)$. □

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