A note on the Runge–Kutta method for stochastic differential equations

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Abstract. In the paper the convergence of a mixed Runge–Kutta method of the first and second orders to a strong solution of the Ito stochastic differential equation is studied under a monotonicity condition.

Keywords: stochastic differential equation, Runge–Kutta method, monotonicity and Lipschitz condition

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1. Introduction.

We consider the Ito stochastic differential equation (SDE)

$$dX_t = f(t, X_t)\,dt + g(t, X_t)\,dW_t$$

with initial condition $X_0 = 0$, where $\{W_t\}$ is a standard Wiener process, $t \in [0, T]$, the random functions f, g are continuous functions of x, predictable with respect to (ω, t) and satisfying the linear growth condition, the function g satisfies the Lipschitz condition and f the monotonicity condition.

The convergence of Runge–Kutta (RK) approximation methods in the case when the function f is subjected to the stronger Lipschitz condition instead of the monotonicity condition is well known [1]. In [2] the convergence of the Euler scheme is proved provided f satisfies the monotonicity condition. The question is whether the RK approximation methods of higher orders can be applied under this condition. Our aim is to show a convergence of a generalized RK iterative scheme. In contradiction to the cited papers we use a second order RK approximation instead of the Euler scheme for the function $g(t, x)$. This result can be apparently generalized to the RK methods of any order n.

2. Results.

Let us have a sampling from $[0, T] : h = T/n$, $t_0 = 0$,

$$t_{i+1} = t_i + h, \quad i = 0, 1, \dots, n-1, \qquad \text{and let } \ \Delta W_i = W_{t_{i+1}} - W_{t_i}\,.$$

Lemma. *Let* $(\Omega, \mathcal{F}, \mathbf{P})$ *be a complete probability space and* $\{\mathcal{F}_t, t \in \mathcal{R}_+\}$ *a standard filtration. Assume that the functions* $f : \Omega \times \mathcal{R}_+ \times \mathcal{R} \to \mathcal{R}$ *and* $g : \Omega \times \mathcal{R}_+ \times \mathcal{R} \to \mathcal{R}$ *are continuous functions of* $x \in \mathcal{R}$, *predictable with respect to* (ω, t), *that* $g(\omega, .\,, .) \in \mathcal{C}^2$ *for any* ω, *and that for every* $\omega \in \Omega$, $t \geq 0$, $x, y \in \mathcal{R}$, *they fulfil the conditions*

(i) $f^2(t,x) + g^2(t,x) \leq k_1(1+x^2)$ — *the linear growth condition,*
(ii) $2(x-y)(f(t,x) - f(t,y)) \leq k_2(x-y)^2$ — *the monotonicity condition,*
(iii) $(g(t,x) - g(t,y))^2 \leq k_3(x-y)^2$ — *the Lipschitz condition.*

Assume further that for every $\omega \in \Omega$, $t \geq 0$, $x, y \in \mathcal{R}$,

(iv) $\frac{\partial}{\partial x} g(t,x)$ *satisfies the linear growth condition*

$$\Big(\frac{\partial}{\partial x} g(t,x)\Big)^2 \leq k_4(1+x^2),$$

(v) $g(t,x)\frac{\partial}{\partial x} g(t,x)$ *satisfies the Lipschitz condition*

$$\Big|g(t,x)\frac{\partial}{\partial x} g(t,x) - g(t,y)\frac{\partial}{\partial x} g(t,y)\Big|^2 \leq k_5(x-y)^2,$$

(vi) $\frac{\partial^2}{\partial x^2} g(t,x)$ *and* $\frac{\partial^2}{\partial t^2} g(t,x)$ *are bounded.*

Then the iterative scheme

$$\begin{aligned} \widehat{X}_{i+1} &= \widehat{X}_i + f(t_i, \widehat{X}_i)h + \frac{1}{2}(g(t_i, \widehat{X}_i) + g(t_{i+1}, \tilde{X}_{i+1}))\,\Delta W_i, \\ \tilde{X}_{i+1} &= \widehat{X}_i + f(t_i, \widehat{X}_i)h + g(t_i, \widehat{X}_i)\,\Delta W_i, \quad \widehat{X}_0 = 0, \quad i = 0, 1, \dots, n-1, \end{aligned} \tag{1}$$

converges in quadratic mean to the solution of the Stratonovich SDE

$$dY_t = f(t, Y_t)\,dt + g(t, Y_t) \circ dW_t, \quad Y_0 = 0, \tag{2}$$

that is

$$\max_{0 \leq i \leq n} \mathbf{E}\big|Y_{t_i} - \widehat{X}_i\big|^2 = \mathbf{O}(h^{1/2})$$

as h *tends to* 0.

Proof: The Stratonovich SDE (2) is equivalent to the Ito SDE

$$dY_t = f(t, Y_t)\,dt + \frac{1}{2} g(t, Y_t)\frac{\partial}{\partial x} g(t, Y_t)\,dt + g(t, Y_t)\,dW_t, \quad Y_0 = 0.$$

We set $\widehat{Y}_0 = 0$ and

$$\widehat{Y}_{i+1} = \widehat{Y}_i + f(t_i, \widehat{Y}_i)h + \frac{1}{2} g(t_i, \widehat{Y}_i)\frac{\partial}{\partial x} g(t_i, \widehat{Y}_i)h + g(t_i, \widehat{Y}_i)\,\Delta W_i.$$

The results of [2] imply that $\mathbf{E}\max_{0\le i\le n}\big|Y_{t_i}-\widehat{Y}_i\big|^2=\mathbf{O}(h^{1/2})$. Since $|Y_{t_i}-\widehat{X}_i|\le |Y_{t_i}-\widehat{Y}_i|+|\widehat{Y}_i-\widehat{X}_i|$, it is sufficient to show that $\mathbf{E}|\widehat{Y}_i-\widehat{X}_i|^2=\mathbf{O}(h)$ for every $i=0,1,\dots,n$. Using the Taylor expansion of $g(t,x)$ we get

$$\begin{aligned}g(t_{i+1},\tilde{X}_{i+1})=g(t_i,\widehat{X}_i)&+\frac{\partial}{\partial x}g(t_i,\widehat{X}_i)(f(t_i,\widehat{X}_i)h+g(t_i,\widehat{X}_i)\,\Delta W_i)+\\&+\frac{1}{2}\frac{\partial^2}{\partial x^2}g(t_i+\alpha_i h,\theta_i)\,(f(t_i,\widehat{X}_i)h+g(t_i,\widehat{X}_i)\,\Delta W_i)^2+\\&+\frac{\partial}{\partial t}g(t_i,\widehat{X}_i)h+\frac{1}{2}\frac{\partial^2}{\partial t^2}g(t_i+\alpha_i h,\theta_i)h^2,\end{aligned}$$

where $\theta_i=\widehat{X}_i+\alpha_i(\tilde{X}_{i+1}-\widehat{X}_i)$, $0<\alpha_i<1$.
We write the difference $\widehat{X}_{i+1}-\widehat{Y}_{i+1}$ in the form

$$\begin{aligned}\widehat{X}_{i+1}-\widehat{Y}_{i+1}=\widehat{X}_i-\widehat{Y}_i&+h(f(t_i,\widehat{X}_i)-f(t_i,\widehat{Y}_i))+\Delta W_i(g(t_i,\widehat{X}_i)-g(t_i,\widehat{Y}_i))+\\&+\frac{1}{2}h(g(t_i,\widehat{X}_i)\frac{\partial}{\partial x}g(t_i,\widehat{X}_i)-g(t_i,\widehat{Y}_i)\frac{\partial}{\partial x}g(t_i,\widehat{Y}_i))+\\&+\frac{1}{2}g(t_i,\widehat{X}_i)\frac{\partial}{\partial x}g(t_i,\widehat{X}_i)(\Delta W_i^2-h)+\frac{1}{2}f(t_i,\widehat{X}_i)\frac{\partial}{\partial x}g(t_i,\widehat{X}_i)h\Delta W_i+\\&+\frac{1}{4}\frac{\partial^2}{\partial x^2}g(t_{i+1}+\alpha_i h,\theta_i)(f(t_i,\widehat{X}_i)h+g(t_i,\widehat{X}_i)\Delta W_i)^2\Delta W_i+\\&+\frac{1}{2}\frac{\partial}{\partial t}g(t_i,\widehat{X}_i)h\Delta W_i+\frac{1}{4}\frac{\partial^2}{\partial t^2}g(t_i+\alpha_i h,\theta_i)h^2\Delta W_i.\end{aligned}$$

We square both sides of the equation and take the expectation. After estimating the members on the right-hand side, as e.g.

$$\mathbf{E}\{2(\widehat{X}_i-\widehat{Y}_i)h(f(t_i,\widehat{X}_i)-f(t_i,\widehat{Y}_i))\}\le hk_2\mathbf{E}(\widehat{X}_i-\widehat{Y}_i)^2$$

and

$$\begin{aligned}&\mathbf{E}\{2\Delta W_i(g(t_i,\widehat{X}_i)-g(t_i,\widehat{Y}_i))\frac{1}{2}g(t_i,\widehat{X}_i)\frac{\partial}{\partial x}g(t_i,\widehat{X}_i)(\Delta W_i^2-h)\}\le\\&\le k_3^{\frac{1}{2}}\mathbf{E}\{|\widehat{X}_i-\widehat{Y}_i|(k_1(1+\widehat{X}_i^2))^{\frac{1}{2}}(k_4(1+\widehat{X}_i^2))^{\frac{1}{2}}\Delta W_i(\Delta W_i^2-h)\}\le\\&\le k\mathbf{E}\{\mathbf{E}\{|\widehat{X}_i-\widehat{Y}_i|(1+\widehat{X}_i^2)\Delta W_i(\Delta W_i^2-h)|\mathcal{F}_i\}\}\le\\&\le k\mathbf{E}\{|\widehat{X}_i-\widehat{Y}_i|(1+\widehat{X}_i^2)\mathbf{E}\{\Delta W_i(\Delta W_i^2-h)|\mathcal{F}_i\}\}\le\\&\le k\mathbf{E}\{|\widehat{X}_i-\widehat{Y}_i|(1+\widehat{X}_i^2)\}\mathbf{E}\{\Delta W_i(\Delta W_i^2-h)\}\le\\&\le c(\mathbf{E}|\widehat{X}_i-\widehat{Y}_i|^2)^{\frac{1}{2}}\mathbf{E}\{\Delta W_i(\Delta W_i^2-h)\}=0\end{aligned}$$

we obtain

$$\mathbf{E}|\widehat{Y}_{i+1}-\widehat{X}_{i+1}|^2\le\mathbf{E}|\widehat{Y}_i-\widehat{X}_i|^2+C_1h\mathbf{E}|\widehat{Y}_i-\widehat{X}_i|^2+C_2h^2.$$

Hence we conclude (see e.g. [2]) that $\mathbf{E}|\widehat{Y}_{i+1}-\widehat{X}_{i+1}|^2=\mathbf{O}(h)$. This completes the proof of the lemma. □

From the lemma one easily deduces the following

Theorem. *If $f(t,x)$ and $g(t,x)$ fulfil the conditions of Lemma then*

$$\sup_{0\le t\le T} \mathbf{E}|Y_t - \widehat{X}_t|^2 = \mathbf{O}(h^{\frac{1}{2}}),$$

where we set

$$\widehat{X}_t = \widehat{X}_i + \frac{t-t_i}{t_{i+1}-t_i}(\widehat{X}_{i+1} - \widehat{X}_i),$$
$$t_i \le t \le t_{i+1}, \quad i = 0,1,\dots,n-1,$$

and $\widehat{X}_i, Y_t$ are defined by (1), (2), respectively.

We note that the conclusions of Lemma and Theorem remain valid, if (iv), (v) are replaced by the assumption that $f(x,t)$, $g(x,t)$ and $\frac{\partial}{\partial x} g(t,x)$ are bounded.

References

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