On superpositionally measurable semi–Carathéodory multifunctions

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Abstract. For multifunctions $F : T \times X \to 2^Y$, measurable in the first variable and semicontinuous in the second one, a relation is established between being product measurable and being superpositionally measurable.

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Introduction.

In various problems, one encounters a superposition of the type $F\big(t, G(t)\big)$, where F and G are, in general, multifunctions and where it is often required that the mentioned superposition is measurable for every measurable multifunction G. A multifunction of such a property is called superpositionally measurable. It is known that under suitable assumptions on the spaces T, X and Y, Carathéodory multifunction $F : T \times X \to 2^Y$, i.e. measurable in t and continuous in x, is superpositionally measurable (see [1], [6], [8], [11], [12]). Unfortunately, when F is semicontinuous (in some sense) in x, such a multifunction, henceforth called semi-Carathéodory, may not be already superpositionally measurable. In this note we discuss the connection between superpositional measurability and product measurability of semi-Carathéodory multifunctions.

Preliminaries.

Thus, given two arbitrary nonempty sets $\mathcal{X}$, $\mathcal{Y}$ and denoting by $2^{\mathcal{Y}}$ the family of all subsets of $\mathcal{Y}$, by a multifunction $\Phi : \mathcal{X} \to 2^{\mathcal{Y}}$ we mean a mapping Φ of a domain $\mathcal{X}$ and a range contained in $2^{\mathcal{Y}}$. Let Σ be a σ-field of subsets of $\mathcal{X}$ and let $\mathcal{Y}$ be a topological space. A multifunction $\Phi : \mathcal{X} \to 2^{\mathcal{Y}}$ is said to be Σ-measurable (resp. weakly Σ-measurable) if the set $\Phi^-(A) = \{x \in \mathcal{X} : \Phi(x) \cap A \neq \emptyset\}$ belongs to Σ for every closed (resp. open) set $A \subset \mathcal{Y}$. It is known (see [2], [3], [13]) that when $(\mathcal{X}, \Sigma)$ is a complete measurable space (i.e. there is a complete σ-finite measure defined on Σ) and $\mathcal{Y}$ is a Polish space (i.e. $\mathcal{Y}$ is separable and metrisable by a complete metric), then these two measurability concepts coincide for a closed values multifunction. Let $\mathcal{X}$ be a topological space, too. A multifunction $\Phi : \mathcal{X} \to 2^{\mathcal{Y}}$ is said to be lower

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(resp. upper) semicontinuous if the set $\Phi^{-}(A)$ is open (resp. closed) in $\mathcal{X}$ for every open (resp. closed) set $A \subset \mathcal{Y}$.

Henceforth we use the following notations:

$(T, \mathcal{A})$ - is a complete measurable space;
X - is Polish space;
$\mathcal{B}(X)$ - is a σ-field of Borel subsets of X;
$\mathcal{A} \otimes \mathcal{B}(X)$ - is a product σ-field on $T \times X$ (i.e. the minimal σ-field containing all products $A \times B$, with $A \in \mathcal{A}$, $B \in \mathcal{B}(X)$);
Y - is a topological space.

Let us consider a multifunction $F : T \times X \to 2^{Y}$. F is called product measurable if it is $\mathcal{A} \otimes \mathcal{B}(X)$-measurable and it is called superpositionally measurable if, for every $\mathcal{A}$-measurable multifunction $G : T \to 2^{X}$ with nonempty closed values, a multifunction $F_G : T \to 2^{Y}$ defined by the superposition $F_G(t) = F(t, G(t))$ is $\mathcal{A}$-measurable, where $F(t, G(t))$ denotes the sum of sets $F(t, x)$ when $x \in G(t)$. Further, we say that F is a lower (resp. upper) semi-Carathéodory multifunction if $F(\cdot, x)$ is $\mathcal{A}$ measurable for each fixed $x \in X$ and $F(t, \cdot)$ is lower (resp. upper) semicontinuous for each fixed $t \in T$.

Main results.

Theorem 1. *Every product measurable multifunction $F : T \times X \to 2^{Y}$ is superpositionally measurable.*

Proof: Let a closed set $A \subset Y$ and an $\mathcal{A}$-measurable multifunction $G : T \to 2^{X}$ with nonempty closed values be given. In view of $\mathcal{A} \otimes \mathcal{B}(X)$-measurability of F, the set $F^{-}(A) = \{(t, x) \in T \times X : F(t, x) \cap A \neq \emptyset\}$ belongs to $\mathcal{A} \otimes \mathcal{B}(X)$. On the other hand, the assumptions on multifunction G imply $\mathcal{A} \otimes \mathcal{B}(X)$-measurability of its graph, i.e. $\mathrm{gr}G = \{(t, x) \in T \times X : x \in G(t)\} \in \mathcal{A} \otimes \mathcal{B}(X)$ (see [3, Theorem 3.5]). Thus,

$$\{(t, x) \in T \times X : F(t, x) \cap A \neq \emptyset,\ x \in G(t)\} = F^{-}(A) \cap \mathrm{gr}G \in \mathcal{A} \otimes \mathcal{B}(X).$$

Hence, using the Projection Theorem, see [2, Theorem III. 23], [10, Theorem 4]), we obtain

$$\begin{aligned} F_G^{-}(A) &= \{t \in T : F_G(t) \cap A \neq \emptyset\} = \{t \in T : F\big(t, G(t)\big) \cap A \neq \emptyset\} = \\ &= \mathrm{proj}_T\big(F^{-}(A) \cap \mathrm{gr}G\big) \in \mathcal{A} \end{aligned}$$

which, in view of the optionality of A and G, means that the multifunction F is superpositionally measurable (here proj_T denotes the projection of $T \times X$ onto T).

In particular, we can see from Theorem 1 that both an upper and a lower semi-Carathéodory product measurable multifunction is superpositionally measurable. The converse implication holds only for the upper semi-Carathéodory multifunction. Namely, we have

Theorem 2. *Every upper semi-Carathéodory superpositionally measurable multifunction is product measurable.*

Proof: Let us first notice that superpositional measurability implies $\mathcal{A}$-measurability of sets $\{t \in T : F(t, B) \cap A \neq \emptyset\}$ for each closed $A \subset Y$ and $B \subset X$. Now, for every $t \in T$ and each closed $A \subset Y$, let us put $\Phi_A(t) = \{x \in X : F(t,x) \cap A \neq \emptyset\}$. In virtue of the upper semicontinuity of $F(t,\cdot)$, the set $\Phi_A(t)$ is closed in X. We claim that thus defined closed valued multifunction $\Phi_A : T \to 2^X$ is $\mathcal{A}$-measurable. Indeed, for every closed $B \subset X$, we have:

$$\Phi_A^-(B) = \{t \in T : \Phi_A(t) \cap B \neq \emptyset\} = \{t \in T : \bigvee_{x \in B} x \in \Phi_A(t)\} =$$
$$= \{t \in T : \bigvee_{x \in B} F(t,x) \cap A \neq \emptyset\} = \{t \in T : F(t,B) \cap A \neq \emptyset\} \in \mathcal{A}.$$

Thus, by [3, Theorem 3.5] its graph $\mathrm{gr}\Phi_A$ belongs to the σ-field $\mathcal{A} \otimes \mathcal{B}(X)$. But

$$F^-(A) = \{(t,x) \in T \times X : F(t,x) \cap A \neq \emptyset\} = \{(t,x) \in T \times X : x \in \Phi_A(t)\} =$$
$$= \mathrm{gr}\Phi_A \in \mathcal{A} \otimes \mathcal{B}(X)$$

which completes the proof of product measurability of F. $\square$

Example. In the case of the lower semi-Carathéodory multifunction the superpositional measurability does not generally imply the product measurability. In order to show it, we shall use the multifunction $\Phi : T \times I \to 2^{\mathbf{R}}$ constructed by A. Kucia in her paper [4, Example]. Let I be the interval $[0,1]$, $\mathcal{A}$ — the σ-field on I generated by one-point sets, let $(T,\mathcal{A}) = (I,\mathcal{A})$. It is easy to see that $(T,\mathcal{A})$ is complete measurable space and that a real-valued function $\varphi : T \to \mathbf{R}$ is measurable if and only if φ is eventually constant, i.e. there exists a countable set $N \subset I$ such that φ is constant on $I \setminus N$. Hence it follows, by "Castaing representation" theorem (see [2, Theorem III. 8]), that a multifunction $G : T \to 2^{\mathbf{R}}$ with nonempty closed values is $\mathcal{A}$-measurable if and only if G is eventually constant. The multifunction Φ is defined as follows:

$$\Phi(t,x) = \begin{cases} \{t\} & \text{if } |t-x| = \frac{1}{n} \text{ for some positive integer } n, \text{ or } t = x, \\ I & \text{in the other case.} \end{cases}$$

A. Kucia showed that such a multifunction Φ is lower semi-Carathéodory and is not $\mathcal{A} \otimes \mathcal{B}(I)$ measurable (see [4, p. 240]). Here we shall prove that Φ is superpositionally measurable. To this end, let us first consider an arbitrary but fixed nonempty closed set $B \subset I$. Two cases are possible: 1° B is countable, or 2° B is uncountable. In the second case the following condition holds:

$$(*) \qquad \forall_{t \in I}\ \exists_{x \in B}\ \ x \neq t \ \text{ and } \ x \neq t \pm \tfrac{1}{n} \ \text{ for } \ n = 1,2,\ldots$$

Indeed, otherwise there exists $\bar{t} \in I$ such that for any $x \in B$ we have $x = \bar{t}$ or $x = \bar{t} + \frac{1}{n}$ or $x = \bar{t} - \frac{1}{n}$ for some $n \in \mathbf{N}$. But then the set B must be countable, which is impossible.

Now let an $\mathcal{A}$-measurable multifunction $G : T \to 2^I$ with nonempty closed values be given. There exist a nonempty closed set $B \subset I$ and a countable set $N \subset I$ such that $G(t) = B$ for $t \in I \setminus N$. For every closed $A \subset \mathbf{R}$ let us denote $N_A = \{t \in N : \Phi_G(t) \cap A \neq \emptyset\} = \{t \in N : \Phi(t, G(t)) \cap A \neq \emptyset\}$. It is obvious that $N_A \in \mathcal{A}$.

Now, if $B = \{b_1, b_2, \dots\}$ then for every closed $A \subset \mathbf{R}$ we have

$$\begin{aligned}\Phi_G^-(A) &= \{t \in T : \Phi_G(t) \cap A \neq \emptyset\} = \{t \in I \setminus N : \Phi_G(t) \cap A \neq \emptyset\} \cup N_A = \\ &= \{t \in I \setminus N : \Phi(t, B) \cap A \neq \emptyset\} \cup N_A = \\ &= \bigcup_{n=1}^{\infty} \{t \in I \setminus N : \Phi(t, b_n) \cap A \neq \emptyset\} \cup N_A \in \mathcal{A}.\end{aligned}$$

If B is uncountable then from $(*)$ and the definition of Φ we get $\Phi(t, B) = I$ for every $t \in I \setminus N$. Hence for every closed $A \subset \mathbf{R}$ we have

$$\begin{aligned}\Phi_G^-(A) &= \{t \in T : \Phi_G(t) \cap A \neq \emptyset\} = \{t \in I \setminus N : \Phi_G(t) \cap A \neq \emptyset\} \cup N_A = \\ &= \{t \in I \setminus N : \Phi(t, B) \cap A \neq \emptyset\} \cup N_A = \{t \in I \setminus N : I \cap A \neq \emptyset\} \cup N_A \in \mathcal{A}\end{aligned}$$

because

$$\{t \in I \setminus N : I \cap A \neq \emptyset\} = \begin{cases} \emptyset & \text{if } I \cap A = \emptyset, \\ T \setminus N & \text{if } I \cap A \neq \emptyset. \end{cases}$$

Finally we can see that $\Phi_G^-(A)$ belongs to $\mathcal{A}$ for every $\mathcal{A}$-measurable multifunction $G : T \to 2^I$ with nonempty closed values and each closed set $A \subset \mathbf{R}$, what completes the proof of the superpositional measurability of the multifunction $\Phi : T \times X \to 2^{\mathbf{R}}$.

Conclusion.

It is known that many multifunctions $F : T \times X \to 2^Y$ which describe the right hand of differential inclusions are exactly semi-Carathéodory multifunctions. Hence, it would also be useful to know if such multifunctions are superpositionally measurable. From **Theorem 1** it follows that a multifunction $F : T \times X \to 2^Y$ is superpositionally measurable, provided it is product measurable. However, in general, the semi-Carathéodory multifunction is not product measurable (see, for instance, [9, p. 31]). Below we give three most often recurring cases when the semi-Carathéodory multifunction is product measurable.

1. ([8, Theorem 3.3], [7, Proposition 2.3])
   $(T, \mathcal{A})$ is a measurable space, X is a separable metric space, Y is a metric space, $F : T \times X \to 2^Y$ is an upper semi-Carathéodory multifunction with nonempty closed values and such that $F(t, \cdot)$ is lower semi-continuous with respect to a Hausdorff topology.
2. ([8, Theorem 3.4])
   $(T, \mathcal{A})$ is a complete measurable space, $X = Y$ is a separable reflexive Banach space, $F : T \times X \to 2^X$ is a lower semi-Carathéodory multifunction with nonempty closed convex values and such that $F(t, \cdot)$ is upper semi-continuous from X to X_ω, where X_ω denotes space X with weak topology.
3. ([14, Theorems 3 and 4])
   $(T, \mathcal{A}, \mu)$ is a measure space with a Hausdorff compact metric space T and

a Borel σ-finite, regular and complete measure μ defined on $\mathcal{A}$, X is a Polish space, Y — a separable metric space, $F : T \times X \to 2^Y$ is a lower (resp. upper) semi-Carathéodory multifunction with nonempty closed values and such that the following condition — due to Scorza–Dragoni — is satisfied:

"for every $\varepsilon > 0$ there exists a closed subset T_ε of T, with $\mu(T \setminus T_\varepsilon) < \varepsilon$, such that $F|_{T_\varepsilon \times X}$ is lower (resp. upper) semi-continuous."

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