Bourbaki's Fixpoint Lemma reconsidered

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Abstract. A constructively valid counterpart to Bourbaki's Fixpoint Lemma for chain-complete partially ordered sets is presented to obtain a condition for one closure system in a complete lattice L to be stable under another closure operator of L. This is then used to deal with coproducts and other aspects of frames.

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A *preclosure operator* on a complete lattice L is a map $k_0 : L \to L$ which preserves the partial order and is upward, that is, $x \leq k_0(x)$ for all $x \in L$. For such k_0, $\mathrm{Fix}(k_0) = \{x \in L \mid k_0(x) = x\}$ is readily seen to be a closure system in L, that is, closed under arbitrary meets in L, and we let k be the associated closure operator. In various contexts, one would like to be able to conclude, for certain subsets $S \subseteq L$, the following

Stability Lemma. *S is k-stable whenever it is k_0-stable.*

Now, one way of describing k is as the stable transfinite iterate of k_0: if one defines, for any $x \in L$, any ordinal α and any limit ordinal λ,

$$k_0^0(x) = x, \quad k_0^{\alpha+1}(x) = k_0(k_0^\alpha(x)), \quad k_0^\lambda(x) = \bigvee\{k_0^\alpha(x) \mid \alpha < \lambda\},$$

then $k = k_0^\gamma$ for the first γ such that $k_0^{\gamma+1} = k_0^\gamma$. Here, one sees by induction that any $\{k_0^\alpha(x) \mid \alpha < \beta\}$ is a chain, and hence the desired result follows for any $S \subseteq L$ closed under taking joins, in L, of (non-void) chains.

The same conclusion can also be obtained, without the use of ordinals, as an application of

Bourbaki's Fixpoint Lemma. *Any upward map of a chain-complete partially ordered set into itself has a fixpoint.*

For any S as above and $a \in S$,

$$P = \{x \in S \mid a \leq x \leq k(a)\}$$

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is chain-complete and is mapped into itself by k_0. For the resulting fixpoint $c = k_0(c)$ in P, $a \leq c \leq k(a)$ implies $c = k(a)$ and thus $k(a) \in S$.

It is an open problem precisely what rules of logic are needed to establish this lemma, and specifically, whether it is constructively valid. The known proofs (for instance, Witt [6]) use, for the partially ordered set in question, that $x \leq y$ implies $x < y$ or $x = y$ for all elements x and y, and that, for certain subsets U, V, W, if $U \subseteq V \cup W$ then $U \subseteq V$ or there exist $x \in U$ such that $x \in W$. These steps are not constructively valid but they do hold in any Boolean topos (Johnstone [2]), and hence so does the Stability Lemma, for any S closed under taking joins of chains.

The purpose of this note is to establish a constructively valid counterpart of Bourbaki's Fixpoint Lemma, to derive a form of the Stability Lemma from this, and to apply the latter to certain considerations concerning the coproducts of frames.

I am much indebted to Japie Vermeulen for a stimulating correspondence on this subject. For a slightly different treatment related to the Stability Lemma, see [4].

Consider, then, any preclosure operator k_0 on a complete lattice L, with associated closure operator k. For any $a \in L$, let W be the smallest downset (= containing all $y \leq x$ with any x) in $\uparrow a = \{x \in L \mid x \geq a\}$ such that

(1) $a \in W$,
(2) W is k_0-stable, and
(3) $\bigvee D \in W$ for any updirected $D \subseteq W$.

Then we have

Lemma 1. $W = \{x \in L \mid a \leq x \leq k(a)\}$.

Proof: Let $V = \{x \in W \mid x \vee y \in W \text{ for all } y \in W\}$. This is a downset since W is. Also, $a \in V$ because $a \vee y = y$ for all $y \in W$. Further, for $x \in V$ and $y \in W$, $k_0(x) \vee y \leq k_0(x \vee y)$, and since $k_0(x \vee y) \in W$ by (2) and the definition of V it follows that $k_0(x) \vee y \in W$, showing that $k_0(x) \in V$. Finally, if $D \subseteq V$ is updirected and $y \in W$ then $E = \{t \vee y \mid t \in D\}$ is an updirected subset of W, hence $\bigvee E \in W$ by (3), but $\bigvee E = (\bigvee D) \vee y$ and therefore $\bigvee D \in V$. It follows now that $V = W$, thus $x \vee y \in W$, for any $x, y \in W$, making W itself updirected so that $s = \bigvee W$ belongs to W. Consequently, by (2), $k_0(s) \leq s$ and hence $s = k_0(s)$. Now, $W \subseteq \{x \in L \mid a \leq x \leq k(a)\}$ since its intersection with the latter still satisfies the conditions (1)–(3), and therefore $a \leq s \leq k(a)$. This implies $s = k(a)$ which proves the lemma. $\square$

We now apply Lemma 1 to obtain a form of the Stability Lemma. For this, a closure system S in a complete lattice L will be called *finitary* if it is closed under taking joins, in L, of arbitrary updirected subsets. Note that, for the closure operator ℓ associated with S, this condition means that ℓ preserves joins of updirected subsets of L.

Lemma 2. *Any finitary closure system in L which is k_0-stable is also k-stable.*

Proof: Let S be the finitary closure system, with associated closure operator ℓ. Then, for all $x \in L$, $k_0(\ell(x)) \in S$, hence $\ell k_0 \ell(x) = k_0 \ell(x)$, and consequently

$\ell k_0(x) \leq k_0\ell(x)$. Now for any $a \in S$, let

$$U = \{x \in L \mid a \leq x, \ell(x) \leq k(a)\}.$$

Then U is a downset in $\uparrow a$. Also, $a \in U$ since $a = \ell(a)$. Further, for any $x \in U$, $\ell(x) \leq k(a)$ implies $k_0\ell(x) \leq k_0(k(a)) = k(a)$ and hence $\ell(k_0(x)) \leq k(a)$, showing that $k_0(x) \in U$. Finally, for any updirected $D \subseteq U$, $\ell[D] \subseteq \downarrow k(a)$, hence $t = \bigvee \ell[D] \leq k(a)$; further, $t \in S$ since S is finitary, and from $\bigvee D \leq t$ it then follows that $\ell(\bigvee D) \leq t \leq k(a)$. This shows $\bigvee D \in U$. As a result, U satisfies the conditions (1)–(3) stated above, hence $W \subseteq U$ and therefore $k(a) \in U$ by Lemma 1. This means that $\ell(k(a)) \leq k(a)$, showing that $k(a) \in S$. □

As an application of Lemma 2, we now give an improved version of the description of frame coproducts presented in Banaschewski [1]. For general facts concerning frames we refer to Johnstone [3].

Recall that, on a frame L, a *nucleus* is a closure operator such that $k(x \wedge y) = k(x) \wedge k(y)$, and a *prenucleus* is a preclosure operator k_0 for which $k_0(x) \wedge y \leq k_0(x \wedge y)$. The significance of these notions lies in the fact that, for any nucleus k on L, $\mathrm{Fix}(k)$ is a frame such that the map $L \to \mathrm{Fix}(k)$ given by k is a frame homomorphism, and for any prenucleus on L, the associated closure operator is a nucleus.

Now, for any family $(L_i)_{i\in I}$ of frames, the coproduct may be obtained by suitable constructs originating from the weak product A of the $(L_i)_{i\in I}$ as meet-semilattices. The first stage in this is the lattice $\mathcal{D}$ of all downsets of A; being closed under arbitrary unions and intersections, $\mathcal{D}$ is certainly a topology and hence a frame. Now, A is not only a meet-semilattice but also has joins, taken componentwise, for arbitrary *updirected* subsets. This suggests the consideration of the Scott-closed subsets of A, that is, the downsets closed under taking joins of arbitrary updirected subsets. These form a closure system $\mathcal{S}$ in $\mathcal{D}$, obviously determined by the preclosure operator σ_0 such that, for any $U \in \mathcal{D}$,

$$\sigma_0(U) = \{\bigvee D \mid D \subseteq U, \textit{updirected}\}.$$

Moreover, σ_0 is a prenucleus, and hence $\mathcal{S}$ is a frame, with frame homomorphism $\mathcal{D} \to \mathcal{S}$ induced by the associated nucleus σ.

For each $i \in I$ we have a map $k_i : L_i \to A$ such that $k_i(x)$ has component x for the index i and the unit of L_j for each index $j \neq i$. Then, the map $L_i \to \mathcal{S}$ taking x to $\downarrow k_i(x)$ preserves all finite meets and updirected joins.

Now, consider a further operator $\tau_0 : \mathcal{D} \to \mathcal{D}$ such that, for each $U \in \mathcal{D}$, $\tau_0(U)$ consists of all $a \wedge k_i(\bigvee Z)$ for any $a \in A$, $i \in I$ and finite $Z \subseteq L_i$ for which all $a \wedge k_i(t) \in U$, $t \in Z$. This is obviously a preclosure operator, but also easily checked to be a prenucleus. Let $\mathcal{T} = \mathrm{Fix}(\tau_0)$ and τ be the associated nucleus. Note that the maps $L_i \to \mathcal{T}$ taking $x \in L_i$ to $\tau(\downarrow k_i(x))$ preserve all finite meets and joins.

We are interested in the relationship between the two nuclei σ and τ. Since the definition of τ_0 makes it obvious that $\mathcal{T}$ is a finitary closure system in $\mathcal{D}$, we can conclude by Lemma 2 that $\mathcal{T}$ is σ-stable provided we show that it is σ_0-stable. For this, we first note that the closure condition defining $\mathcal{T}$ can be checked by just taking the cases $Z = \oslash$ and $Z = \{s,t\}$ for the finite set Z involved — the general case then resulting by obvious induction. Here, the condition for $Z = \oslash$ requires that all $a \in A$ for which some component is zero belong to the $U \in \mathcal{T}$, and since $U \subseteq \sigma_0(U)$ this also holds for $\sigma_0(U)$. Hence, in order to see that $\sigma_0(U) \in \mathcal{T}$ for any $U \in \mathcal{T}$ it remains to deal with the case $Z = \{s,t\}$. Let, then, $a \wedge k_i(s)$ and $a \wedge k_i(t)$ belong to $\sigma_0(U)$ for some $a \in A$, $i \in I$, and $s,t \in L_i$, and take, accordingly, updirected $D, E \subseteq U$ such that $a \wedge k_i(s) = \bigvee D$ and $a \wedge k_i(t) = \bigvee E$. Now, for any $x = (x_i)_{i \in I}$ in A, define $\bar{x} = \bigwedge\{k_j(x_j) \mid j \neq i\}$ and note that $x = \bar{x} \wedge k_i(x_i)$. Then, for each $x \in D$ and $y \in E$,

$$\bar{x} \wedge \bar{y} \wedge k_i(x_i) \quad \text{and} \quad \bar{x} \wedge \bar{y} \wedge k_i(y_i)$$

belong to U so that

$$\bar{x} \wedge \bar{y} \wedge k_i(x_i \vee y_i) \in U$$

since $U \in \mathcal{T}$. Now, the set of these elements is again updirected and hence

$$b = \bigvee\{\bar{x} \wedge \bar{y} \wedge k_i(x_i \vee y_i) \mid x \in D,\ y \in E\} \in \sigma_0(U).$$

Finally, since directed joins in A are taken componentwise,

$$\begin{aligned} b &= \bigvee\{\bar{x} \wedge \bar{y} \mid x \in D,\ y \in E\} \wedge k_i\big(\bigvee\{x_i \vee y_i \mid x \in D,\ y \in E\}\big) \\ &= \bar{a} \wedge k_i(a_i \wedge (s \vee t)) = a \wedge k_i(s \vee t), \end{aligned}$$

showing that the latter elements also belongs to $\sigma_0(U)$, as desired.

The result thus obtained shows that $\tau\sigma\tau = \sigma\tau$, which in turn implies that $\sigma\tau$ is idempotent and therefore a nucleus on $\mathcal{D}$. Now, Banaschewski [1] describes the coproduct of a family $(L_i)_{i \in I}$ of frames as the closure system $\mathcal{L}$ in $\mathcal{S}$ given by the condition that corresponds to the definition of τ_0. It follows that $\mathcal{L} = \mathcal{S} \cap \mathcal{T}$, and in all this proves:

Proposition. *$\sigma\tau$ is a nucleus on $\mathcal{D}$ such that $\mathrm{Fix}(\sigma\tau)$ is the coproduct of the family $(L_i)_{i \in I}$ of frames, with coproduct maps $L_i \to \mathrm{Fix}(\sigma\tau)$ taking x to $\sigma\tau(\downarrow k_i(x))$, for each $x \in L_i$ and $i \in I$.*

Remark. A crucial stage in the proof in [1] of the localic Tychonoff Theorem that the coproduct of compact frames is compact was the result that, for any family of frames, *the nucleus on $\mathcal{S}$ determining $\mathcal{L} = \mathcal{S} \cap \mathcal{T}$ is finitary.* Here, this follows from the trivial fact that the nucleus τ on $\mathcal{D}$ is finitary, given that, by the proposition, the nucleus in question is the restriction of $\sigma\tau$. We note that it was at this stage that Bourbaki's Fixpoint Lemma was used in [1]. The argument here replaces this by Lemma 1 and hence is constructively valid. This amendment makes the results of [1] concerning frame coproducts valid in any topos, provided the family $(L_i)_{i \in I}$ has *decidable* index set I. The latter restriction enters because the arguments involved here do make use of the condition that $i \neq j$ or $i = j$ for any $i, j \in I$.

As a further application of Lemma 1 we derive an important lemma due to Vermeulen [5].

For any frames L and M, let $\mathcal{D}$ be the frame of all downsets of $L \times M$, σ and τ the nuclei considered earlier, $\varrho = \sigma\tau$, and $\mathcal{K} = \mathrm{Fix}(\varrho)$. Thus $\mathcal{K}$ is the coproduct of L and M, with coproduct maps $L \to \mathcal{K}$ and $M \to \mathcal{K}$ given, respectively, by

$$x \rightsquigarrow \varrho(\downarrow (x, e)) \quad \text{and} \quad y \rightsquigarrow \varrho(\downarrow (e, y)).$$

We put

$$x \oplus y = \varrho(\downarrow (x, e)) \cap \varrho(\downarrow (e, y))$$

and note that, for any $U \in \mathcal{D}$,

$$\bigvee\{x \oplus y \mid (x, y) \in U\} = \varrho(U).$$

Further, $U \in \mathcal{D}$ will be called closed under *first* (or *second*) *slice joins* whenever $X \times \{b\} \subseteq U$ implies $(\bigvee X, b) \in U$, for any $X \subseteq L$ and $b \in M$ (or $\{a\} \times Y \subseteq U$ implies $(a, \bigvee Y) \in U$, for any $a \in L$ and $Y \subseteq M$). If X or Y in this condition are restricted to *finite* sets, we refer to *finitary* slice joins.

The result to be proved now is

Vermeulen's Lemma. *For any compact frame L and arbitrary frame M, if $S \in \mathcal{D}$ is closed under finitary first and arbitrary second slice joins then $e \oplus a \leq \bigvee\{x \oplus y \mid (x, y) \in S\}$ implies $(e, a) \in S$.*

PROOF: Consider the set $\mathcal{M}$ of all $U \in \mathcal{D}$ such that $S \subseteq U$ and $(e, z) \in U$ implies $(e, z) \in S$, for all $z \in M$. Clearly, $\mathcal{M}$ is a downset in $\uparrow S$ and $S \in \mathcal{M}$. Further, for any $U \in \mathcal{M}$, let $(e, z) \in \sigma_0(U)$. Then, $(e, z) = \bigvee D$ for some updirected $D \subseteq U$, hence by compactness there exists $(e, t_0) \in D$, and then $z = \bigvee\{t \in M \mid (e, t_0) \in D\}$. Here all $(e, t) \in U$ but since $U \in \mathcal{M}$ also $(e, t) \in S$, and therefore $(e, z) \in S$ by hypothesis on S. This shows that $\sigma_0(U) \in \mathcal{M}$ for all $U \in \mathcal{M}$. Finally, $\bigcup \mathcal{A} \in \mathcal{M}$ for any updirected $\mathcal{A} \subseteq \mathcal{M}$, immediately from the definition of $\mathcal{M}$. It now follows by Lemma 1 that $\sigma(S) \in \mathcal{M}$. Moreover, since S is closed under finitary first and second slice joins, $\tau(S) = S$ and hence $\varrho(S) = \sigma(S)$ so that, in fact, $\varrho(S) \in \mathcal{M}$. Now $e \oplus a \leq \bigvee\{x \oplus y \mid (x, y) \in S\}$ means $(e, a) \in \varrho(S)$, and we conclude $(e, a) \in S$, as desired. □

Remark. It might be worth noting that the above proof does not use the full force of the hypothesis on S. It is actually sufficient to have that $S = \tau(S)$, that is, S is closed under all finitary slice joins, and that $\{e\} \times Y \subseteq S$ implies $(e, \bigvee Y) \in S$.

We conclude with a presentation, in slightly different language, of two applications Vermeulen [5] makes of his lemma.

For this, recall that the frame version of the Hausdorff separation axiom for topological spaces is the condition that the codiagonal map $\nabla : L \oplus L \to L$, given by $\nabla(x \oplus y) = x \wedge y$ be closed, that is, induce an isomorphism $\uparrow s \to L$ where

$$s = \bigvee\{U \in L \oplus L \mid \nabla(U) = 0\} = \bigvee\{x \oplus y \mid x \wedge y = 0\}.$$

We shall call a frame L *separated* if it satisfies this (although elsewhere such L are also called strongly Hausdorff). It is easy to see that a frame L is separated iff $(e \oplus a) \vee s = (a \oplus e) \vee s$ for all $a \in L$.

Now, the results in question are as follows, with emphasis on the fact that their proofs are constructively valid [5]:

(R) *Every compact separated frame is regular.*

(I) *Any dense homomorphism from a separated frame onto a compact frame is an isomorphism.*

Proof of (R): Since $(e \oplus a) \vee s = (a \oplus e) \vee s$ one has, for any $a \in L$,

$$\begin{aligned} e \oplus a &\leq \bigvee\{x \oplus y \mid x \leq a \ \text{ or } \ x \wedge y = 0\} \\ &\leq \bigvee\{x \oplus y \mid x \leq a \vee y^*\} \leq \bigvee\{x \oplus y \mid (x,y) \in S\} \end{aligned}$$

where $(\ )^*$ stands for pseudocomplement and

$$S = \{(x,y) \mid y \leq \bigvee\{t \mid x \leq a \vee t^*\}\}.$$

Now, S is clearly a downset, closed under arbitrary second slice joins. Moreover it obviously contains $(0,e)$, and if $(x,y),(z,y) \in S$ then

$$y \leq \bigvee\{u \wedge v \mid x \leq a \vee u^*\} \leq \bigvee\{t \mid x \vee z \leq a \vee t^*\}$$

since $u^* \vee v^* \leq (u \wedge v)^*$, and hence $(x \vee z, y) \in S$. This shows S is also closed under finitary first slice joins, and Vermeulen's Lemma then implies that $(e,a) \in S$, meaning

$$a = \bigvee\{t \mid e = a \vee t^*\},$$

which just expresses the regularity of L. □

Proof of (I): For separated L and compact M, let $h : L \to M$ be dense onto. Further, let $k : L \oplus L \to L \oplus M$ be the homomorphism determined by id_L and h, and $s = \bigvee\{x \oplus y \mid x \wedge y = 0\}$ in $L \oplus L$. Then, for any $a, b \in L$,

$$a \oplus e \leq (e \oplus a) \vee s \ \text{ and } \ (e \oplus b) \leq (b \oplus e) \vee s$$

in $L \oplus L$ since L is separated. Now, let $h(a) = h(b)$. Acting k on these two inequalities, one obtains

$$a \oplus e \leq (e \oplus h(a)) \vee k(s) = (e \oplus h(b)) \vee k(s) \leq (b \oplus e) \vee k(s)$$

in $L \oplus M$, and therefore

$$a \oplus e \leq \bigvee\{x \oplus h(y) \mid x \leq b \ \text{ or } \ x \wedge y = 0\} \leq \bigvee\{x \oplus h(y) \mid x \leq b \vee y^*\},$$

where y^* is the pseudocomplement of y. Here, $S = \{(x, h(y)) \mid x \leq b \vee y^*\}$ is a downset in $L \times M$, closed under arbitrary first and finitary second slice joins, the latter since (e, o) clearly belongs to S, and if $x \leq b \vee y^*$ and $x \leq b \vee z^*$ then

$$x \leq (b \vee y^*) \wedge (b \vee z^*) = b \vee (y \vee z)^*.$$

Hence Vermeulen's Lemma implies that $(a, e) \in S$, meaning there exist $y \in L$ such that $h(y) = e$ and $a \leq b \vee y^*$. Now

$$0 = h(y \wedge y^*) = h(y) \wedge h(y^*) = h(y^*)$$

shows $y^* = 0$ since h is dense, hence $a \leq b$, and thus $a = b$ by symmetry, as desired. □

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