Čech complete nearness spaces

H.L. Bentley, W.N. Hunsaker

Dedicated to the memory of Zdeněk Frolík

Abstract. We study Čech complete and strongly Čech complete topological spaces, as well as extensions of topological spaces having these properties. Since these two types of completeness are defined by means of covering properties, it is quite natural that they should have a convenient formulation in the setting of nearness spaces and that in that setting these formulations should lead to new insights and results. Our objective here is to give an internal characterization of (and to study) those nearness structures which are induced by topological extensions of the two above mentioned types.

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Top denotes the category of topological spaces and Near denotes the category of all nearness spaces. We assume familiarity with nearness spaces and with how they can be used to study extensions of topological spaces (see [BH76]). See also Herrlich's 1974 papers on nearness spaces [He74a], [He74b], his more recent survey [He83], and book [He88].

Every nearness space has an underlying topological space whose structure is determined by the closure operator cl defined by :

$$x \in \operatorname{cl} A \qquad \Longleftrightarrow \qquad \{\{x\}, A\} \text{ is near in } X.$$

There arises the functor $\mathsf{T} : \mathsf{Near} \to \mathsf{Top}$. Its image is not all of Top, rather it is the subcategory $\mathsf{Top_S}$ of all symmetric topological spaces, i.e., those which satisfy the axiom of Šanin [Ša43]:

$$x \in \operatorname{cl}\{y\} \text{ if and only if } y \in \operatorname{cl}\{x\}.$$

(These spaces have also been called R_0 spaces and essentially T_1 spaces.)

The functor $\mathsf{T} : \mathsf{Near} \to \mathsf{Top_S}$ has a right inverse $\mathsf{Top_S} \to \mathsf{Near}$ which turns out to be a full embedding of $\mathsf{Top_S}$ as a bicoreflective subcategory of Near; we shall assume this embedding is an inclusion, an assumption which is tantamount to assuming that a symmetric topological space has its structure given by the set of open covers, i.e., a symmetric topological space is a nearness space whose uniform covers are precisely those covers which are refined by some open cover.

Herrlich [He74b], [He88] has shown that a nearness space can have its structure given via any one of the three concepts: uniform cover, near collection, micromeric (or Cauchy) collection. We assume the reader to be familiar with the relationships between these concepts but we do remark that Cauchy filters are definable in terms of uniform covers formally in the same way as in uniform spaces, i.e., every uniform cover has a member which is also a member of the filter.

We need to recall the ideas surrounding the completion of a nearness space (see [BH76]). A **cluster** is a nonempty maximal near collection. A nearness space is said to be **complete** provided each cluster $\mathcal{A}$ has an adherence point, i.e., $\cap \operatorname{cl} \mathcal{A} \neq \emptyset$ where

$$\operatorname{cl} \mathcal{A} = \left\{ \operatorname{cl} A \;\middle|\; A \in \mathcal{A} \right\}.$$

Under regularity, completeness reduces to the usual concept: A regular nearness space is complete iff each Cauchy filter converges.

A uniformly continuous map $f : X \to Y$ is initial iff each uniform cover $\mathcal{A}$ of X is refined by one of the form

$$f^{-1}\mathcal{B} = \left\{ f^{-1}B \;\middle|\; B \in \mathcal{B} \right\}$$

with $\mathcal{B}$ a uniform cover of Y. If $f : X \to Y$ is an initial map and $A \subset X$ then $\operatorname{op}_f A$ denotes the largest open subset G of Y such that $\operatorname{int}_X A = f^{-1}G$, i.e., $\operatorname{op}_f A = Y \setminus \operatorname{cl}_Y f[X \setminus A]$. An initial uniformly continuous map $f : X \to Y$ is called **strict** provided

$$\left\{ \operatorname{op}_f \mathcal{A} \;\middle|\; \mathcal{A} \text{ is a uniform cover of } X \right\}$$

is a base for the uniform covers of Y, where

$$\operatorname{op}_f \mathcal{A} = \left\{ \operatorname{op}_f A \;\middle|\; A \in \mathcal{A} \right\}.$$

By a **dense** map we mean a uniformly continuous map $f : X \to Y$ for which the image fX is dense in the topological space $\mathsf{T}Y$. An initial map $f : X \to Y$ into a regular Y is a strict map iff it is a dense map. Surjective initial maps are always strict. Observe that if $f : X \to Y$ is a dense map then for each subset A of X we have $\operatorname{op}_f A \subset \operatorname{cl}_Y fA$.

Every nearness space has a completion $f : X \to Y$ which is characterized by the following properties:

(1) $f : X \to Y$ is a strict dense injection.
(2) Y is complete.
(3) For each $y \in Y \setminus fX$, $\{y\}$ is a closed subset of the topological space $\mathsf{T}Y$.

If $f : X \to Y$ satisfies the above conditions, then $f : X \to Y$ will be called the **completion** of X; it is determined by (1), (2), and (3) up to isomorphism.

We shall be dealing with open filters: a filter $\mathcal{F}$ on a topological space is, as usual, said to be **open** provided every member of $\mathcal{F}$ contains some open member of $\mathcal{F}$. This condition can be expressed by saying that

$$\operatorname{int} \mathcal{F} = \left\{ \operatorname{int} F \;\middle|\; F \in \mathcal{F} \right\}$$

corefines $\mathcal{F}$.

1. Proposition. *Let X and Y be nearness spaces and let $f : X \to Y$ be an initial dense map. Then*

(1) *If $\mathcal{A}$ is a collection of subsets of X such that $\operatorname{int}\mathcal{A}$ corefines $\mathcal{A}$ then $\mathcal{A}$ is near in X iff $\operatorname{op}_f \mathcal{A}$ is near in Y.*

(2) *If $\mathcal{F}$ is an open filter on X then $\operatorname{op}_f \mathcal{F}$ is an open filterbase on Y, i.e., it generates an open filter.*

Proof: (1): If $\mathcal{A}$ is near and $\operatorname{int}\mathcal{A}$ corefines $\mathcal{A}$ then $\operatorname{int}\mathcal{A}$ is near. Since f is uniformly continuous, $f \operatorname{int}\mathcal{A}$ is near in Y. Since f is an initial map, $\operatorname{op}_f \mathcal{A}$ corefines $f \operatorname{int}\mathcal{A}$, and it follows that $\operatorname{op}_f \mathcal{A}$ is near in Y.

(2): Since f is a dense map, $\operatorname{cl}_Y f\mathcal{A}$ corefines $\operatorname{op}_f \mathcal{A}$. Thus, if $\operatorname{op}_f \mathcal{A}$ is near in Y then so is $\operatorname{cl}_Y f\mathcal{A}$. Y being a nearness space, it then follows that $f\mathcal{A}$ is near in Y. Since f is initial, $f\mathcal{A}$ being near in Y implies that $\mathcal{A}$ is near in X. □

Čech complete and strongly Čech complete spaces

Frolík [Fr60] introduced the concept of a set of open covers α being complete. His definition required those open filters which are α-Cauchy to have an adherence point. Frolík showed that spaces possessing such a set of open covers have many interesting properties, and that among Tychonoff spaces, these are precisely the Čech complete spaces. Fletcher and Lindgren [FL72] modified Frolík's definition by requiring that the open α-Cauchy filters should converge. Furthermore, they called a set α of open covers of the latter type "complete" and those which Frolík called complete, they called "weakly complete". Frolík's use of the term "complete" was natural for him since he used only the one concept, and Fletcher's and Lindgren's use of different terminology was natural for them since they were concerned with both concepts. Since, as Fletcher and Lindgren showed, these two concepts do not coincide (even for paracompact Hausdorff spaces), one must be very careful about terminology. In order to avoid creating confusion, we use the terminology of Fletcher and Lindgren.

2. Definition. *Let X be a topological space and let α be a collection of nonempty covers of X.*

(1) *A filter (or filterbase) $\mathcal{F}$ on X is said to be α-**Cauchy** iff for every cover $\mathcal{U} \in \alpha$ there exist $U \in \mathcal{U}$ and $F \in \mathcal{F}$ with $F \subset U$.*

(2) *The collection α is said to be **complete** iff every open α-Cauchy filter converges.*

(3) *The collection α is said to be **weakly complete** iff every open α-Cauchy filter has an adherence point.*

Clearly, in the preceding conditions (1) and (2), we equivalently could have required every open α-Cauchy filterbase to converge (respectively, to have an adherence point).

3. Definition.

(1) *A topological space X is said to be **Čech-complete** iff there exists a countable, weakly complete collection α of open covers of X.*

(2) *A topological space X is said to be* **strongly Čech-complete** *iff there exists a countable, complete collection α of open covers of X.*

Since every complete collection is also weakly complete, it follows that every strongly Čech complete space is also Čech complete.

Note that we have defined "Čech completeness" without any restriction that the space involved be Tychonoff. Since we will be concerned with quite general spaces, any restriction to Tychonoffness would be unnaturally hampering.

Čech complete extensions: constrained spaces

We are concerned with the following general type of problem:

> Let P be a property of topological spaces. Characterize those spaces X that have an extension Y with property P.

Solving problems of this type usually involves introducing other types of structure on X, e.g., metrics, uniformities, proximities, nearnesses, etc. Bentley and Herrlich [BH76] used nearness structures to solve several problems of this type; they considered extensions with the properties: Hausdorffness, regularity, compactness, paracompactness, topological completeness, and others. Brandenburg [Br77], and independently, Carlson [Ca80] solved the problem for developable spaces and for complete Moore spaces. This is only a partial listing; during the last fifteen years quite a few problems of this type have been solved using nearness structures [Be77], [Ca79], [Ca81], [He88], [BHO89], [Be91].

We need to make matters more precise.

Let a symmetric topological space Y be an extension of a topological space X, i.e., X is a dense subspace of Y. Then a nearness structure on X is determined by defining a collection $\mathcal{A}$ of subsets X to be near iff $\mathcal{A}$ has an adherence point in Y, i.e., iff

$$\cap\{ \operatorname{cl}_Y A \mid A \in \mathcal{A} \} \neq \emptyset.$$

This nearness structure is said to be **the nearness structure on X induced by the extension Y**.

Nearness spaces induced by an extension are always subtopological, i.e., every near collection is contained in a near grill. The somewhat stronger condition that every near collection is contained in a cluster is satisfied when the nearness space is induced by a strict extension. In that case, the induced nearness space is said to be **concrete**. The concrete nearness spaces are precisely those whose completion is topological. For an explanation of these ideas, see [BH76], [Be75], [He74b], [He83], [Be76], and especially [He88].

4. Definition. *A nearness space X is said to be* **constrained** *iff there exists a countable set α of open uniform covers of X such that every open α-Cauchy filter is near in X ("open" here means open in the underlying topological space $\mathsf{T}X$ of X).*

Equivalently, we could have required that every open α-Cauchy filterbase be near.

In order to shorten the language, whenever α is a countable set of open uniform covers of a nearness space X such that every open α-Cauchy filter is near in X, then we shall say that α **demonstrates the constrainedness of** X, or equivalently, that X is **constrained by** α.

We let Cnstra denote the full subcategory of Near consisting of all constrained spaces.

5. Proposition. *A symmetric topological space is Čech complete iff as a nearness space it is constrained.*

The underlying topological space $\mathsf{T}X$ of a constrained nearness space X can fail to be constrained (i.e., Čech complete). (See Example 1.)

In the next few propositions, we shall be exhibiting the fact that constrainedness is preserved under certain constructions.

6. Proposition. *Let X and Y be nearness spaces and let $f : X \to Y$ be a strict dense map. Then X is constrained iff Y is constrained.*

Proof: Assume that X is constrained by α. We will show that the countable set of open uniform covers

$$\operatorname{op}_f \alpha = \left\{ \operatorname{op}_f \mathcal{U} \;\middle|\; \mathcal{U} \in \alpha \right\}$$

demonstrates the constrainedness of Y. Let $\mathcal{G}$ be an open $(\operatorname{op}_f \alpha)$-Cauchy filter on Y. Then the filter $\mathcal{F}$ on X generated by

$$\left\{ f^{-1}G \;\middle|\; G \in \mathcal{G} \right\}$$

is an open filter on X which is easily seen to be α-Cauchy. Therefore $\mathcal{F}$ is near in X and it follows easily that $\mathcal{G}$ is near in Y.

For the proof in the converse direction, assume that Y is constrained by β. For every $\mathcal{U} \in \beta$ select $\mathcal{G}_{\mathcal{U}}$, an open uniform cover of X such that $\operatorname{op}_f \mathcal{U}$ refines $\mathcal{U}$, and let

$$\alpha = \left\{ \mathcal{G}_{\mathcal{U}} \;\middle|\; \mathcal{U} \in \beta \right\}.$$

It follows from Proposition 1 that α demonstrates the constrainedness of X. □

7. Corollary. *A nearness space is constrained if and only if its completion is constrained.*

The following theorem is one of our main results; it is a corollary to the foregoing results.

8. Theorem. *Let X be a nearness space. Then the following are equivalent:*

(1) *X carries the nearness structure induced by a Čech complete strict extension $\mathsf{T}X \to Y$ with Y a symmetric topological space.*
(2) *X is concrete and constrained.*
(3) *The completion of X is both topological and Čech complete.*

The next few propositions address the preservation of constrainedness under products and sums.

9. Proposition. Cnstra *is countably productive in* Near.

Proof: Let $(X_i)_{i\in I}$ be a family of constrained nearness spaces with I a countable set and let $(\pi_i : X \to X_i)_{i\in I}$ be the Near product. For each $i \in I$ let α_i demonstrate the constrainedness of X_i. We shall show that

$$\alpha = \left\{ \pi_i^{-1}\mathcal{B}_i \;\middle|\; i \in I \text{ and } \mathcal{B}_i \in \alpha_i \right\}$$

demonstrates the constrainedness of X. Clearly, α is a countable set of open uniform covers of X. Let $\mathcal{F}$ be an open α-Cauchy filter on X. We need to show that $\mathcal{F}$ is near in X. Let $\mathcal{G}$ be a maximal open filter on X such that $\mathcal{F} \subset \mathcal{G}$. Clearly, $\mathcal{G}$ is α-Cauchy. It is sufficient to show that $\mathcal{G}$ is near in X. For each $i \in I$, $\pi_i\mathcal{G}$ is a maximal open filter on X and it is easy to see that $\pi_i\mathcal{G}$ is α_i-Cauchy. Therefore, $\pi_i\mathcal{G}$ is near in X_i. In order to show that $\mathcal{G}$ is near in X, let $\mathcal{U}$ be a uniform cover of X. There exists a finite subset J of I and there exists a family $(\mathcal{B}_i)_{i\in J}$ such that $\mathcal{B}_i$ is a uniform cover of X_i and

$$\textstyle\bigwedge_{i\in J} \pi_i^{-1}\mathcal{B}_i \quad \text{refines} \quad \mathcal{U}.$$

For each $i \in I$ there exists $B_i \in \mathcal{B}_i$ which meets every member of $\pi_i\mathcal{G}$. There exists $U \in \mathcal{U}$ such that

$$\bigcap_{i\in J} \pi_i^{-1} B_i \subset U$$

The maximality of $\pi_i\mathcal{G}$ guarantees that $B_i \in \pi_i\mathcal{G}$. For each $i \in J$, select $G_i \in \mathcal{G}$ such that $B_i = \pi_i G_i$ and define $G = \bigcap_{i\in J} G_i$. Then $G \in \mathcal{G}$, and since $G \subset U$, $U \in \mathcal{G}$. Therefore, U meets every member of $\mathcal{G}$ and it follows that $\mathcal{G}$ is near in X. □

10. Proposition. Cnstra *is finitely summable in* Near.

Proof: Let $(f_i : X_i \to X)_{i\in I}$ be a sum in Near with each X_i constrained and with I finite. For each $i \in I$ let α_i demonstrate the constrainedness of X_i. Then the set

$$\alpha = \left\{ \bigcup_{i\in I} f_i\mathcal{B}_i \;\middle|\; (\mathcal{B}_i)_{i\in I} \in \prod_{i\in I} \alpha_i \right\}$$

is a countable set of open uniform covers of X which demonstrates the constrainedness of X.

□

We end this section with a proposition which is an improvement on Proposition 6 in case we restrict our attention to weakly regular nearness spaces. Recall that a nearness space X is called **weakly regular** provided that whenever $\mathcal{A}$ is far in X then $\mathcal{A}$ is corefined by some far collection of sets which are open. (For a comparison of weak regularity and other forms of regularity, see [BLC91].)

11. Proposition. *If X is a weakly regular space which is constrained, then every subspace of X is constrained as well.*

Proof: Let Y be a subspace of X and let α demonstrate the constrainedness of X. We show that

$$\beta = \left\{ \{Y\} \wedge \mathcal{U} \;\middle|\; \mathcal{U} \in \alpha \right\}$$

demonstrates the constrainedness of Y. Let $\mathcal{G}$ be an open β-Cauchy filter on Y and define

$$\mathcal{F} = \left\{ U \subset X \;\middle|\; U \text{ is open in } X \text{ and } G \subset U \text{ for some } G \in \mathcal{G} \right\}.$$

One easily sees that $\mathcal{F}$ is α-Cauchy and from that it follows that $\mathcal{F}$ is near in X. Since X is weakly regular, it follows that $\mathcal{G}$ is near in X, hence also in Y. □

Strongly Čech complete extensions: controlled spaces

12. Definition. *A nearness space X is said to be* **controlled** *iff there exists a countable set α of open uniform covers of X such that every open α-Cauchy filter is Cauchy in X (as before "open" means open in the underlying topological space $\mathsf{T}X$ of X).*

We let Cntrol denote the full subcategory of Near consisting of all controlled spaces.

13. Proposition. *A symmetric topological space is strongly Čech complete iff as a nearness space it is controlled.*

The underlying topological space $\mathsf{T}X$ of a controlled nearness space X can fail to be controlled (i.e., strongly Čech complete). (See Example 2.)

Proofs of the propositions appearing in this section are omitted since they are either simpler than or somewhat analogous to the ones on constrained spaces.

14. Proposition. *Let X and Y be nearness spaces and let $f : X \to Y$ be a strict dense map. Then X is controlled iff Y is controlled.*

The following proposition shows that controlled spaces are better behaved than constrained ones (cf. Propositions 6 and 11).

15. Proposition. Cntrol *is hereditary in* Near.

16. Corollary. *A nearness space is controlled if and only if its completion is controlled.*

The above results make evident the proof of the following theorem, which is the second one of our main results.

17. Theorem. *Let X be a nearness space. Then the following are equivalent:*

(1) *X carries the nearness structure induced by a strongly Čech complete strict extension $\mathsf{T}X \to Y$ with Y a symmetric topological space.*
(2) *X is concrete and controlled.*
(3) *The completion of X is both topological and strongly Čech complete.*

Controlled merotopic spaces

By relaxing the close tie with topological spaces and moving our investigation to the setting of merotopic spaces, we can get some additional insight into the controlled spaces.

Katětov [Ka62], [Ka65] originally axiomatized merotopic spaces by means of the micromeric (or Cauchy) collections, but he showed that they can be obtained equivalently by means of uniform covers. A merotopic space is a set X together with a collection of covers that satisfy all of the axioms for a uniform structure [Tu40] except (possibly) the star refinement axiom. Herrlich [He74a] has shown that merotopic structures can be equivalently described by axiomatizing the concept of near collections. We let Mer denote the category of all merotopic spaces.

Fil denotes the category of all filter spaces, i.e., those merotopic spaces X such that every micromeric collection is corefined by some Cauchy (i.e., micromeric) filter (equivalently, every near collection is contained in some near grill [Ro75], [BHR76]). Recall that Fil is bicoreflective in Mer: for each merotopic space X, the Fil bicoreflection $\mathsf{Fil}\,X \to X$ is the identity map on the set level and $\mathsf{Fil}\,X$ is defined by requiring that a collection $\mathcal{A}$ is micromeric in $\mathsf{Fil}\,X$ if and only if there exists a Cauchy filter $\mathcal{F}$ on X which corefines $\mathcal{A}$.

The following characterization of initial maps in Fil appears in [Ka65] (note that Katětov used the terminology "projective" instead of "initial"). See also [He74b] and [LC89]).

18. Proposition. *Let X and Y be filter spaces and let $f : X \to Y$ be a uniformly continuous map. Then the following are equivalent:*

(1) *$f : X \to Y$ is initial in* Fil.
(2) *For any filter $\mathcal{F}$ on X, if $f\mathcal{F}$ is micromeric in Y then $\mathcal{F}$ is a Cauchy filter on X.*

We lift the concept of controlledness to Mer.

19. Definition. *A merotopic space X is said to be* **controlled** *iff there exists a countable set α of uniform covers of X such that every α-Cauchy filter is Cauchy in X.*

In order to shorten the language, whenever α is a countable set of covers of a merotopic space X such that every α-Cauchy filter is Cauchy in X, then we shall say that α **demonstrates the controlledness of** X, or equivalently, that X is **controlled by** α.

20. Proposition. *A nearness space is a controlled nearness space* (*as defined in Definition* 12) *if and only if it is a controlled merotopic space* (*as defined in Definition* 19).

Proof: The proof can be easily given once the following observations are made regarding a nearness space X.

(1) If α is a set of uniform covers of X then

$$\alpha' = \left\{ \operatorname{int}\mathcal{U} \;\middle|\; \mathcal{U} \in \alpha \right\}$$

is a set of open uniform covers of X such that every open α'-Cauchy filter is also α-Cauchy.

(2) If α is a set of open uniform covers of X and $\mathcal{F}$ is an α-Cauchy filter then

$$\mathcal{G} = \left\{ G \subset X \;\middle|\; \text{for some } F \in \mathcal{F},\, F \subset \operatorname{int} G \right\}$$

is an open α-Cauchy filter such that $\mathcal{G} \subset \mathcal{F}$. □

The following theorem is quite important in that it frequently enables one to produce category theoretic proofs of propositions about controlled spaces (e.g., in Propositions 24 and 27 below); it is preceded by a lemma.

21. Lemma. *Let X be a controlled merotopic space. Then there exists a collection α of uniform covers of X which demonstrates the controlledness of X such that*

$$\mathcal{A}, \mathcal{B} \in \alpha \qquad \Longrightarrow \qquad \mathcal{A} \wedge \mathcal{B} \in \alpha.$$

22. Theorem. *For any merotopic space X the following are equivalent:*

(1) *X is controlled.*
(2) *There exists a merotopic space Y having the same underlying set as X so that Y has a countable base, with $\mathsf{Fil}\, X = \mathsf{Fil}\, Y$, and with $\mathrm{id} : X \to Y$ being uniformly continuous.*
(3) *There exists a merotopic space Y having a countable base and there exists a uniformly continuous map $f : X \to Y$ with its Fil reflection $f : \mathsf{Fil}\, X \to \mathsf{Fil}\, Y$ an initial map in Fil.*

23. Corollary. *If X is controlled, then so is every X' with $\mathsf{Fil}\,(X') = \mathsf{Fil}\, X$ and with $\mathrm{id} : X' \to X$ uniformly continuous.*

24. Proposition. *A merotopic space with a countable base is controlled.*

Proof: Let X have a countable base and let $Y = X$. Observe that $\mathrm{id} : X \to Y$ is uniformly continuous and $\mathrm{id} : \mathsf{Fil}\, X \to \mathsf{Fil}\, Y$ is initial. □

25. Corollary. *Every discrete space and every indiscrete space is controlled.*

Proof: If X is discrete then $\{\mathcal{A}\}$ is a base for uniform covers of X where $\mathcal{A} = \big\{ \{x\} \,|\, x \in X \big\}$. If X is indiscrete then $\big\{\{X\}\big\}$ is a base for uniform covers of X. □

We recall the nature of final sinks and initial sources in Mer. A sink $(f_i : X_i \to X)_{i \in I}$ is final in Mer provided a cover $\mathcal{A}$ of X is a uniform cover of X iff for all $i \in I$, $f_i^{-1}\mathcal{A}$ is a uniform cover of X_i. A source $(f_i : X \to X_i)_{i \in I}$ is initial in Mer provided a cover $\mathcal{A}$ of X is a uniform cover of X iff there exists a finite subset J of I and a family $(\mathcal{B}_i)_{i \in J}$ with $\mathcal{B}_i$ a uniform cover of X_i and with

$$\bigwedge_{i \in J} (f_i^{-1}\mathcal{B}_i) \qquad \text{a refinement of} \qquad \mathcal{A}.$$

26. Proposition. *The property of having a countable base is hereditary, countably productive, and finitely summable in* Mer*.*

Proof: The proof is given in [He88]: for hereditary see 3.1.11 (7), for countable productivity see 3.2.12, and for finite summability see 3.3.13. □

27. Proposition. *In* Mer*, the property of being controlled is hereditary, countably productive, and finitely summable.*

Proof: The proof that controlledness is hereditary is straightforward. For finite summability, one can give a direct proof analogous to the proof of Proposition 10. Alternatively, one can construct a category theoretic proof using Theorem 22 and Proposition 26.

For countable productivity, we will present a category theoretic proof. Let $(f_i : X \to X_i)_{i \in I}$ be an initial source in Mer with each X_i controlled. By Theorem 22, there exists uniformly continuous maps $g_i : X_i \to Y_i$ with Y_i having a countable base and with each single map $g_i : \mathsf{Fil}\, X_i \to \mathsf{Fil}\, Y_i$ being initial in Fil. Therefore,

$$\bigl(f_i : \mathsf{Fil}\, X \to \mathsf{Fil}\, X_i\bigr)_{i \in I}$$

is initial in Fil. By the preservation of initiality under composition, it follows that

$$\bigl(g_i \circ f_i : \mathsf{Fil}\, X \to \mathsf{Fil}\, Y_i\bigr)_{i \in I}$$

is initial in Fil. Let Y denote the Mer product of the spaces $(Y_i)_{i \in I}$ and let $u : X \to Y$ be the product of the maps $(g_i \circ f_i)_{i \in I}$. Then it is not difficult to show that $u : \mathsf{Fil}\, X \to \mathsf{Fil}\, Y$ is an initial map in Fil, and the proof is complete. □

Controlled nearness spaces revisited

Because of its close relationship with Top as exemplified in the possibility of studying extensions internally (Theorems 8 and 17), we are interested more in the category Near than in the category Mer. Therefore, we now present the ideas which enable us to modify the characterization of controlled merotopic spaces given in Theorem 22 to obtain a similar characterization of a wide class of controlled nearness spaces.

The following notion, due to Brandenburg [Br77], is crucial.

28. Definition. *Let β be a collection of covers of a set X.*

(1) *For $A \subset X$, we define*

$$\operatorname{int}_\beta A = \left\{ x \in X \;\middle|\; \operatorname{star}(x, \mathcal{B}) \subset A \text{ for some } \mathcal{B} \in \beta \right\}.$$

(2) *For $\mathcal{A}$ a collection of subsets of X, we define*

$$\operatorname{int}_\beta \mathcal{A} = \left\{ \operatorname{int}_\beta A \;\middle|\; A \in \mathcal{A} \right\}.$$

(3) *The collection β is called* **kernel-normal** *iff for each $\mathcal{A} \in \beta$ there exists $\mathcal{B} \in \beta$ such that $\mathcal{B}$ refines $\operatorname{int}_\beta \mathcal{A}$.*

29. Proposition. *If α is a kernel normal collection of covers of a set X, then*

$$\beta = \left\{ \bigwedge_{i=1}^{n} \mathcal{A}_i \;\middle|\; \mathcal{A}_1, \cdots, \mathcal{A}_n \in \alpha \right\}$$

is a kernel normal collection of covers of X.

We let Subtop denote the subcategory of Near whose objects are the subtopological nearness spaces, i.e., those nearness spaces which are a subspace of some symmetric topological space. Robertson has proved [Ro75], [BHR76], that

$$\mathsf{Subtop} = \mathsf{Near} \cap \mathsf{Fil}.$$

It is known [Be75] that Subtop is bicoreflective in Near and [Ro75] that its bicoreflector is the restriction of the bicoreflector $\mathsf{Mer} \to \mathsf{Fil}$. Also, Subtop is bireflective in Fil and its bireflector is the restriction of the bireflector $\mathsf{Mer} \to \mathsf{Near}$.

We need the following result.

30. Proposition. *Let X and Y be subtopological nearness spaces and let $f : X \to Y$ be a map. Then $f : X \to Y$ is initial in Subtop iff $f : X \to Y$ is initial in Fil.*

The following theorem is our desired analogue of Theorem 22; it is preceded by a lemma.

31. Lemma. *Let X be a nearness space. Then the following are equivalent:*

(1) *X is controlled by a kernel normal collection of uniform covers of X.*
(2) *X is controlled by a kernel normal collection of open uniform covers of X.*
(3) *X is controlled by a kernel normal collection α of open uniform covers of X such that*

$$\mathcal{A}, \mathcal{B} \in \alpha \qquad \Longrightarrow \qquad \mathcal{A} \wedge \mathcal{B} \in \alpha.$$

32. Theorem. *For any nearness space, the following are equivalent:*

(1) *X is controlled by some α which is kernel-normal.*
(2) *There exists a nearness space Y having the same underlying set as X so that Y has a countable base, with $\mathsf{S}X = \mathsf{S}Y$, and with $\mathrm{id} : X \to Y$ being uniformly continuous.*
(3) *There exists a nearness space Y with a countable base and there exists a uniformly continuous map $f : X \to Y$ such that $f : SX \to SY$ is an initial map in Subtop.*

33. Question. *Do there exist nearness spaces which are controlled but which are not controlled by a* **kernel normal** *collection of uniform covers?*

Čech complete nearness spaces

We introduce the concept of Čech complete nearness spaces. These spaces have a kind of dual citizenship, i.e., Čech completeness is a hybrid concept partaking partly of the notion of a constrained nearness space and partly of the notion of the underlying topological space being Čech complete. Other than to point out the obvious relationships, we do not study these spaces here.

34. Definition. *We say that a nearness space X is* **Čech complete** *provided there exists a countable collection of uniform covers of X which is a weakly complete collection of open covers of $\mathsf{T}X$.*

35. Proposition. *If X is a Čech complete nearness space then its underlying topological space $\mathsf{T}X$ is a Čech complete topological space.*

36. Proposition. *Every Čech complete nearness space is constrained.*

Strongly Čech complete nearness spaces

Analogous to the Čech complete nearness spaces there are the strongly Čech complete nearness spaces. The remarks we made in the preceding section about Čech complete nearness spaces apply as well to strongly Čech complete nearness spaces.

37. Definition. *We say that a nearness space X is* **strongly Čech complete** *provided there exists a countable collection of uniform covers of X which is a complete collection of open covers of $\mathsf{T}X$.*

Clearly every strongly Čech complete nearness space is also Čech complete.

38. Proposition. *If X is a strongly Čech complete nearness space then its underlying topological space $\mathsf{T}X$ is a strongly Čech complete topological space.*

39. Proposition. *Every strongly Čech complete nearness space is controlled.*

Examples

Example 1. *There exists a constrained space X with $\mathsf{T}X$ not constrained. Let Z be a Tychonoff space which is not Čech complete* (*e.g., the space of rational numbers with its usual topology*)*. By Proposition* 5*, Z is not constrained. Let X be the nearness space with the same underlying set as Z and having the structure induced by the Stone-Čech compactification $Z \to \beta Z$. Then X is a nearness subspace of βZ; in fact, βZ is the completion of X. βZ, being a compact Hausdorff space, is Čech complete, hence is constrained. By Proposition* 7*, X is constrained. But $\mathsf{T}X = Z$ is not constrained.*

Example 2. *There exists a controlled space X with $\mathsf{T}X$ not controlled. Let Z be a Tychonoff space which is not strongly Čech complete* (*again, e.g., the rational numbers*) *and proceed as in Example* 1*.*

References

[Be75] Bentley H.L., *Nearness spaces and extensions of topological spaces*, Studies in Topology, Edited by Nick M. Stavrakas and Keith R. Allen, Academic Press, New York, 1975, pp. 47–66.

[Be76] ______, *The role of nearness spaces in topology*, Categorical Topology, Edited by E. Binz and H. Herrlich, Lecture Notes in Math., vol. 540, Springer Verlag, Berlin, 1976, pp. 1–22.

[Be77] ______, *Normal nearness spaces*, Quaestiones Math. **2** (1977), 23–43.

[Be91] ______, *Paracompact spaces*, Topology and its Appl. **39** (1991), 283–297.

[BH76] Bentley H.L., Herrlich H., *Extensions of topological spaces*, Topology, Edited by Stanley P. Franklin and Barbara V. Smith Thomas, Lect. Notes in Pure and Appl. Math., vol. 24, Marcel Dekker, New York, 1976, pp. 129–184.

[BHO89] Bentley H.L., Herrlich H., Ori R.G., *Zero sets and complete regularity for nearness spaces*, Categorical Topology, Edited by Jiří Adámek and Saunders MacLane,, World Scientific, Singapore, 1989, pp. 446–461.

[BHR76] Bentley H.L., Herrlich H., Robertson W.A., *Convenient categories for topologists*, Comm. Math. Univ. Carolinae **17** (1976), 207–227.

[BLC91] Bentley H.L., Lowen-Colebunders E., *Completely regular spaces*, Comm. Math. Univ. Carolinae **32** (1991), 129–153.

[Br77] Brandenburg H., *On a class of nearness spaces and the epireflective hull of developable topological spaces*, Proc. of the Int. Topology Symp., Belgrade, 1977.

[Ca79] Carlson John W., *H-Closed and countably compact extensions*, Pacific J. Math. **81** (1979), 317–326.

[Ca80] ______, *Developable spaces and nearness structures*, Proc. Amer. Math. Soc. **78** (1980), 573–579.

[Ca81] ______, *Baire space and second category extensions*, Topology and its Appl. **12** (1981), 135–140.

[En77] Engelking R., *General Topology*, Heldermann Verlag, Berlin, 1989.

[FL72] Fletcher P., Lindgren W.F., *Orthocompactness and strong Čech completeness in Moore spaces*, Duke. Math. Jour. **4** (1972), 753–766.

[Fr60] Frolík Z., *Generalizations of the G_δ-property of complete metric spaces*, Czech. Math. Jour. **10** (1960), 359–379.

[He74a] Herrlich H., *A concept of nearness*, Gen. Topology Appl. **4** (1974), 191–212.

[He74b] ______, *Topological structures*, Topological Structures, vol. 52, Math. Centre Tracts, 1974, pp. 59–122.

[He83] ______, *Categorical topology 1971-1981*, General Topology and its Relations to Modern Analysis and Algebra V, Proceedings of the Fifth Prague Topological Symposium 1981, Heldermann Verlag, Berlin, 1983, pp. 279–383.

[He88] ______, *Topologie II: Uniforme Räume*, Heldermann Verlag, Berlin, 1988.

[Ka62] Katětov M., *Allgemeine Stetigkeitsstrukturen*, Proc. Inter. Congr. Math. Stockholm 1962, 1963, pp. 473–479.

[Ka65] ______, *On continuity structures and spaces of mappings*, Comment. Math. Univ. Carol. **6** (1965), 257–278.

[LC89] Lowen-Colebunders E., *Function Classes of Cauchy Continuous Maps*, Marcel Dekker, Inc., New York, 1989.

[Ro75] Robertson W.A., *Convergence as a Nearness Concept*, Thesis, Carlton University, Ottawa, 1975.

[Ša43] Šanin N.A., *On separation in topological spaces*, Dokl. Akad. Nauk SSSR **38** (1943), 110–113.

[Tu40] Tukey J.W., *Convergence and uniformity in topology*, Ann. Math. Studies, vol. 2, Princeton Univ. Press, Princeton, 1940.

University of Toledo, Toledo, OH 43560, USA

Southern Illinois University, Carbondale, IL 62901, USA

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