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Homology theory in the AST II
Basic concepts, Eilenberg-Steenrod's axioms

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Abstract. Homology functor in the spirit of the AST is defined, its basic properties are studied. Eilenberg-Steenrod axioms for this functor are formulated and established.

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0. Introduction.

In this paper which is the direct continuation of [G1] we start with developing of one kind of a homological theory in the AST which can be based on algebraic results from [G1].

It is necessary to emphasize the fact that topological phenomena in the AST are studied by means of indiscernibility relations (or at least π-symmetries) and not by means of topological structures (i.e. open and closed sets, convergence etc.).

There is no doubt that a figure in some indiscernibility relation can be considered as a topological object, but this very different approach to topological phenomena allows a different and very natural possibility to develop a homology theory. Our approach is based on ideas of the simplicial theory, relevantly reformulated into the AST. We start from simplexes which are infinitesimally small (i.e. which lie in one monad) and without any (linear) structure. A nontrivial difference is that we need infinite summation (to cover more than a part of finite number of monads) and this is the main reason for the whole paper [G1] in which we prepared the most important algebraic facts for the first steps in a homological theory (in the AST). This approach seems to be one of the most natural for the AST . Of course there are (except direct transferring of some classical approaches to the AST) also other possibilities but (as I believe) we shall discuss them later.

1. Generalized symmetries.

The notion of the generalized symmetry plays a similar role in our considerations as that of the geometrical simplicial complex in classical homology theories.

Definition 1.1. *Class* $\mathbf{R}$ *is said to be a* **generalized symmetry on $\mathbf{A}$** *iff*

(1) $\emptyset \notin \mathbf{R}$
(2) $(\emptyset \neq \mathbf{u} \subseteq \mathbf{v} \ \& \ \mathbf{v} \in \mathbf{R}) \Rightarrow \mathbf{u} \in \mathbf{R}$
(3) $(\forall x \in \mathbf{A})\,(\{x\} \in \mathbf{R})$.

Lemma 1.2. *Let $\mathbf{S}$ be a relation such that $(\forall x \in \mathbf{A})\,([x,x] \in \mathbf{S})$. Then $\mathbf{r}(\mathbf{S}) = \left\{\mathbf{u}; \mathbf{u}^2 \subseteq \mathbf{S}\right\} \setminus \{\emptyset\}$ is a generalized symmetry on $\mathbf{A}$.*

PROOF: Clearly, the property which defines $\mathbf{r}(-)$ is hereditary on subsets. Therefore (2) of 1.1 is satisfied. The rest of 1.1 is obvious.

□

Lemma 1.3. *Let $\mathcal{M}$ be a codable class of generalized symmetries on $\mathbf{A}$. Then $\bigcup \mathcal{M}$ and $\bigcap \mathcal{M}$ are also generalized symmetries.*

Theorem 1.4. *Let $\mathbf{R}$ be a generalized symmetry on $\mathbf{A}$ which is a π-class. Then there is a sequence $\{\mathbf{R}_n; n \in \mathbf{FN}\}$ such that for each n, $\mathbf{R}_n$ is set-definable class and generalized symmetry on $\mathbf{A}$, $\mathbf{R}_{n+1} \subseteq \mathbf{R}_n$ and finally $\mathbf{R} = \bigcap \{\mathbf{R}_n; n \in \mathbf{FN}\}$.*

PROOF: As $\mathbf{R}$ is a π-class, there is a sequence $\{\mathbf{S}_n; n \in \mathbf{FN}\}$ of Sd-classes such that $\mathbf{S}_{n+1} \subseteq \mathbf{S}_n$. Then (1) and (3) of 1.1 are satisfied for all n. Denote $\mathbf{R}_n = \{\mathbf{u}; \mathbf{u} \neq \emptyset \ \& \ (\exists \mathbf{v} \in \mathbf{S}_n)\,(\mathbf{u} \subseteq \mathbf{v})\}$. Each $\mathbf{R}_n$ is an Sd-class and also it is clear that for each n $\mathbf{R}_{n+1} \subseteq \mathbf{R}_n$ and (1) and (3) of 1.1 are satisfied. All we have to prove is that $\mathbf{R} = \bigcap \{\mathbf{R}_n; n \in \mathbf{FN}\}$. Clearly $\mathbf{R} = \bigcap \{\mathbf{S}_n; n \in \mathbf{FN}\} \subseteq \bigcap \{\mathbf{R}_n; n \in \mathbf{FN}\}$ (because $\mathbf{S}_n \subseteq \mathbf{R}_n$). Let $\mathbf{u} \in \bigcap \{\mathbf{R}_n; n \in \mathbf{FN}\}$. It means that for each n there is $\mathbf{u}_n \in \mathbf{S}_n$ such that $\mathbf{u} \subseteq \mathbf{u}_n$. By the axiom of prolongation and the theorem on prolongation of a countable sequence of Sd-classes, we obtain $\alpha \in \mathbf{N} \setminus \mathbf{FN}$ such that $\mathbf{u} \subseteq \mathbf{u}_\alpha \in \mathbf{S}_\alpha \subseteq \mathbf{R}$. Hence $\mathbf{u} \subseteq \mathbf{u}_\alpha \in \mathbf{R}$. As $\mathbf{R}$ is a generalized symmetry, we have $\mathbf{u} \in \mathbf{R}$, which proves the last inclusion. □

Theorem 1.5. *Let $\mathbf{S}$ be a π-symmetry, $\{\mathbf{S}_n; n \in \mathbf{FN}\}$ be its generating sequence (i.e. $\mathrm{Sd}(\mathbf{S}_n), \mathbf{S}_{n+1} \subseteq \mathbf{S}_n, \mathbf{S} = \bigcap \{\mathbf{S}_n; n \in \mathbf{FN}\}$). Then $\mathbf{r}(\mathbf{S}) = \bigcap \{\mathbf{r}(\mathbf{S}_n); n \in \mathbf{FN}\}$.*

PROOF: Take $\mathbf{u} \in \mathbf{r}(\mathbf{S})$. We have $\mathbf{u} \neq \emptyset$ and $\mathbf{u}^2 \subseteq \mathbf{S}$. Therefore for each n $\mathbf{u}^2 \subseteq \mathbf{S}_n$ and hence $\mathbf{u} \in \mathbf{r}(\mathbf{S}_n)$. So $\mathbf{u} \in \bigcap \{\mathbf{r}(\mathbf{S}_n); n \in \mathbf{FN}\}$. Let $\mathbf{u} \in \bigcap \{\mathbf{r}(\mathbf{S}_n); n \in \mathbf{FN}\}$. Then for every n $\mathbf{u}^2 \subseteq \mathbf{S}_n$ and hence $\mathbf{u}^2 \subseteq \bigcap \{\mathbf{S}_n; n \in \mathbf{FN}\} = \mathbf{S}$. Therefore $\mathbf{u} \in \mathbf{r}(\mathbf{S})$. □

Lemma 1.2. and Theorem 1.5. show that any generalized symmetry can be obtained from every symmetry and that the given construction commutes with countable intersections.

2. Definition of a homology theory.

In this section, we shall assume that $\mathbf{R}$ would be a generalized symmetry which is a π-class.

First of all let us remember that there is a set-definable (even without parameters) bijection $\mathbf{F} : \mathbf{N} \to \mathbf{V}$.

Definition 2.1. *Denote $\mathbf{R}^{(\nu)} = \{\mathbf{u}; \mathbf{u} \in \mathbf{R} \ \& \ \mathrm{card}(\mathbf{u}) = \nu + 1\}$. The elements of $\mathbf{R}^{(\nu)}$ are said to be* **ν-dimensional (unordered) simplexes of $\mathbf{R}$**.

Denote

$$\mathbf{R}_{\mathbf{F}}^{[\nu]} = \big\{[x_0, x_1, \dots, x_\nu]; \{x_0, x_1, \dots, x_\nu\} \in \mathbf{R} \ \& \\ \& \ (\forall \iota, \psi \in \nu + 1)\left(\iota < \psi \Rightarrow \mathbf{F}^{-1}(x_\iota) < \mathbf{F}^{-1}(x_\psi)\right)\big\}.$$

The elements of $\mathbf{R}_{\mathbf{F}}^{[\nu]}$ *are said to be* **ordered** ν**-dimensional simplexes of R ordered by F**.

Clearly $\mathbf{R}^{(\nu)}$ and $\mathbf{R}_{\mathbf{F}}^{[\nu]}$ are π-classes. Therefore $\left(\mathcal{F}\left(\mathbf{R}_{\mathbf{F}}^{[\nu]}\right),\oplus\right)$ is a commutative π-group with injective projections (with i.p. — cf. [G1, Definition 3.2]). This fact allows the following definition.

Definition 2.2. *Let* $\mathbf{G}$ *be a commutative* π*-group with i.p. Then we shall define a chain complex of groups* $\{\mathbf{C}_\nu(\mathbf{R},\mathbf{F},\mathbf{G}),\partial_\nu,\nu\in\mathbf{Z}\}$ *as follows:*

$$\mathbf{C}_\nu(\mathbf{R},\mathbf{F},\mathbf{G})=\mathbf{G}^{\mathbf{R}_{\mathbf{F}}^{[\nu]}}=\left(\mathcal{F}\left(\mathbf{R}_{\mathbf{F}}^{[\nu]}\right),\oplus\right)\otimes\mathbf{G}.$$

The elements of $\mathbf{C}_\nu(\mathbf{R},\mathbf{F},\mathbf{G})$ *are said to be* ν**-dimensional chains in the ordering F with coefficients in G**.

A boundary operator $\partial_\nu:\mathbf{C}_\nu(\mathbf{R},\mathbf{F},\mathbf{G})\to\mathbf{C}_{\nu-1}(\mathbf{R},\mathbf{F},\mathbf{G})$ *will be defined first for* $\mathbf{G}=(\mathbf{Z},+)$*. So that we give a definition for* $\mathbf{C}_\nu(\mathbf{R},\mathbf{F},\mathbf{Z})=\left(\mathcal{F}\left(\mathbf{R}_{\mathbf{F}}^{[\nu]}\right),\oplus\right)$*.*

Let $[x_0,x_1,\ldots,x_\nu]\in\mathbf{R}_{\mathbf{F}}^{[\nu]}$*. Put*

$$\partial_\nu([x_0,\ldots,x_\nu])=\sum_{\beta=0}^{\nu}(-1)^\beta\left[x_0,\ldots,\widehat{x_\beta},\ldots,x_\nu\right] \tag{1}$$

$\left(\left[x_0,\ldots,\widehat{x_\beta},\ldots,x_\nu\right]\right.$ *stands here as usual for* $\left.\left[x_0,\ldots,x_{\beta-1},x_{\beta+1},\ldots,x_\nu\right]\right)$*.*

$\mathbf{C}_{\nu-1}(\mathbf{R},\mathbf{F},\mathbf{G})$ *is a commutative* π*-group and therefore the sum on the left side of* (1) *is its correctly defined element and moreover the map* ∂_ν *can be linearly extended to a total homomorphism (for which we shall use the same assignment). If* $\mathbf{G}\neq(\mathbf{Z},+)$*, we use the boundary operator*

$$\partial'_\nu=\partial_\nu\otimes\mathbf{Id}_{\mathbf{G}}:\left(\mathcal{F}\left(\mathbf{R}_{\mathbf{F}}^{[\nu]}\right),\oplus\right)\otimes\mathbf{G}\to\left(\mathcal{F}\left(\mathbf{R}_{\mathbf{F}}^{[\nu-1]}\right),\oplus\right)\otimes\mathbf{G}. \tag{2}$$

Obviously, for $\nu<0$ *it holds that* $\mathbf{R}_{\mathbf{F}}^{[\nu]}=\emptyset$*, hence* $\left(\mathcal{F}\left(\mathbf{R}_{\mathbf{F}}^{[\nu]}\right),\oplus\right)$ *is a trivial group in this case and we use also trivial homomorphisms* $\partial_0,\partial_{-1},\ldots$ *(i.e. for all admissible* x $\partial_0(x)=0,\ldots$*).*

Remark. We shall use the sign $\mathbf{0}$ instead of $\{0\}$ for trivial groups.

An easy computation shows that if $\mathbf{G}=(\mathbf{Z},+)$, we have for each $\nu\in\mathbf{Z}$ $\partial_{\nu-1}\circ\partial_\nu=0$. Therefore also $\partial'_{\nu-1}\circ\partial'_\nu=(\partial_{\nu-1}\otimes\mathbf{Id}_{\mathbf{G}})\circ(\partial_\nu\otimes\mathbf{Id}_{\mathbf{G}})=(\partial_{\nu-1}\circ\partial_\nu)\otimes\mathbf{Id}_{\mathbf{G}}=0\otimes\mathbf{Id}_{\mathbf{G}}=0$ in a general case. These equalities allow the following definition. (If it makes no confusion, we should use the assignment ∂_ν also instead of ∂'_ν.)

Definition 2.3. *Put*

$$\mathbf{Z}_\nu(\mathbf{R},\mathbf{F},\mathbf{G})=ker(\partial_\nu)=\{x\in\mathbf{C}_\nu(\mathbf{R},\mathbf{F},\mathbf{G});\partial_\nu(x)=0\}.$$

The elements of this group are said to be ν**-dimensional cycles**.

Put

$$\mathbf{B}_\nu(\mathbf{R},\mathbf{F},\mathbf{G}) = Im(\partial_{\nu+1}) = \\ = \{x \in \mathbf{C}_\nu(\mathbf{R},\mathbf{F},\mathbf{G}); (\exists y \in \mathbf{C}_{\nu+1}(\mathbf{R},\mathbf{F},\mathbf{G}))(\partial_{\nu+1}(y) = x)\}$$

The elements of this group are said to be ν**-dimensional boundaries**.

By the above assertion, $\mathbf{B}_\nu(\mathbf{R},\mathbf{F},\mathbf{G})$ *is a subgroup of* $\mathbf{Z}_\nu(\mathbf{R},\mathbf{F},\mathbf{G})$ *and we can put*

$$\mathbf{H}_\nu(\mathbf{R},\mathbf{F},\mathbf{G}) = \mathbf{Z}_\nu(\mathbf{R},\mathbf{F},\mathbf{G}) / \mathbf{B}_\nu(\mathbf{R},\mathbf{F},\mathbf{G}) \tag{3}$$

and call it the ν**-th homology group (of the generalized symmetry R in the ordering F with coefficients in G)**. *If* $\mathbf{c},\mathbf{d} \in \mathbf{C}_\nu(\mathbf{R},\mathbf{F},\mathbf{G})$ *are chains such that* $\mathbf{c}-\mathbf{d} \in \mathbf{B}_\nu(\mathbf{R},\mathbf{F},\mathbf{G})$, *we shall write* $\mathbf{c} \simeq \mathbf{d}$ (*or* $\mathbf{c}-\mathbf{d} \simeq 0$) *which is the abbreviation of* $\mathbf{c}$ *is* **homological to d**, *or that* $\mathbf{c}$ *and* $\mathbf{d}$ *are* **homological**.

Now we shall prove that $\mathbf{H}_\nu(\mathbf{R},\mathbf{F},\mathbf{G})$ is "independent" on the Sd-ordering $\mathbf{F}$. First we give a proof for $\mathbf{G} = (\mathbf{Z},+)$. We shall write $\mathbf{C}_\nu(\mathbf{R},\mathbf{F})$, $\mathbf{Z}_\nu(\mathbf{R},\mathbf{F})$, $\mathbf{B}_\nu(\mathbf{R},\mathbf{F})$ and finally $\mathbf{H}_\nu(\mathbf{R},\mathbf{F})$ instead of $\mathbf{C}_\nu(\mathbf{R},\mathbf{F},(\mathbf{Z},+))$, ...

We give one combinatorial lemma first.

Lemma 2.4. *Let* ν *be a natural number,* $\mathbf{g} = [g_0, g_1, \dots, g_\nu]$ *be a permutation of the set* $\nu+1$ (*i.e.* $\mathbf{g}$ *is a bijection* $\mathbf{g} : \nu+1 \to \nu+1$, $\mathbf{g}(\alpha) = g_\alpha$). *Let* $\mathbf{h} = \mathbf{g}^{-1}$. *Denote* $i\langle\mathbf{g}\rangle = i\langle[g_0, g_1, \dots, g_\nu]\rangle$ *and* $i\langle\mathbf{g},\gamma\rangle = i\left\langle\left[g_0, g_1, \dots, \widehat{g_{\mathbf{h}(\gamma)}}, \dots, g_\nu\right]\right\rangle$ *for* $0 \le \gamma \le \nu$ *number of inversions* (*i.e. the number of such pairs* $\iota < \psi$ *that* $\mathbf{g}_\psi > \mathbf{g}_\iota$) *in the permutation* $\mathbf{g}$ *and* $\langle\mathbf{g},\gamma\rangle$ *respectively.* (*By* $\langle\mathbf{g},\gamma\rangle$ *we mean the function* $\langle\mathbf{g},\gamma\rangle : \nu \to (\nu+1) \setminus \{\gamma\}$ *with values as it is indicated above.*) *Denote by*

$$\begin{aligned} \varphi(\mathbf{g},\gamma) &\quad \text{the formula} \quad (i\langle\mathbf{g},\gamma\rangle - i\langle\mathbf{g}\rangle \equiv \mathbf{h}(\gamma) - \gamma \ mod\ 2) \ \ and \\ \psi(\mathbf{g}) &\quad \text{the formula} \quad (\forall\gamma \in dom(\mathbf{g}))\,\varphi(\mathbf{g},\gamma). \end{aligned}$$

Then for each permutation $\mathbf{g}$, $\psi(\mathbf{g})$.

Proof: First of all we prove a simpler version of this lemma.

Let $\mathbf{g}$ be a permutation such that $\psi(\mathbf{g})$. Let $\mathbf{f}$ be a transposition such that $\mathrm{dom}(\mathbf{f}) = \mathrm{dom}(\mathbf{g})$. Then $\psi(\mathbf{f} \circ \mathbf{g})$.

Take $0 \le \alpha < \beta \le \nu$ and let $\mathbf{f} = [0, \dots, \beta, \dots, \alpha, \dots, \nu] = (\alpha\beta)$. Let us remember that $\mathbf{f}^{-1} = \mathbf{f}$ and that $\mathbf{h} = \mathbf{g}^{-1}$. Now let $\gamma \notin \{\alpha,\beta\}$. Then $\mathbf{h}(\mathbf{f}(\gamma)) = \mathbf{h}(\gamma)$ and for $\mathbf{g} = [g_0, \dots, g_a, \dots, g_b, \dots, g_\nu]$ where $g_a = \alpha$ and $g_b = \beta$ we have

$$\mathbf{f} \circ \mathbf{g} = [g_0, \dots, g_{a-1}, g_b, g_{a+1}, \dots, g_{b-1}, g_a, g_{b+1}, \dots, g_\nu].$$

But $\gamma = g_c$ for some $c \in \{0, 1, \dots, \nu\} \setminus \{a, b\}$ and

$$i\langle[g_0, \dots, \widehat{g_c}, \dots, g_a, \dots, g_b, \dots, g_\nu]\rangle \not\equiv \\ \not\equiv i\langle[g_0, \dots, \widehat{g_c}, \dots, g_b, \dots, g_a, \dots, g_\nu]\rangle \mod 2 \tag{4}$$

or in a more precise and shorter expression $i\langle \mathbf{g},\gamma\rangle \not\equiv i\langle \mathbf{f}\circ\mathbf{g},\gamma\rangle \bmod 2$. (By the first expression one can think only about the case $c<a<b$, but its meaning is quite general.) Indeed both $\langle \mathbf{g},\gamma\rangle$ and $\langle \mathbf{f}\circ\mathbf{g},\gamma\rangle$ are permutations of the set $(\nu+1)-\{\gamma\}$ in this case. Being a transposition $\mathbf{f}$ changes the parity of the number $i\langle \mathbf{g},\gamma\rangle$. Therefore (4) holds. Now we have

$$i\langle \mathbf{f}\circ\mathbf{g},\gamma\rangle - i\langle \mathbf{f}\circ\mathbf{g}\rangle \equiv i\langle \mathbf{f}\circ\mathbf{g},\gamma\rangle - i\langle \mathbf{g},\gamma\rangle + i\langle \mathbf{g},\gamma\rangle - i\langle \mathbf{g}\rangle + i\langle \mathbf{g}\rangle - i\langle \mathbf{f}\circ\mathbf{g}\rangle \equiv$$
$$\equiv 1+\mathbf{h}(\gamma)-\gamma+1 \equiv \mathbf{h}(\mathbf{k}(\gamma))-\gamma \bmod 2$$

because $\psi(\mathbf{g})$. Hence $\varphi(\mathbf{g},\gamma)$ in this case.

Now let $\gamma=\alpha$. Clearly the difference

$$i\langle \mathbf{g},\gamma\rangle - i\langle \mathbf{f}\circ\mathbf{g},\gamma\rangle = i\langle[\mathbf{g}_0,\dots,\widehat{\mathbf{g}_a},\dots,\mathbf{g}_b,\dots,\mathbf{g}_\nu]\rangle - i\langle[\mathbf{g}_0,\dots,\mathbf{g}_b,\dots,\widehat{\mathbf{g}_a},\dots,\mathbf{g}_\nu]\rangle$$

depends only on the part $\mathbf{g}_{a+1},\dots,\mathbf{g}_{b-1}$ and on the value of $\mathbf{g}_b$. In this moment we consider the case $\mathbf{g}_a=\alpha$, $\mathbf{g}_b=\beta$. The second case is quite similar. The permutation $\langle \mathbf{g},\gamma\rangle$ can be obtained from $\langle \mathbf{f}\circ\mathbf{g},\gamma\rangle$ by means of sequence of the transpositions $(\mathbf{g}_{b-1},\mathbf{g}_b),(\mathbf{g}_{b-2},g_b),\dots,(\mathbf{g}_{a+1},\mathbf{g}_b)$. The number of these transpositions is $(b-1)-(a+1)+1=b-a+1$, hence

$$i\langle \mathbf{g},\gamma\rangle - i\langle \mathbf{f}\circ\mathbf{g},\gamma\rangle \equiv b-a+1 \bmod 2$$

Hence we have

$$i\langle \mathbf{f}\circ\mathbf{g},\gamma\rangle - i\langle \mathbf{f}\circ\mathbf{g}\rangle \equiv i\langle \mathbf{f}\circ\mathbf{g},\gamma\rangle - i\langle \mathbf{g},\gamma\rangle + i\langle \mathbf{g},\gamma\rangle - i\langle \mathbf{g}\rangle + i\langle \mathbf{g}\rangle - i\langle \mathbf{f}\circ\mathbf{g}\rangle \equiv$$
$$\equiv b-a+1+\mathbf{h}(\gamma)-\gamma+1 \equiv b-a+\mathbf{h}(\gamma)-\gamma \equiv \mathbf{h}(\mathbf{k}(\gamma))-\gamma \bmod 2$$

because $\mathbf{h}(\mathbf{k}(\gamma))=b$, $\mathbf{h}(\gamma)=a$ and $i\langle \mathbf{g}\rangle \not\equiv i\langle \mathbf{f}\circ\mathbf{g}\rangle \bmod 2$. Therefore $\psi(\mathbf{f}\circ\mathbf{g})$.

The simpler version of the lemma is proved. It is clear that if $\mathbf{I}=[0,1,\dots,\nu]$ is the identical permutation, then $\psi(\mathbf{I})$. Because for each permutation holds the equality $\mathbf{g}\circ\mathbf{I}=\mathbf{g}$, we have $\psi(\mathbf{g})$ for each transposition. And because each set permutation can be written as a set composition of transpositions, the lemma follows by the induction. □

Theorem 2.5. *Let* $\mathbf{R}$ *be a* π*-symmetry, let* $\mathbf{F}_1,\mathbf{F}_2:\mathbf{N}\to\mathbf{V}$ *be two Sd-bijections, let* $\mathbf{G}$ *be a commutative* π*-group with i.p. Then* $\mathbf{H}_\nu(\mathbf{R},\mathbf{F}_1,\mathbf{G}) \cong \mathbf{H}_\nu(\mathbf{R},\mathbf{F}_2,\mathbf{G})$ *for every* $\nu\in\mathbf{N}$.

Proof: There are Sd bijections $\mathbf{H},\mathbf{G}:\mathbf{N}\to\mathbf{N}$ such that $\mathbf{H}(i)=j$ if and only if $\mathbf{F}_1(i)=\mathbf{F}_2(j)$ and $\mathbf{G}(i)=j$ if and only if $\mathbf{F}_1(j)=\mathbf{F}_2(i)$. From this definition it follows that $\mathbf{F}_1(i)=\mathbf{F}_2(\mathbf{H}(i))$ and $\mathbf{F}_1(\mathbf{G}(i))=\mathbf{F}_2(i)$.

Let us define maps η and ω as follows:

$$\eta:\{\mathbf{C}_\nu(\mathbf{R},\mathbf{F}_1),\partial_\nu,\nu\in\mathbf{Z}\}\to\{\mathbf{C}_\nu(\mathbf{R},\mathbf{F}_2),\partial_\nu,\nu\in\mathbf{Z}\}$$

by the equality

$$\eta(\langle v\rangle\mathbf{F}_2)=(-1)^{i\langle[\mathbf{G}(j_0),\dots,\mathbf{G}(j_\nu)]\rangle}\langle v\rangle\mathbf{F}_2 \text{ where}$$
$$\langle v\rangle\mathbf{F}_2=[\mathbf{F}_2(j_0),\dots,\mathbf{F}_2(j_n)]\dots(\text{ i.e. } j_0<j_1<\dots<j_\nu) \text{ and}$$

$$\omega : \{\mathbf{C}_\nu(\mathbf{R}, \mathbf{F}_2), \partial_\nu, \nu \in \mathbf{Z}\} \to \{\mathbf{C}_\nu(\mathbf{R}, \mathbf{F}_1), \partial_\nu, \nu \in \mathbf{Z}\}$$

by the equality

$$\begin{aligned} \omega(\langle v\rangle \mathbf{F}_2) &= (-1)^{i\langle[\mathbf{H}(j_0),\dots,\mathbf{H}(j_\nu)]\rangle} \langle v\rangle \mathbf{F}_1, \text{ where} \\ \langle v\rangle \mathbf{F}_1 &= [\mathbf{F}_1(j_0), \dots, \mathbf{F}_1(j_\nu)] \end{aligned}$$

and by means of the linear extension [G1, Theorem 1.8], of course. We have to prove that both these maps commute with boundary operators, i.e. that

$$\partial \circ \eta_\nu = \eta_{\nu-1} \circ \partial \qquad \text{and} \qquad \partial \circ \omega_\nu = \omega_{\nu-1} \circ \partial. \tag{1}$$

Let $\langle v\rangle \mathbf{F}_1 = [\mathbf{F}_1(i_0), \dots, \mathbf{F}_1(i_\nu)]$ and at the same time let $\langle v\rangle \mathbf{F}_2 = [\mathbf{F}_2(j_0), \dots, \mathbf{F}_2(j_\nu)]$. Then

$$\begin{aligned} \eta_{\nu-1} \circ \partial(\langle v\rangle \mathbf{F}_1) &= \eta_{\nu-1}\left(\sum_{\beta=0}^{\nu} (-1)^\beta \left[\mathbf{F}_1(i_0), \dots, \widehat{\mathbf{F}_1(i_\beta)}, \dots, \mathbf{F}_1(i_\nu)\right]\right) = \\ &= \sum_{\beta=0}^{\nu} (-1)^\beta (-1)^{i\left\langle\left[\mathbf{G}(j_0),\dots,\widehat{\mathbf{G}(\mathbf{H}(i_\beta)}),\dots,\mathbf{G}(j_\nu)\right]\right\rangle} \left\langle v \setminus \left\{\mathbf{F}_1(i_\beta)\right\}\right\rangle \mathbf{F}_2 \end{aligned}$$

because $\mathbf{F}_1(i_\beta) = \mathbf{F}_2(\mathbf{H}(i_\beta))$ and

$$\begin{aligned} \partial \circ \eta_\nu(\langle v\rangle \mathbf{F}_1) &= \partial\left((-1)^{i\langle[\mathbf{G}(j_0),\dots,\mathbf{G}(j_\nu)]\rangle} \langle v\rangle \mathbf{F}_2\right) = \\ &= (-1)^{i\langle[\mathbf{G}(j_0),\dots,\mathbf{G}(j_\nu)]\rangle}\left(\sum_{\beta=0}^{\nu} (-1)^\beta \left[\mathbf{F}_2(j_0), \dots, \widehat{\mathbf{F}_2(j_\beta)}, \dots, \mathbf{F}_2(j_\nu)\right]\right). \end{aligned}$$

Now it is enough to prove that

$$\begin{aligned} &(-1)^{i\left\langle\left[\mathbf{G}(j_0),\dots,\mathbf{G}(\widehat{\mathbf{H}(i_\beta)}),\dots,\mathbf{G}(j_\nu)\right]\right\rangle+\beta} \left\langle v \setminus \left\{\mathbf{F}_1(i_\beta)\right\}\right\rangle \mathbf{F}_2 = \\ &\qquad = (-1)^{i\langle[\mathbf{G}(j_0),\dots,\mathbf{G}(j_\nu)]\rangle+I(\mathbf{H}(i_\beta))} \left[\mathbf{F}_2(j_0), \dots, \widehat{\mathbf{F}_2(j_\beta)}, \dots, \mathbf{F}_2(j_\nu)\right] \end{aligned}$$

or equivalently that

$$\begin{aligned} i\left\langle\left[\mathbf{G}(j_0), \dots, \mathbf{G}(\widehat{\mathbf{H}(i_\beta)}), \dots, \mathbf{G}(j_\nu)\right]\right\rangle + \beta &\equiv \\ &\equiv i\langle[\mathbf{G}(j_0), \dots, \mathbf{G}(j_\nu)]\rangle + I(\mathbf{H}(i_\beta)) \mod 2, \end{aligned} \tag{2}$$

where $I(\mathbf{H}(i_\beta)) = k$ iff $\mathbf{H}(i_\beta) = j_k$.

Because $i_0 < i_1 < \dots < i_\nu$ and $j_0 < j_1 < \dots < j_\nu$, they are increasing sequences. Therefore it holds for each permutation $\alpha_0 < \alpha_1 < \dots < \alpha_\nu$ of the set $\{0, 1, \dots, \nu\}$ that

$$i\langle[\alpha_0, \dots, \alpha_\nu]\rangle = i\langle[i_{\alpha_0}, \dots, i_{\alpha_\nu}]\rangle = i\langle[j_{\alpha_0}, \dots, j_{\alpha_\nu}]\rangle .$$

Therefore it is clear that it is enough to prove (2) for the case when $\mathbf{G}$, $\mathbf{H}$ are permutations of the set $\{0,1,\ldots,\nu\}$ such that $\mathbf{H} = \mathbf{G}^{-1}$. But it holds that $I\left(\mathbf{H}\left(i_\beta\right)\right) = \mathbf{H}\left(i_\beta\right)$ in this case and hence (2) holds according to 2.4. Hence $\partial \circ \eta_\nu = \eta_{\nu-1} \circ \partial$. A proof for ω is similar.

Now we can also assume that $[i_0,\ldots,i_\nu] = [j_0,\ldots,j_\nu] = [0,\ldots,\nu]$ (because each of these sequences is increasing). It holds that

$$\begin{aligned}\omega_\nu \circ \eta_\nu \left(\langle v\rangle \mathbf{F}_1\right) =& \omega_\nu \left((-1)^{i\langle[\mathbf{G}(j_0),\ldots,\mathbf{G}(j_\nu)]\rangle} \langle v\rangle \mathbf{F}_2\right) = \\ &= (-1)^{i\langle[\mathbf{G}(j_0),\ldots,\mathbf{G}(j_\nu)]\rangle + i\langle[\mathbf{H}(j_0),\ldots,\mathbf{H}(j_\nu)]\rangle} \langle v\rangle \mathbf{F}_1 \qquad \text{and also} \\ & i\left\langle\left[\mathbf{G}\left(j_0\right),\ldots,\mathbf{G}\left(j_\nu\right)\right]\right\rangle = i\left\langle\left[\mathbf{H}\left(j_0\right),\ldots,\mathbf{H}\left(j_\nu\right)\right]\right\rangle\end{aligned}$$

because $\mathbf{H}$ and $\mathbf{G}$ are inverse maps.

Hence

$$\omega_\nu \circ \eta_\nu = \mathbf{Id} \restriction \mathbf{C}_\nu\left(\mathbf{R},\mathbf{F}_1\right).$$

Using similar arguments we can also prove that

$$\eta_\nu \circ \omega_\nu = \mathbf{Id} \restriction \mathbf{C}_\nu\left(\mathbf{R},\mathbf{F}_2\right).$$

So that

$$\eta : \left\{\mathbf{C}_\nu\left(\mathbf{R},\mathbf{F}_1\right),\partial_\nu,\nu\in\mathbf{Z}\right\} \to \left\{\mathbf{C}_\nu\left(\mathbf{R},\mathbf{F}_2\right),\partial_\nu,\nu\in\mathbf{Z}\right\}$$

is a chain isomorphism and therefore it holds for every $\nu \in \mathbf{N}$ that

$$\mathbf{H}_\nu\left(\mathbf{R},\mathbf{F}_1\right) \cong \mathbf{H}_\nu\left(\mathbf{R},\mathbf{F}_2\right).$$

Now we can obtain general result for arbitrary commutative π-group with i.p. of coefficients using extensions of the maps ω and η by $\otimes\mathbf{Id}_{\mathbf{G}}$ (cf. [G1, Theorem 3.12]). $\square$

In the view of the previous theorem we shall omit an assignment for an ordering in the assignments of $\mathbf{C}_\nu(\mathbf{R},\mathbf{F},\mathbf{G})$, $\mathbf{Z}_\nu(\mathbf{R},\mathbf{F},\mathbf{G})$, $\mathbf{B}_\nu(\mathbf{R},\mathbf{F},\mathbf{G})$ and $\mathbf{H}_\nu(\mathbf{R},\mathbf{F},\mathbf{G})$ as well. We shall write usually only $\mathbf{C}_\nu(\mathbf{R},\mathbf{G})$, $\mathbf{Z}_\nu(\mathbf{R},\mathbf{G})$, $\mathbf{B}_\nu(\mathbf{R},\mathbf{G})$ and finally $\mathbf{H}_\nu(\mathbf{R},\mathbf{G})$ or even only $\mathbf{C}_\nu(\mathbf{R})$, $\mathbf{Z}_\nu(\mathbf{R})$, $\mathbf{B}_\nu(\mathbf{R})$ and $\mathbf{H}_\nu(\mathbf{R})$.

Definition 2.6. *Let* $\mathbf{R}$ *and* $\mathbf{S}$ *be two generalized symmetries. Let* $\mathbf{G}$ *be an Sd map such that* $(\forall \mathbf{u}\in\mathbf{R})\left(\mathbf{G}''\mathbf{u}\in\mathbf{S}\right)$ *and* $\bigcup\mathbf{R} \subseteq \mathrm{dom}\,(\mathbf{G})$. *Then* $\mathbf{G}$ *is said to be a* **simplicial map from R to S**.

Let $\mathbf{R}$ and $\mathbf{S}$ be two generalized symmetries (which are π-classes). Let $\mathbf{F}_1$, $\mathbf{F}_2$ be two Sd bijections $\mathbf{F}_1,\mathbf{F}_2 : \mathbf{N} \to \mathbf{V}$. Let $\mathbf{G}$ be a simplicial map from $\mathbf{R}$ to $\mathbf{S}$ such that $\mathbf{G} \restriction \bigcup\mathbf{R}$ is the "nondecreasing" map between the given orderings $\mathbf{F}_1$ and $\mathbf{F}_2$ (precisely if $\iota,\kappa \in \mathbf{N}$ are such that $\mathbf{F}_1(\iota),\mathbf{F}_1(\kappa) \in \bigcup\mathbf{R}$ and $\iota < \kappa$ then $\mathbf{F}_2^{-1}\left(\mathbf{G}\left(\mathbf{F}_1(\iota)\right)\right) \le \mathbf{F}_2^{-1}\left(\mathbf{G}\left(\mathbf{F}_1(\kappa)\right)\right)$).

Let us put

$$\overline{\mathbf{G}}\left(\langle\mathbf{v}\rangle\mathbf{F}_1\right) = \begin{cases} \left\langle\mathbf{G}''\mathbf{v}\right\rangle\mathbf{F}_2 & \text{if } \mathrm{card}\,(\mathbf{v}) = \mathrm{card}\left(\mathbf{G}''\mathbf{v}\right) \\ 0 & \text{otherwise.}\end{cases}$$

This map $\overline{\mathbf{G}}$ can be [G1, Theorem 1.8] linearly extended to the total homomorphism (which we shall call $\overline{\mathbf{G}}$ as well) $\overline{\mathbf{G}} : \mathbf{C}_\nu\left(\mathbf{R},\mathbf{F}_1\right) \to \mathbf{C}_\nu\left(\mathbf{R},\mathbf{F}_2\right)$.

Theorem 2.7. *The homomorphism* $\overline{\mathbf{G}}$ *from the previous considerations is a chain homomorphism between the chain complexes* $\{\mathbf{C}_\nu(\mathbf{R},\mathbf{F}_1),\partial_\nu,\nu\in\mathbf{Z}\}$ *and* $\{\mathbf{C}_\nu(\mathbf{R},\mathbf{F}_2),\partial_\nu,\nu\in\mathbf{Z}\}$.

Proof: Take $\nu\in\mathbf{N}$, let $\operatorname{card}\left(\mathbf{G}''\mathbf{u}\right)\le\nu-1=\operatorname{card}(\mathbf{u})-2$. Clearly $\partial\circ\overline{\mathbf{G}}\left(\langle\mathbf{u}\rangle\mathbf{F}_1\right)=0=\overline{\mathbf{G}}\circ\partial\left(\langle\mathbf{u}\rangle\mathbf{F}_1\right)$ in this case. It is obvious that $\partial\circ\overline{\mathbf{G}}\left(\langle\mathbf{u}\rangle\mathbf{F}_1\right)=0=\overline{\mathbf{G}}\circ\partial\left(\langle\mathbf{u}\rangle\mathbf{F}_1\right)$ provided $\operatorname{card}\left(\mathbf{G}''\mathbf{u}\right)=\operatorname{card}(\mathbf{u})$ as well. Finally let $\operatorname{card}\left(\mathbf{G}''\mathbf{u}\right)=\operatorname{card}(\mathbf{u})-1=\nu$. Let $\langle\mathbf{u}\rangle\mathbf{F}_1=[\mathbf{F}_1(i_0),\dots,\mathbf{F}(i_\nu)]$. There is just one pair of indexes $\{j,k\}$ $(j\neq k)$ with $\mathbf{G}\left(\mathbf{F}_1\left(i_j\right)\right)=\mathbf{G}\left(\mathbf{F}_1\left(i_k\right)\right)$ in this case. By $\overline{\mathbf{G}}\left(\langle\mathbf{u}\rangle\mathbf{F}_1\right)=0$, we have $\partial\circ\overline{\mathbf{G}}\left(\langle\mathbf{u}\rangle\mathbf{F}_1\right)=0$. It holds that

$$\begin{aligned}\partial\left(\langle\mathbf{u}\rangle\mathbf{F}_1\right)=(-1)^j&\left[\mathbf{F}_1(i_0),\dots,\widehat{\mathbf{F}_1\left(i_j\right)},\dots,\mathbf{F}_1(i_\nu)\right]+\\&+(-1)^k\left[\mathbf{F}_1(i_0),\dots,\widehat{\mathbf{F}_1\left(i_k\right)},\dots,\mathbf{F}_1(i_\nu)\right]+\mathbf{s}\end{aligned}$$

where $\mathbf{s}$ is a sum (with both possible signs + and -) of ordered simplexes in which both vertices $\mathbf{F}_1\left(i_j\right)$ and $\mathbf{F}_1\left(i_k\right)$ occur. Hence $\overline{\mathbf{G}}(\mathbf{s})=0$. By the assumption that $\mathbf{G}$ is nondecreasing, it follows that j and k are such that $|j-k|=1$. Moreover

$$\left[\mathbf{F}_1(i_0),\dots,\widehat{\mathbf{F}_1\left(i_j\right)},\dots,\mathbf{F}_1(i_\nu)\right]=\left[\mathbf{F}_1(i_0),\dots,\widehat{\mathbf{F}_1\left(i_k\right)},\dots,\mathbf{F}_1(i_\nu)\right].$$

Inasmuch

$$\begin{aligned}(-1)^j&\left[\mathbf{F}_1(i_0),\dots,\widehat{\mathbf{F}_1\left(i_j\right)},\dots,\mathbf{F}_1(i_\nu)\right]+\\&+(-1)^k\left[\mathbf{F}_1(i_0),\dots,\widehat{\mathbf{F}_1\left(i_k\right)},\dots,\mathbf{F}_1(i_\nu)\right]=0\end{aligned}$$

and $\overline{\mathbf{G}}\left(\partial\left(\langle\mathbf{u}\rangle\mathbf{F}_1\right)\right)=0$. □

Composing nondecreasing $\mathbf{G}$ with a chain homomorphism of the kind η (cf. the proof of the Theorem 2.5.) we can prove 2.7 provided $\mathbf{G}$ be any simplicial map. An easy computation shows that $\overline{\mathbf{G}}\otimes\mathbf{Id}_{\mathbf{H}}$ is a chain homomorphism between chain complexes $\{\mathbf{C}_\nu(\mathbf{R},\mathbf{F}_1,\mathbf{H}),\partial_\nu,\nu\in\mathbf{Z}\}$ and $\{\mathbf{C}_\nu(\mathbf{R},\mathbf{F}_2,\mathbf{H}),\partial_\nu,\nu\in\mathbf{Z}\}$ ($\mathbf{H}$ is a commutative π-group with i.p.). Hence any simplicial map $\mathbf{G}$ from $\mathbf{R}$ to $\mathbf{S}$ induces the homomorphism (more precisely a sequence of homomorphisms) $\mathbf{G}_*:\mathbf{H}_\nu(\mathbf{R},\mathbf{F}_1,\mathbf{H})\to\mathbf{H}_\nu(\mathbf{S},\mathbf{F}_2,\mathbf{H})$.

Theorem 2.8. *For any* $\mathbf{R}$, $\mathbf{Id}_*:\mathbf{H}_\nu(\mathbf{R},\mathbf{F},\mathbf{G})\to\mathbf{H}_\nu(\mathbf{R},\mathbf{F},\mathbf{G})$ *is an identity homomorphism.*

Theorem 2.9. *Let* $\mathbf{R}$, $\mathbf{S}$ *and* $\mathbf{U}$ *be generalized symmetries, let* $\mathbf{G}$ *be a simplicial map from* $\mathbf{R}$ *to* $\mathbf{S}$ *and* $\mathbf{H}$ *a simplicial map from* $\mathbf{S}$ *to* $\mathbf{U}$. *Then* $\mathbf{H}\circ\mathbf{G}$ *is a simplicial map from* $\mathbf{R}$ *to* $\mathbf{U}$ *and* $(\mathbf{H}\circ\mathbf{G})_*=\mathbf{H}_*\circ\mathbf{G}_*$.

Proof: It is clear that $\mathbf{H}\circ\mathbf{S}$ is a simplicial map. To prove the second assertion it is enough to prove that $\overline{(H\circ G)}=\overline{\mathbf{H}}\circ\overline{\mathbf{G}}$. For this, we shall make the following computation:

$$\overline{(H \circ G)}\left(\langle \mathbf{u}\rangle \mathbf{F}_1\right) = \left\langle (\mathbf{H}\circ\mathbf{G})'' \mathbf{u}\right\rangle \mathbf{F}_3 = \left\langle \mathbf{H}''\mathbf{G}''\mathbf{u}\right\rangle \mathbf{F}_3 = \overline{\mathbf{H}}\left(\left\langle \overline{\mathbf{G}}''\mathbf{u}\right\rangle \mathbf{F}_2\right) =$$
$$= \overline{\mathbf{H}}\left(\overline{\mathbf{G}}\left(\langle \mathbf{u}\rangle \mathbf{F}_1\right)\right) = \overline{\mathbf{H}} \circ \overline{\mathbf{G}}\left(\langle \mathbf{u}\rangle \mathbf{F}_1\right) .$$

□

As the result of this section we have constructed a functor from the category of generalized symmetries which are π-classes with simplicial maps to the category of abelian groups with homomorphisms.

3. Invariance theorem for homology.

The main aim of this paper is to introduce homology theory for π-symmetries and π-equivalencies. Homology functor created so far can be used to this end by means of the operator $\mathbf{r}(-)$ defined in 1.2. Let $\mathbf{G}$ be a commutative π-group with i.p. in the whole section. First of all we give one important lemma.

Lemma 3.1. *Let $\mathbf{H}_1, \mathbf{H}_2 : \mathbf{V} \to \mathbf{V}$ be two Sd functions. Let $\mathbf{u} = [x_0, ..., x_\nu]$ be an ν-tuple. Let us put*

$$\mathbf{D}(\mathbf{u}) = \sum_0^\nu (-1)^j \left[\mathbf{H}_1(x_0), ..., \mathbf{H}_1(x_j), \mathbf{H}_2(x_j), ..., \mathbf{H}_2(x_\nu)\right] .$$

This expression defines (by the linear extension) homomorphisms between groups $\mathbf{C}_\nu(\mathbf{R}, \mathbf{F}_1, \mathbf{G})$ and $\mathbf{C}_{\nu+1}(\mathbf{S}, \mathbf{F}_2, \mathbf{G})$ provided $\mathbf{H}_1''\mathbf{v} \cup \mathbf{H}_2''\mathbf{v} \in \mathbf{S}$ for every $\mathbf{v} \in \mathbf{R}$.

Then for every chain $c \in \mathbf{C}_\nu(\mathbf{R}, \mathbf{F}_1, \mathbf{G})$, the equality $\overline{\mathbf{H}}_2(c) - \overline{\mathbf{H}}_1(c) = (\mathbf{D}\circ\partial + \partial\circ\mathbf{D})(c)$ holds.

PROOF: By an assumption both $\mathbf{H}_1$ and $\mathbf{H}_2$ are simplicial maps. The rest of the proof can be found in [A,pp.285-287] (we must apply operator $\otimes\mathbf{Id}_\mathbf{G}$ in a general case). □

Theorem 3.2. *Let $\mathbf{S}$ be a π-symmetry, let $\mathbf{u}$, $\mathbf{v}$ be sets such that $\mathbf{u} \subseteq \mathbf{v} \subseteq \mathbf{S}''\mathbf{u}$. Let $\mathbf{g} : \mathbf{v} \to \mathbf{u}$ be a function such that $\mathbf{g} \restriction \mathbf{u} = \mathbf{Id}_\mathbf{u}$, $(\forall x, y \in \mathbf{v})(x\mathbf{S}y \Rightarrow (x\mathbf{S}\mathbf{g}(y)\,\&\,\mathbf{g}(x)\mathbf{S}\mathbf{g}(y)))$. Let $\mathbf{F} : \mathbf{N} \to \mathbf{V}$ be an Sd bijection. Then*
$\mathbf{H}_\nu\left(\mathbf{r}\left(\mathbf{S}\cap\mathbf{u}^2\right), \mathbf{F}, \mathbf{G}\right) \simeq \mathbf{H}_\nu\left(\mathbf{r}\left(\mathbf{S}\cap\mathbf{v}^2\right), \mathbf{F}, \mathbf{G}\right)$ for every $\nu \in \mathbf{N}$.

PROOF: By 2.5 we need not care about an ordering. The identity $\mathbf{i} : \mathbf{u} \to \mathbf{v}$ is a simplicial map from $\mathbf{r}\left(\mathbf{S}\cap\mathbf{u}^2\right)$ to $\mathbf{r}\left(\mathbf{S}\cap\mathbf{v}^2\right)$. We state that $\mathbf{i}_* : \mathbf{H}_\nu\left(\mathbf{r}\left(\mathbf{S}\cap\mathbf{u}^2\right), \mathbf{F}, \mathbf{G}\right) \to \mathbf{H}_\nu\left(\mathbf{r}\left(\mathbf{S}\cap\mathbf{v}^2\right), \mathbf{F}, \mathbf{G}\right)$ is an isomorphism.
Injectivity: Let $c, d \in \mathbf{Z}_\nu\left(\mathbf{r}\left(\mathbf{S}\cap\mathbf{u}^2\right), \mathbf{F}, \mathbf{G}\right)$ be two chains nonhomological in $\mathbf{C}_\nu\left(\mathbf{r}\left(\mathbf{S}\cap\mathbf{u}^2\right), \mathbf{F}, \mathbf{G}\right)$. We claim that they are nonhomological in $\mathbf{C}_\nu\left(\mathbf{r}\left(\mathbf{S}\cap\mathbf{v}^2\right), \mathbf{F}, \mathbf{G}\right)$ as well. For the contrary, let $e \in \mathbf{C}_{\nu+1}\left(\mathbf{r}\left(\mathbf{S}\cap\mathbf{v}^2\right), \mathbf{F}, \mathbf{G}\right)$ be a chain such that $\partial(e) = c - d$. By assumptions, $\mathbf{g}$ is a simplicial map from $\mathbf{r}\left(\mathbf{S}\cap\mathbf{v}^2\right)$ to $\mathbf{r}\left(\mathbf{S}\cap\mathbf{u}^2\right)$, hence $\overline{\mathbf{g}}$ is a chain homomorphism. Because $\mathbf{g} \restriction \mathbf{u} = \mathbf{Id}_\mathbf{u}$, we have $\overline{\mathbf{g}}(c) = c$ and $\overline{\mathbf{g}}(d) = d$. Therefore

$$c - d = \overline{\mathbf{g}}(c - d) = \overline{\mathbf{g}} \circ \partial(e) = \partial(\overline{\mathbf{g}}(e)).$$

This means that c and d are homological in $\mathbf{C}_\nu\left(\mathbf{r}\left(\mathbf{S}\cap\mathbf{u}^2\right),\mathbf{F},\mathbf{G}\right)$ — a contradiction. Hence $\mathbf{i}_*$ is an injection.
Surjectivity: It is enough to prove that for any cycle $z\in\mathbf{Z}_\nu\left(\mathbf{r}\left(\mathbf{S}\cap\mathbf{v}^2\right),\mathbf{F},\mathbf{G}\right)$ there is a cycle $d\in\mathbf{Z}_\nu\left(\mathbf{r}\left(\mathbf{S}\cap\mathbf{u}^2\right),\mathbf{F},\mathbf{G}\right)$ such that z and d are homological in $\mathbf{C}_\nu\left(\mathbf{r}\left(\mathbf{S}\cap\mathbf{v}^2\right),\mathbf{F},\mathbf{G}\right)$. Let $d=\overline{\mathbf{g}}(z)$. Because $\overline{\mathbf{g}}$ is a chain homomorphism, it holds $\partial(d)=\partial(\overline{\mathbf{g}}(z))=\overline{\mathbf{g}}(\partial(z))=\overline{\mathbf{g}}(0)=0$. Therefore d is a cycle in $\mathbf{Z}_\nu\left(\mathbf{r}\left(\mathbf{S}\cap\mathbf{u}^2\right),\mathbf{F},\mathbf{G}\right)$. We are about to find $e\in\mathbf{C}_{\nu+1}\left(\mathbf{r}\left(\mathbf{S}\cap\mathbf{u}^2\right),\mathbf{F},\mathbf{G}\right)$ such that $d-z=\partial(e)$. Take $\mathbf{H}_1=\mathbf{Id}$ and $\mathbf{H}_2=\mathbf{g}$ in the Lemma 3.1. According to the assumptions these maps fulfil the assumptions of 3.1 and so we have a homomorphism $\mathbf{D}$. Let us put $e=\mathbf{D}(z)$. We have

$$d-z=\overline{\mathbf{g}}(z)-z=(\mathbf{D}\circ\partial+\partial\circ\mathbf{D})(z)=\mathbf{D}(\partial(z))+\partial(\mathbf{D}(z))=\mathbf{D}(0)+\partial(e)=\partial(e)$$

so that z and d are homological (in $\mathbf{C}_\nu\left(\mathbf{r}\left(\mathbf{S}\cap\mathbf{v}^2\right),\mathbf{F},\mathbf{G}\right)$). Hence $\mathbf{i}_*$ is a surjective map. □

Corollary 3.3 (Invariance theorem)**.** *Let* $\mathbf{S}$ *be a* π*-equivalence, let* $\mathbf{u}$, $\mathbf{v}$ *be such that* $\mathrm{Fig}^{\mathbf{S}}(\mathbf{u})=\mathrm{Fig}^{\mathbf{S}}(\mathbf{v})$. *Then* $\mathbf{H}_\nu\left(\mathbf{r}\left(\mathbf{S}\cap\mathbf{u}^2\right),\mathbf{F},\mathbf{G}\right)\cong\mathbf{H}_\nu\left(\mathbf{r}\left(\mathbf{S}\cap\mathbf{v}^2\right),\mathbf{F},\mathbf{G}\right)$ *for every* $\nu\in\mathbf{Z}$.

Proof: By the assumptions, $\mathbf{u}\subseteq\mathbf{u}\cup\mathbf{v}\subseteq\mathbf{S}''\mathbf{u}$ and the π-relation $\mathbf{R}=\{[x,y];y\in\mathbf{u}\ \&\ x\mathbf{S}y\}$ has $\mathrm{dom}\,(\mathbf{R})\supseteq\mathbf{u}\cup\mathbf{v}$. Therefore there is a selector $\mathbf{f}$ of the relation $\mathbf{R}$ such that $\mathrm{dom}\,(\mathbf{f})=\mathbf{v}\setminus\mathbf{u}$. Let us put

$$\mathbf{g}(x)=\begin{cases}\mathbf{f}(x) & \text{if}\quad x\in\mathbf{v}\setminus\mathbf{u}\\ x & \text{if}\quad x\in\mathbf{u}\,.\end{cases}$$

By the definition of $\mathbf{g}$ we have $x\mathbf{S}y\Rightarrow\mathbf{g}(x)\mathbf{S}x\mathbf{S}y\mathbf{S}\mathbf{g}(y)$. As $\mathbf{S}$ is an equivalence, we have $x\mathbf{S}y\Rightarrow\mathbf{g}(x)\mathbf{S}\mathbf{g}(y)$. $\mathbf{g}\restriction\mathbf{u}=\mathbf{Id}_{\mathbf{u}}$ as well. Finally it holds that $x\mathbf{S}\mathbf{g}(x)$ so that $x\mathbf{S}y\Rightarrow x\mathbf{S}y\mathbf{S}\mathbf{g}(y)$. Again because $\mathbf{S}$ is an equivalence we have $x\mathbf{S}y\Rightarrow x\mathbf{S}\mathbf{g}(y)$. Therefore $\mathbf{S}$, $\mathbf{u}$, $\mathbf{u}\cup\mathbf{v}$ and the function $\mathbf{g}$ fulfil the assumptions of 3.2 and therefore

$$\mathbf{H}_\nu\left(\mathbf{r}\left(\mathbf{S}\cap\mathbf{u}^2\right),\mathbf{F},\mathbf{G}\right)\cong\mathbf{H}_\nu\left(\mathbf{r}\left(\mathbf{S}\cap(\mathbf{u}\cup\mathbf{v})^2\right),\mathbf{F},\mathbf{G}\right)\,.$$

Similarly we can prove that

$$\mathbf{H}_\nu\left(\mathbf{r}\left(\mathbf{S}\cap\mathbf{v}^2\right),\mathbf{F},\mathbf{G}\right)\cong\mathbf{H}_\nu\left(\mathbf{r}\left(\mathbf{S}\cap(\mathbf{u}\cup\mathbf{v})^2\right),\mathbf{F},\mathbf{G}\right).$$

Hence

$$\mathbf{H}_\nu\left(\mathbf{r}\left(\mathbf{S}\cap\mathbf{u}^2\right),\mathbf{F},\mathbf{G}\right)\cong\mathbf{H}_\nu\left(\mathbf{r}\left(\mathbf{S}\cap\mathbf{v}^2\right),\mathbf{F},\mathbf{G}\right).$$

□

Corollary 3.4. *Let* $\mathbf{S}$ *be an indiscernibility relation, let* $\mathbf{u}$ *be a set. Then for every* $\nu\in\mathbf{N}\setminus\mathbf{FN}$,

$$\mathbf{H}_\nu\left(\mathbf{r}\left(\mathbf{S}\cap\mathbf{u}^2\right),\mathbf{F},\mathbf{G}\right)\cong\mathbf{0}.$$

Proof: Take $\gamma\notin\mathbf{FN}$. Let $\mathbf{v}$ be such that $\mathrm{Fig}^{\mathbf{S}}(\mathbf{v})=\mathrm{Fig}^{\mathbf{S}}(\mathbf{u})$ and $\mathrm{card}(\mathbf{v})<\gamma$. It means that $\mathbf{C}_\nu\left(\mathbf{r}\left(\mathbf{S}\cap\mathbf{v}^2\right),\mathbf{F},\mathbf{G}\right)=\mathbf{0}$ for $\nu\geq\gamma$. Therefore $\mathbf{H}_\gamma\left(\mathbf{r}\left(\mathbf{S}\cap\mathbf{v}^2\right),\mathbf{F},\mathbf{G}\right)$ $\cong\mathbf{0}$ and by 4.3 $\quad\mathbf{H}_\nu\left(\mathbf{r}\left(\mathbf{S}\cap\mathbf{u}^2\right),\mathbf{F},\mathbf{G}\right)\cong\mathbf{0}$ as well. □

4. Eilenberg-Steenrod's axioms.

We are going to define relative homology theory (i.e. theory of couples figure-subfigure). It is well known that such a theory would satisfy seven so called Eilenberg-Steenrod axioms. Therefore first of all we give precise formulations and proofs of these axioms. This will enable us to use many useful and well known facts and results about homology theories.

We shall consider the category of pairs of sets $(\mathbf{u},\mathbf{v})$ satisfying $\mathbf{v}\subseteq\mathbf{u}$ for our aim. Morphisms of this category are functions $f:(\mathbf{u},\mathbf{v})\to(\mathbf{z},\mathbf{w})$ — it means that f is the function $f:\mathbf{u}\to\mathbf{z}$ satisfying $f''\mathbf{v}\subseteq\mathbf{w}$. We shall write $\mathbf{u}$ and $f:\mathbf{u}\to\mathbf{z}$ instead of $(\mathbf{u},\emptyset)$ and $f:(\mathbf{u},\emptyset)\to(\mathbf{z},\emptyset)$, respectively.

First we have to define relative homology groups. In this section, we shall work with the indiscernibility relation $\mathbf{R}$ on the set $\mathbf{u}$ unless something else is explicitly stated. Therefore we assume that $\mathbf{R}=\mathbf{R}\cap\mathbf{u}^2$ and moreover that whenever we take the generating sequence $\{R_n;n\in\mathbf{FN}\}$ then $R_n\subseteq\mathbf{u}^2$ holds for every $n\in\mathbf{FN}$. Restricting to some subset $\mathbf{z}\subseteq\mathbf{u}$, we shall write $\mathrm{Int}_{\mathbf{z}}(A)$ (or $\mathring{A}_{\mathbf{z}}$) instead of $\mathrm{Int}\,(A)\cap\mathbf{z}$, $\bar{A}_{\mathbf{z}}$ instead of $\bar{A}\cap\mathbf{z}$ etc. In addition we shall use some fixed commutative π-group G with i.p. and some fixed ordering $\mathbf{F}$ of $\mathbf{V}$.

The basic idea is to use a construction like $\mathbf{C}_\nu\left(\mathbf{r}\left(\mathbf{R}\right)\right)/\mathbf{C}_\nu\left(\mathbf{r}\left(\mathbf{R}\cap\mathbf{a}^2\right)\right)$ for $\mathbf{a}\subseteq\mathbf{u}$. It is important to emphasize that each group of this kind allows the infinite set summation in our case. For this, let us recall that $\mathbf{C}_\nu\left(\mathbf{r}\left(X\right)\right)=\mathbf{G}^{\mathbf{r}(X)^{[\nu]}_F}$. Now let $\{c_\lambda;\lambda\in\alpha\}$ be a set of elements of $\mathbf{C}_\nu\left(\mathbf{r}\left(\mathbf{R}\right)\right)$, let $f:\alpha\to\mathbf{Z}$ be a set function. Put

$$\begin{aligned}\{b_\lambda;\lambda\in\alpha\}=&\left\{c_\lambda\restriction\mathbf{r}\left(\mathbf{R}\cap\mathbf{a}^2\right)^{[\nu]}_F;\lambda\in\alpha\right\}=\\ =&\Big\{b_\lambda;(x\in\mathrm{dom}\,(b_\lambda)\Leftrightarrow[x\in\mathrm{dom}\,(c_\lambda)\,\&(\forall y\in x)(y\in\mathbf{a})])\,\&\\ &\&\,(\forall x\in\mathrm{dom}\,(b_\lambda))\,(b_\lambda(x)=c_\lambda(x))\Big\}.\end{aligned}$$

Next let us put

$$\begin{aligned}\{a_\lambda;\lambda\in\alpha\}=\Big\{&a_\lambda;(x\in\mathrm{dom}\,(a_\lambda)\Leftrightarrow[x\in\mathrm{dom}\,(c_\lambda)\,\&(\exists y\in x)(y\notin\mathbf{a})])\,\&\\ &\&(\forall x\in\mathrm{dom}\,(a_\lambda))(a_\lambda(x)=c_\lambda(x))\Big\}.\end{aligned}$$

According to the above definitions one can see that both $\{a_\lambda;\lambda\in\alpha\}$ and $\{b_\lambda;\lambda\in\alpha\}$ are the sets of elements of $\mathbf{C}_\nu\left(\mathbf{r}\left(\mathbf{R}\right)\right)$. Obviously for each $\lambda\in\alpha$ $\quad c_\lambda=a_\lambda+b_\lambda$ and finally

$$\sum_{\lambda\in\alpha}f(\lambda)c_\lambda=\sum_{\lambda\in\alpha}f(\lambda)a_\lambda+\sum_{\lambda\in\alpha}f(\lambda)b_\lambda$$

as well. It is also easy to see that this summation is independent on the "selection of representants".

Let $\mathbf{a}\subseteq\mathbf{u}$. We shall define relative homology groups of the couple $(\mathbf{u},\mathbf{a})$.

Definition 4.1. *Let* $\mathbf{a} \subseteq \mathbf{u}$. *Let us put*

$$\mathbf{C}_\nu(\mathbf{R}, \mathbf{u}, \mathbf{a}) = \mathbf{C}_\nu(\mathbf{r}(\mathbf{R})) / \mathbf{C}_\nu\left(\mathbf{r}\left(\mathbf{R} \cap \mathbf{a}^2\right)\right).$$

Therefore, "simplexes" of $\mathbf{C}_\nu(\mathbf{R}, \mathbf{u}, \mathbf{a})$ *are those "simplexes" of* $\mathbf{C}_\nu(\mathbf{r}(\mathbf{R}))$ *at least one vertex of which does not belong to* $\mathbf{a}$. *The boundary operator*

$$\partial_\nu : \mathbf{C}_\nu(\mathbf{r}(\mathbf{R})) \to \mathbf{C}_{\nu-1}(\mathbf{r}(\mathbf{R}))$$

maps the subgroup $\mathbf{C}_\nu\left(\mathbf{r}\left(\mathbf{R} \cap \mathbf{a}^2\right)\right)$ *into the subgroup* $\mathbf{C}_{\nu-1}\left(\mathbf{r}\left(\mathbf{R} \cap \mathbf{a}^2\right)\right)$ *and therefore it induces the homomorphism*

$$\bar{\partial}_\nu : \mathbf{C}_\nu(\mathbf{R}, \mathbf{u}, \mathbf{a}) \to \mathbf{C}_{\nu-1}(\mathbf{R}, \mathbf{u}, \mathbf{a}) .$$

This homomorphism is a boundary operator in the new chain complex of groups $\left\{\mathbf{C}_\nu(\mathbf{R}, \mathbf{u}, \mathbf{a}), \bar{\partial}_\nu, \nu \in \mathbf{Z}\right\}$. *We shall frequently write* ∂ *instead of* $\bar{\partial}$. *The mentioned chain complex of groups we shall frequently write as* $\mathbf{C}(\mathbf{R}, \mathbf{u}, \mathbf{a})$. *We shall write* $\mathbf{H}_\nu(\mathbf{R}, \mathbf{u}, \mathbf{a})$ *for homology groups of this chain complex of groups or simply* $\mathbf{H}_\nu(\mathbf{u}, \mathbf{a})$.

Let us write $\mathbf{C}(\mathbf{R})$ instead of the chain complex of groups $\{\mathbf{C}_\nu(\mathbf{r}(\mathbf{R})), \partial_\nu, \nu \in \mathbf{Z}\}$ and $\mathbf{C}\left(\mathbf{R} \cap \mathbf{a}^2\right)$ instead of $\left\{\mathbf{C}_\nu\left(\mathbf{r}\left(\mathbf{R} \cap \mathbf{a}^2\right)\right), \partial_\nu, \nu \in \mathbf{Z}\right\}$.

If $\bar{i}$ and $\bar{p}$ are homomorphisms of chain complexes of groups induced by the inclusion map and by the natural projection respectively, then

$$\mathbf{0} \longrightarrow \mathbf{C}\left(\mathbf{R} \cap \mathbf{a}^2\right) \xrightarrow{\bar{i}} \mathbf{C}(\mathbf{R}) \xrightarrow{\bar{p}} \mathbf{C}(\mathbf{R}, \mathbf{u}, \mathbf{a}) \longrightarrow \mathbf{0}$$

is a short exact sequence.

Now we shall prove the following

Theorem 4.2. *Let* $\mathbf{a} \subseteq \mathbf{b} \subseteq \mathbf{u}$ *be such that* $\operatorname{Fig}(\mathbf{a}) = \operatorname{Fig}(\mathbf{b})$. *Then the short exact sequences*

$$(*) \qquad \mathbf{0} \to \mathbf{C}\left(\mathbf{r}\left(\mathbf{R} \cap \mathbf{a}^2\right)\right) \xrightarrow{\bar{i}} \mathbf{C}(\mathbf{r}(\mathbf{R})) \xrightarrow{\overline{p}_1} \mathbf{C}(\mathbf{R}, \mathbf{u}, \mathbf{a}) \to \mathbf{0} \quad \textit{and}$$

$$(**) \qquad \mathbf{0} \to \mathbf{C}\left(\mathbf{r}\left(\mathbf{R} \cap \mathbf{b}^2\right)\right) \xrightarrow{\bar{i}} \mathbf{C}(\mathbf{r}(\mathbf{R})) \xrightarrow{\overline{p}_2} \mathbf{C}(\mathbf{R}, \mathbf{u}, \mathbf{b}) \to \mathbf{0}$$

are homotopically equivalent. ($\bar{i}$ *are inclusions,* $\overline{p}_1$ *and* $\overline{p}_2$ *are competent projections.*)

Proof: According to the theorem on the selection of π-relations there are the following functions:

$$\mathbf{f} : \mathbf{b} \to \mathbf{a} \quad \text{such that} \quad x\mathbf{R}\mathbf{f}(x) \quad \& \quad \mathbf{f} \restriction \mathbf{a} = \mathbf{Id}_{\mathbf{a}}$$
$$\mathbf{h} : \mathbf{u} \to \mathbf{u} \quad \text{such that} \quad \mathbf{h} \restriction \mathbf{a} \cup (\mathbf{u} \setminus \mathbf{b}) = \mathbf{Id}_{\mathbf{a} \cup (\mathbf{u} \setminus \mathbf{b})}$$

with $\mathbf{h} \restriction \mathbf{b} = \mathbf{f}$.

Let i_0, i_1, i_2 and i be convenient inclusions. Let us consider the following diagram:

$$\begin{array}{ccccccccc}
\mathbf{0} & \longrightarrow & \mathbf{C}(\mathbf{r}(\mathbf{R}\cap\mathbf{a}^2)) & \xrightarrow{\bar{i}_1} & \mathbf{C}(\mathbf{r}(\mathbf{R})) & \xrightarrow{\overline{\mathbf{p}}_1} & \mathbf{C}(\mathbf{R},\mathbf{u},\mathbf{a}) & \longrightarrow & \mathbf{0} \\
 & & \overline{\mathbf{f}}\uparrow \quad \downarrow\bar{i}_0 & & \overline{\mathbf{h}}\uparrow \quad \downarrow\bar{i} & & \overline{\mathbf{w}}\uparrow \quad \downarrow\overline{\mathbf{v}} & & \\
\mathbf{0} & \longrightarrow & \mathbf{C}(\mathbf{r}(\mathbf{R}\cap\mathbf{b}^2)) & \xrightarrow{\bar{i}_2} & \mathbf{C}(\mathbf{r}(\mathbf{R})) & \xrightarrow{\overline{\mathbf{p}}_2} & \mathbf{C}(\mathbf{R},\mathbf{u},\mathbf{b}) & \longrightarrow & \mathbf{0}\,.
\end{array}$$

One can easily verify that the left square of this diagram commutes in both directions. Moreover, following the Theorem 3.2 we have the next relations:

$$(1)\quad \overline{\mathbf{f}}\circ\bar{i}_0 \simeq \overline{\mathbf{Id}} \qquad (2)\quad \bar{i}_0\circ\overline{\mathbf{f}} \simeq \overline{\mathbf{Id}} \qquad (3)\quad \overline{\mathbf{h}}\circ\bar{i} \simeq \overline{\mathbf{Id}} \qquad (4)\quad \bar{i}\circ\overline{\mathbf{h}} \simeq \overline{\mathbf{Id}}.$$

We consider identities with convenient domains in the previous relations.

Next let us define $\overline{\mathbf{v}}_\nu : \mathbf{C}_\nu\left(\mathbf{R},\mathbf{u},\mathbf{a}\right) \to \mathbf{C}_\nu\left(\mathbf{R},\mathbf{u},\mathbf{b}\right)$ as follows:

$$\overline{\mathbf{v}}_\nu\left(x+\mathbf{C}_\nu\left(\mathbf{r}\left(\mathbf{R}\cap\mathbf{a}^2\right)\right)\right) = \bar{i}(x)+\mathbf{C}_\nu\left(\mathbf{r}\left(\mathbf{R}\cap\mathbf{b}^2\right)\right)$$

and similarly $\overline{\mathbf{w}}_\nu : \mathbf{C}_\nu\left(\mathbf{R},\mathbf{u},\mathbf{b}\right) \to \mathbf{C}_\nu\left(\mathbf{R},\mathbf{u},\mathbf{a}\right)$ as follows:

$$\overline{\mathbf{w}}_\nu\left(x+\mathbf{C}_\nu\left(\mathbf{r}\left(\mathbf{R}\cap\mathbf{b}^2\right)\right)\right) = \overline{\mathbf{h}}_\nu(x)+\mathbf{C}_\nu\left(\mathbf{r}\left(\mathbf{R}\cap\mathbf{a}^2\right)\right).$$

Now we verify correctness of these definitions.

Let $x-x' \in \mathbf{C}_\nu\left(\mathbf{r}\left(\mathbf{R}\cap\mathbf{a}^2\right)\right)$, i.e. $x' = x+y$ where $y \in \mathbf{C}_\nu\left(\mathbf{r}\left(\mathbf{R}\cap\mathbf{a}^2\right)\right)$. Then

$$\begin{aligned}
\overline{\mathbf{v}}_\nu\left(x'+\mathbf{C}_\nu\left(\mathbf{r}\left(\mathbf{R}\cap\mathbf{a}^2\right)\right)\right) &= \bar{i}\left(x'\right)+\mathbf{C}_\nu\left(\mathbf{r}\left(\mathbf{R}\cap\mathbf{b}^2\right)\right) = \\
&= \bar{i}(x)+\bar{i}(y)+\mathbf{C}_\nu\left(\mathbf{r}\left(\mathbf{R}\cap\mathbf{b}^2\right)\right) = \\
&= \bar{i}(x)+y+\mathbf{C}_\nu\left(\mathbf{r}\left(\mathbf{R}\cap\mathbf{b}^2\right)\right) = \\
&= \bar{i}(x)+\mathbf{C}_\nu\left(\mathbf{r}\left(\mathbf{R}\cap\mathbf{b}^2\right)\right) = \\
&= \overline{\mathbf{v}}_\nu\left(x+\mathbf{C}_\nu\left(\mathbf{r}\left(\mathbf{R}\cap\mathbf{a}^2\right)\right)\right).
\end{aligned}$$

Similarly, let $x-x' \in \mathbf{C}_\nu\left(\mathbf{r}\left(\mathbf{R}\cap\mathbf{b}^2\right)\right)$, i.e. $x' = x+y$, where $y \in \mathbf{C}_\nu\left(\mathbf{r}\left(\mathbf{R}\cap\mathbf{b}^2\right)\right)$. Then

$$\begin{aligned}
\overline{\mathbf{w}}_\nu\left(x'+\mathbf{C}_\nu\left(\mathbf{r}\left(\mathbf{R}\cap\mathbf{b}^2\right)\right)\right) &= \overline{\mathbf{h}}_\nu\left(x'\right)+\mathbf{C}_\nu\left(\mathbf{r}\left(\mathbf{R}\cap\mathbf{a}^2\right)\right) = \\
&= \overline{\mathbf{h}}_\nu(x)+\overline{\mathbf{h}}_\nu(y)+\mathbf{C}_\nu\left(\mathbf{r}\left(\mathbf{R}\cap\mathbf{a}^2\right)\right) = \\
&= \overline{\mathbf{h}}_\nu(x)+\mathbf{C}_\nu\left(\mathbf{r}\left(\mathbf{R}\cap\mathbf{a}^2\right)\right) = \\
&= \overline{\mathbf{w}}_\nu\left(x+\mathbf{C}_\nu\left(\mathbf{r}\left(\mathbf{R}\cap\mathbf{b}^2\right)\right)\right).
\end{aligned}$$

Hence the maps $\overline{\mathbf{v}}$ and $\overline{\mathbf{w}}$ are defined correctly. Moreover they have been defined just in order to make the right square in the diagram commutating in both directions.

Because

$$\overline{\mathbf{v}} \circ \overline{\mathbf{w}} \left(x + \mathbf{C}_\nu \left(\mathbf{r}\left(\mathbf{R} \cap \mathbf{b}^2\right)\right)\right) = \overline{i} \circ \overline{\mathbf{h}}(x) + \mathbf{C}_\nu \left(\mathbf{r}\left(\mathbf{R} \cap \mathbf{b}^2\right)\right) \text{ and}$$
$$\overline{\mathbf{w}} \circ \overline{\mathbf{v}} \left(x + \mathbf{C}_\nu \left(\mathbf{r}\left(\mathbf{R} \cap \mathbf{a}^2\right)\right)\right) = \overline{\mathbf{h}} \circ \overline{i}(x) + \mathbf{C}_\nu \left(\mathbf{r}\left(\mathbf{R} \cap \mathbf{a}^2\right)\right),$$

it is clear that chain homotopies assuring relations (3) and (4) induce through the projections chain homotopies which assure the relations $\overline{\mathbf{w}} \circ \overline{\mathbf{v}} \simeq \overline{\mathbf{Id}}$ and $\overline{\mathbf{v}} \circ \overline{\mathbf{w}} \simeq \overline{\mathbf{Id}}$. □

Corollary 4.3. *Let* $\mathbf{a}, \mathbf{b} \subseteq \mathbf{u}$ *be such that* $\mathrm{Fig}\,(\mathbf{a}) = \mathrm{Fig}\,(\mathbf{b})$. *Then the exact sequences of chain complexes* $(*)$ *and* $(**)$ *are homotopically equivalent.*

Proof: This follows from the fact that if $\mathrm{Fig}\,(\mathbf{a}) = \mathrm{Fig}\,(\mathbf{b})$, then $\mathrm{Fig}\,(\mathbf{a} \cup \mathbf{b}) = \mathrm{Fig}\,(\mathbf{a}) = \mathrm{Fig}\,(\mathbf{b})$ and from the previous theorem. □

A special part of this corollary is the next

Corollary 4.4. *Let* $\mathbf{a}$ *and* $\mathbf{b}$ *be such that* $\mathrm{Fig}\,(\mathbf{a}) = \mathrm{Fig}\,(\mathbf{b})$. *Then for every* $n \in \mathbf{Z}$ $\mathbf{H}_n(\mathbf{u}, \mathbf{a})$ *and* $\mathbf{H}_n(\mathbf{u}, \mathbf{b})$ *are isomorphic.*

The corollary enables us to define $\mathbf{H}_n(\mathbf{u}, \mathrm{Fig}\,(\mathbf{a}))$ to be equal to $\mathbf{H}_n(\mathbf{u}, \mathbf{a})$ so that the $\mathbf{H}_n(\mathbf{u}, \mathrm{Fig}\,(\mathbf{a}))$ is (up to an isomorphism) independent on the choice of $\mathbf{a}$. The theorem itself carries a more wide information, of course.

Let us define a homomorphism $\partial_* : \mathbf{H}_n\,(\mathbf{u}, \mathbf{a}) \to \mathbf{H}_{n-1}\,(\mathbf{a})$ as follows: let $c \in \mathbf{Z}_n\,(\mathbf{R}, \mathbf{u}, \mathbf{a})$ and let $[c]$ be its homology class. As the projection p is mapping onto $\mathbf{C}_n\,(\mathbf{R}, \mathbf{u}, \mathbf{a})$, there is $b \in \mathbf{C}_n\,(\mathbf{r}\,(\mathbf{R}))$ such that $p\,(b) = c$. We have $p \circ \partial\,(b) = \partial \circ p\,(b) = \partial\,(c) = 0$, hence by the exactness of $(*)$ it follows that there exists $d \in \mathbf{C}_{n-1}\left(\mathbf{r}\left(\mathbf{R} \cap \mathbf{a}^2\right)\right)$ with $\overline{i}\,(d) = \partial\,(b)$. Moreover, because $\overline{i} \circ \partial\,(d) = \partial \circ \overline{i}\,(d) = \partial \circ \partial\,(b) = 0$ and because $\overline{i}$ is an injection, we have $d \in \mathbf{Z}_{n-1}\left(\mathbf{r}\left(\mathbf{R} \cap \mathbf{a}^2\right)\right)$. For correctness of this definition see e.g. [H-W, Theorem 5.5.1].

It is obvious that for every x it holds that $H_n\,(x, \emptyset) = H_n\,(x)$. Using inclusion maps $j : (\mathbf{u}, \emptyset) \to (\mathbf{u}, \mathbf{a})$ and $i : \mathbf{a} \to \mathbf{u}$, we can see that there is so called homology sequence of the pair

$$\cdots \longleftarrow \mathbf{H}_{n-1}\,(\mathbf{a}) \xleftarrow{\partial_*} \mathbf{H}_n\,(\mathbf{u}, \mathbf{a}) \xleftarrow{j_*} \mathbf{H}_n\,(\mathbf{u}) \xleftarrow{i_*} \mathbf{H}_n\,(\mathbf{a}) \longleftarrow \cdots$$

Let $\mathbf{R}$ be an indiscernibility equivalence on $\mathbf{u}$, i.e. $\mathbf{R} = \mathbf{R} \cap \mathbf{u}^2$ and let $\mathbf{v} \subseteq \mathbf{u}$. Let $\mathbf{S}$ be an indiscernibility equivalence on $\mathbf{w}$, i.e. $\mathbf{S} = \mathbf{S} \cap \mathbf{w}^2$ and let $\mathbf{z} \subseteq \mathbf{w}$. Let $f : (\mathbf{u}, \mathbf{v}) \to (\mathbf{w}, \mathbf{z})$ be a continuous map from $\mathbf{R}$ to $\mathbf{S}$ (it means also that $f''\mathbf{v} \subseteq \mathbf{z}$). The map f induces a homomorphism from the short exact sequence $\mathbf{0} \to \mathbf{C}(\mathbf{r}(\mathbf{R} \cap \mathbf{v}^2)) \xrightarrow{\overline{i}} \mathbf{C}(\mathbf{r}(\mathbf{R})) \xrightarrow{\overline{p}_1} \mathbf{C}(\mathbf{R}, \mathbf{u}, \mathbf{v}) \to \mathbf{0}$ to the short exact sequence $\mathbf{0} \to \mathbf{C}(\mathbf{r}(\mathbf{S} \cap \mathbf{z}^2)) \xrightarrow{\overline{i}} \mathbf{C}(\mathbf{r}(\mathbf{S})) \xrightarrow{\overline{p}_1} \mathbf{C}(\mathbf{S}, \mathbf{w}, \mathbf{z}) \to \mathbf{0}$ in the analogical way as it is done in the Definition 2.6. Of course this induced homomorphism consists of three homomorphisms $(\overline{f \restriction \mathbf{v}}, \overline{f}, \overline{f'})$ such that the following diagram commutes:

$$
(+)\qquad
\begin{array}{ccccccccc}
\mathbf{0} & \longrightarrow & \mathbf{C}(\mathbf{r}(\mathbf{R}\cap\mathbf{v}^2)) & \xrightarrow{\overline{i}} & \mathbf{C}(\mathbf{r}(\mathbf{R})) & \xrightarrow{\overline{p}_1} & \mathbf{C}(\mathbf{R},\mathbf{u},\mathbf{v}) & \longrightarrow & \mathbf{0} \\
 & & \Big\downarrow{\scriptstyle \overline{f\restriction\mathbf{v}}} & & \Big\downarrow{\scriptstyle \overline{f}} & & \Big\downarrow{\scriptstyle \overline{f'}} & & \\
\mathbf{0} & \longrightarrow & \mathbf{C}(\mathbf{r}(\mathbf{S}\cap\mathbf{z}^2)) & \xrightarrow{\overline{i}} & \mathbf{C}(\mathbf{r}(\mathbf{S})) & \xrightarrow{\overline{p}_1} & \mathbf{C}(\mathbf{S},\mathbf{w},\mathbf{z}) & \longrightarrow & \mathbf{0}\,.
\end{array}
$$

The triple of homomorphisms $(\overline{f \restriction \mathbf{v}}, \overline{f}, \overline{f'})$ induces the triple of homomorphisms $((f \restriction \mathbf{v})_*, f_*, f'_*)$ between convenient homological sequences (cf. [H-W, Chapter I, §§ 5.5–5.6]). Let us denote $\mathbf{R_v} = \mathbf{R}\cap\mathbf{v}^2$, $\mathbf{S_z} = \mathbf{S}\cap\mathbf{z}^2$. Then the above statement is expressed by the following commutative diagram:
(++)

$$
\begin{array}{ccccccccc}
\cdots \leftarrow & \mathbf{H}_{n-1}(\mathbf{r}(\mathbf{R_v})) & \xleftarrow{\partial_*} & \mathbf{H}_n(\mathbf{R},\mathbf{u},\mathbf{v}) & \xleftarrow{j_*} & \mathbf{H}_n(\mathbf{r}(\mathbf{R})) & \xleftarrow{i_*} & \mathbf{H}_n(\mathbf{r}(\mathbf{R_v})) & \leftarrow \cdots \\
 & \Big\downarrow{\scriptstyle (f\restriction\mathbf{v})_*} & & \Big\downarrow{\scriptstyle f_*} & & \Big\downarrow{\scriptstyle f'_*} & & \Big\downarrow{\scriptstyle (f\restriction\mathbf{v})_*} & \\
\cdots \leftarrow & \mathbf{H}_{n-1}(\mathbf{r}(\mathbf{S_z})) & \xleftarrow{\partial_*} & \mathbf{H}_n(\mathbf{S},\mathbf{w},\mathbf{z}) & \xleftarrow{j_*} & \mathbf{H}_n(\mathbf{r}(\mathbf{S})) & \xleftarrow{i_*} & \mathbf{H}_n(\mathbf{r}(\mathbf{S_z})) & \leftarrow \cdots
\end{array}
$$

It is clear that the following relative forms of Theorems 2.8 and 2.9 hold:

Theorem 4.5.

(1) $\mathbf{Id}_*$ *is an identical isomorphism of a relative homology group*
(2) *let f be as above, let $g : (\mathbf{w},\mathbf{z}) \to (\mathbf{s},\mathbf{t})$ be a map continuous from* $\mathbf{S}$ *to* $\mathbf{T}$ *— here* $\mathbf{T}$ *is an indiscernibility equivalence on* $\mathbf{s}$, *i.e.* $\mathbf{T} = \mathbf{T}\cap\mathbf{s}^2$. *Then* $(f\circ g)_* = f_*\circ g_*$.

Commutativity of the diagram (++) yields to

Theorem 4.6. $(f \restriction \mathbf{v})_* \circ \partial_* = \partial_* \circ f_*$.

According to the [H-W, Chapter I, §§ 5.5–5.6] we have

Theorem 4.7. *The homology sequence*

$$\cdots \leftarrow \mathbf{H}_{n-1}\left(\mathbf{r}\left(\mathbf{R}\cap\mathbf{v}^2\right)\right) \xleftarrow{\partial_*} \mathbf{H}_n(\mathbf{R},\mathbf{u},\mathbf{v}) \xleftarrow{j_*} \mathbf{H}_n\left(\mathbf{r}\left(\mathbf{R}\cap\mathbf{u}^2\right)\right) \leftarrow \cdots$$

is exact.

Combining this theorem with "five-lemma" and with Corollary 3.3 it again yields to an isomorphism as in Corollary 4.4.

Homotopy axiom.

Before starting with the definition we shall give one preliminary statement.

Theorem 4.8. *Let $f, g : \mathbf{u} \to \mathbf{w}$, let $\mathbf{R}$ be a symmetry such that $\{[x,x]; x \in U\} \subseteq \mathbf{R} = \mathbf{R} \cap \mathbf{u}^2$ and let $\mathbf{S}$ be an equivalence such that $\mathbf{S} = \mathbf{S} \cap \mathbf{w}^2$. Then the following conditions are equivalent:*

(1) $(\forall x \in \mathbf{u})(f(x)\mathbf{S}g(x))$
(2) $(\forall \mathbf{z} \subseteq \mathbf{u})(\mathrm{Fig}^{\mathbf{S}}(f''\mathbf{z}) = \mathrm{Fig}^{\mathbf{S}}(g''\mathbf{z}))$
(3) $(\forall \mathbf{z} \in \mathbf{r}(\mathbf{R}))(\mathrm{Fig}^{\mathbf{S}}(f''\mathbf{z}) = \mathrm{Fig}^{\mathbf{S}}(g''\mathbf{z}))$.

Proof: The relation $\mathbf{r}(\mathbf{R}) \subseteq \mathcal{P}(\mathbf{u})$ (the set of all subsets of $\mathbf{u}$) assures the implication (2) $\Rightarrow$ (3). Similarly the fact that $\mathbf{r}(\mathbf{R})$ contains all one-element subsets of $\mathbf{u}$ assures the implication (3) $\Rightarrow$ (1). To prove the implication (1) $\Rightarrow$ (2), let $\mathbf{z} \subseteq \mathbf{u}$. Then

$$\mathrm{Fig}^{\mathbf{S}}\left(f''\mathbf{z}\right) = \bigcup\{\mathrm{Fig}^{\mathbf{S}}\left(f(x)\right); x \in \mathbf{z}\} = \bigcup\{\mathrm{Fig}^{\mathbf{S}}\left(g(x)\right); x \in \mathbf{z}\} = \mathrm{Fig}^{\mathbf{S}}\left(g''\mathbf{z}\right)$$

To explain the equality between two unions, let $y \in \bigcup\{\mathrm{Fig}^{\mathbf{S}}\left(f(x)\right); x \in \mathbf{z}\}$. It means that there is $x \in \mathbf{z}$ such that $y\mathbf{S}f(x)$. By (1) we have $y\mathbf{S}f(x)\mathbf{S}g(x)$ and because $\mathbf{S}$ is an equivalence, we have $y\mathbf{S}g(x)$. Hence $y \in \bigcup\{\mathrm{Fig}^{\mathbf{S}}\left(g(x)\right); x \in \mathbf{z}\}$. □

In the view of our definition of simplexes, one can see that the condition (3) of the previous theorem exactly corresponds to the notion of the **contiguity** (cf. [E-S, VI.3.1]) in a classical algebraic topology. But it seems to be better to give the definition also for the case when $\mathbf{S}$ is a symmetry.

Definition 4.9. *Let $\mathbf{R}$ and $\mathbf{S}$ be symmetries, let $\mathbf{u}$ and $\mathbf{w}$ be sets with $\mathbf{R} = \mathbf{R} \cap \mathbf{u}^2$ and $\mathbf{S} = \mathbf{S} \cap \mathbf{w}^2$. Let $f, g : \mathbf{u} \to \mathbf{w}$ be two maps continuous from $\mathbf{R}$ to $\mathbf{S}$. The maps f, g are said to be* **contiguous** *— we shall write $f \simeq g$ — iff*

$$(\forall \mathbf{z} \in \mathbf{r}(\mathbf{R}))(f''\mathbf{z} \cup g''\mathbf{z} \in \mathbf{r}(\mathbf{S})). \tag{$*$}$$

Remark. We require moreover that f, g are maps between the same pairs of sets $(f, g : (\mathbf{u}, \mathbf{v}) \to (\mathbf{w}, \mathbf{z}))$ for the relative case. This is necessary for f, g in order to induce the homorphisms $f_*, g_* : \mathbf{H}_n(\mathbf{R}, \mathbf{u}, \mathbf{v}) \to \mathbf{H}_n(\mathbf{S}, \mathbf{w}, \mathbf{z})$.

Lemma 4.10. *If $\mathbf{S}$ in the previous definition is an equivalence then $(*)$ is equivalent to each of the conditions* (1)–(3) *of Theorem* 4.8.

Proof: We shall prove the equivalence of $(*)$ and (3). Let $(*)$ hold. Let $\mathbf{z} \in \mathbf{r}(\mathbf{R})$. We have $f''\mathbf{z} \cup g''\mathbf{z} \in \mathbf{r}(\mathbf{S})$. Now let $x, y \in \mathbf{z}$. We have $[f(x), f(y)] \in \mathbf{S}$, i.e. $\mathrm{Fig}\left(f''\mathbf{z}\right) = \mathrm{Mon}(f(x))$. Similarly $\mathrm{Fig}\left(g''\mathbf{z}\right) = \mathrm{Mon}(g(x))$. But also $f(x)\mathbf{S}g(x)$ so that $\mathrm{Mon}(f(x)) = \mathrm{Mon}(g(x))$. It means that (3) holds.

Now let $\mathbf{z} \in \mathbf{r}(\mathbf{R})$ again. According to the continuity of f and g it holds that $(f''\mathbf{z})^2 \subseteq \mathbf{S}$ and $(g''\mathbf{z})^2 \subseteq \mathbf{S}$. Hence if $x \in \mathbf{z}$ then $\mathrm{Fig}\left(f''\mathbf{z}\right) = \mathrm{Mon}(f(x))$ and also $\mathrm{Fig}\left(g''\mathbf{z}\right) = \mathrm{Mon}(g(x))$. Using (3) now we have $\mathrm{Fig}\left(f''\mathbf{z} \cup g''\mathbf{z}\right) = \mathrm{Mon}(f(x))$ so that $f''\mathbf{z} \cup g''\mathbf{z} \in \mathbf{r}(\mathbf{S})$. □

The lemma yields to the fact that if $\mathbf{S}$ is a π-equivalence we can define the notion "to be contiguous" with help of any of the conditions (1)–(3). It follows from (1) that if $\mathbf{S}$ is a π-equivalence, omitting the word "continuous", the relation $\simeq$ becomes to be a π-class. Indeed, $\forall\pi \leftrightarrow \pi$.

If $\mathbf{S}$ is a symmetry then $\simeq$ is a symmetry as well, if $\mathbf{S}$ is an equivalence then $\simeq$ is also an equivalence.

We are ready for the definition of the homotopy.

Definition 4.11. *Let $f, g : (\mathbf{u}, \mathbf{v}) \to (\mathbf{w}, \mathbf{z})$, $\mathbf{R}$ and $\mathbf{S}$ be as Definition 4.9 requires. The maps f and g are said to be* **homotopic** *iff there is the set $\{h_i; i \in \nu + 1\}$ ($\nu \in \mathbf{N}$) of continuous functions $h_i : (\mathbf{u}, \mathbf{v}) \to (\mathbf{w}, \mathbf{z})$ such that $f = h_0$, $g = h_\nu$ and $(\forall\, i \in \nu) h_i \simeq h_{i+1}$.*

Remark. One can obtain the usual (absolute) definition of homotopy considering $(\mathbf{u}, \emptyset)$ and $(\mathbf{w}, \emptyset)$ is the same as $\mathbf{u}$ and $\mathbf{w}$, respectively.

Obviously the homotopy is an equivalence relation.

Theorem 4.12 (Homotopy axiom)**.** *Let f, g, $(\mathbf{u}, \mathbf{v})$, $(\mathbf{w}, \mathbf{z})$, $\mathbf{R}$ and finally $\mathbf{S}$ be as it is required in Definition 4.9. Moreover let $\mathbf{R}$ and $\mathbf{S}$ be indiscernibility relations. If f and g are homotopic maps then for each $n \in \mathbf{N}$ $\quad f_* = g_* : \mathbf{H}_n(\mathbf{R}, \mathbf{u}, \mathbf{v}) \to \mathbf{H}_n(\mathbf{S}, \mathbf{w}, \mathbf{z})$.*

Proof: Let us define chain homotopy $\mathbf{D}'$ between $\overline{f}$ and $\overline{g}$ as follows: take the ordered simplex $\langle \mathbf{s} \rangle = [x_0, x_1, \ldots, x_n]$ of $\mathbf{C}_n(\mathbf{r}(\mathbf{R}))$. Let us put

$$\mathbf{D}'(\langle \mathbf{s} \rangle) = \sum_{i=0}^{\nu-1} \sum_{j=0}^{n} (-1)^j [h_i(x_0), \ldots, h_i(x_j), h_{i+1}(x_j), \ldots, h_{i+1}(x_n)].$$

$\mathbf{D}'(\langle \mathbf{s} \rangle)$ is a chain in $\mathbf{C}_{n+1}(\mathbf{r}(\mathbf{S}))$ because $h_i \simeq h_{i+1}$. Let us take the linear extension of $\mathbf{D}'$ to the entire group $\mathbf{C}_n(\mathbf{r}(\mathbf{R}))$ — for this extension we shall write also $\mathbf{D}'$. Combining Lemma 3.1 with the induction we can see that $\mathbf{D}'$ is a chain homotopy between $\overline{f}$ and $\overline{g}$, i.e. $\mathbf{D}' \circ \partial + \partial \circ \mathbf{D}' = \overline{g} - \overline{f}$. Hence $f_* = g_*$. □

The last proof is so simple due to our definition of the homotopy. Moreover this definition seems to be — at least in our opinion — very natural.

Excision axiom.

Dealing with this axiom we must be careful and we must do some restrictions. The notion of A-full class has the fundamental significance for our approach to the excision axiom.

Definition 4.13. *Let $A \subseteq \mathbf{u}$ $\quad(A \subseteq \mathbf{z})$ be a closed figure. A set $\mathbf{a} \subseteq \mathbf{u}$ $\quad(\mathbf{w} \subseteq \mathbf{z})$ is said to be A-**full** ($A_{\mathbf{z}}$-**full**), if $\mathrm{Int}(A) \subseteq \mathbf{a} \subseteq \bar{A}$ $\quad(\mathrm{Int}_{\mathbf{z}}(A) \subseteq \mathbf{w} \subseteq \bar{A}_{\mathbf{z}})$ and $\mathrm{Fig}(\mathbf{a}) = \mathrm{Fig}(A)$ $\quad(\mathrm{Fig}(\mathbf{w}) = \mathrm{Fig}(A)_{\mathbf{z}})$. (According to our agreement made at the beginning of this section it holds that $\mathrm{Int}(A) \subseteq \mathbf{u}$, $\bar{A} \subseteq \mathbf{u}, \ldots$)*

Lemma 4.14. *Let $A \subseteq \mathbf{u}$ be a closed figure. Then there is A-full set $\mathbf{a}$.*

Proof: A and $\mathbf{u} \setminus \mathring{A}$ are closed figures, hence there are sets $\mathbf{a}'$ and $\mathbf{b}$ such that $\mathrm{Fig}(\mathbf{a}') = A$ and $\mathrm{Fig}(\mathbf{b}) = \mathbf{u} \setminus \mathring{A}$. So we have $\mathrm{Fig}(\mathbf{a}') \cup \mathrm{Fig}(\mathbf{b}) = \mathbf{u}$. Let $\{R_\nu; \nu \in \alpha\}$

be a set prolongation of the generating sequence $\{R_n; n \in \mathbf{FN}\}$ of $\mathbf{R}$. Then there is $\nu \in \alpha \setminus \mathbf{FN}$ such that

$$R''_\nu \mathbf{a}' \cup R''_\nu \mathbf{b} = \mathbf{u}\,. \tag{1}$$

Let us put $\mathbf{a} = R''_\nu \mathbf{a}'$. We are going to prove that $\mathbf{a}$ is an A-full set.
Obviously, $\mathbf{a} = R''_\nu \mathbf{a}' \subseteq \mathrm{Fig}\left(\mathbf{a}'\right) = A$ so that $\mathbf{a} \subseteq A$. According to (1) it holds for $x \in \mathring{A}$ either $x \in R''_\nu \mathbf{a}'$ or $x \in R''_\nu \mathbf{b}$. Moreover

$$x \in \mathring{A} \quad \Longrightarrow \quad x \notin \mathbf{u} \setminus \mathring{A} \quad \Longrightarrow \quad x \notin \mathrm{Fig}\,(\mathbf{b})\ .$$

But because $R''_\nu \mathbf{b} \subseteq \mathrm{Fig}\,(\mathbf{b})$, $x \notin R''_\nu \mathbf{b}$ holds as well. Hence we have $x \in \mathbf{a} = R''_\nu \mathbf{a}'$. So that $\mathring{A} \subseteq \mathbf{a} \subseteq A$.
Now let $x \in A$. Then $x \in \mathrm{Fig}\left(\mathbf{a}'\right)$ according to the definition of $\mathbf{a}'$. Hence $x \in \mathrm{Fig}\left(R''_\nu \mathbf{a}'\right) = \mathrm{Fig}\,(\mathbf{a})$, which concludes the proof. □

For the rest of the section, let $U \subseteq \mathbf{u}$ be an open figure, let $A \subseteq \mathbf{u}$ be a closed figure such that $\overline{U} \subseteq \mathring{A}$.

So that $\mathbf{u} \setminus U$ is a closed figure and therefore there is a $(\mathbf{u} \setminus U)$-full set $\mathbf{z}$ with $\mathrm{Fig}\,(\mathbf{z}) = \mathbf{u} \setminus U$.

Lemma 4.15. *Let $\mathbf{a} \subseteq \mathbf{u}$ be an A-full set. Then $\mathbf{a} \cap \mathbf{z}$ is an $(A \setminus U)_{\mathbf{z}}$-full set.*

Proof: Let $x \in \mathrm{Int}_{\mathbf{z}}(A \setminus U)$. Then $x \in \mathrm{Int}\,(A \setminus U)$ and $x \in \mathbf{z}$ hence $x \in \mathbf{a}$ and $x \in \mathbf{z}$. This yields to $x \in \mathbf{a} \cap \mathbf{z}$. Next $x \in \mathbf{a} \cap \mathbf{z}$ means that $x \in \mathbf{a}$ and $x \in \mathbf{z}$. Hence $x \in \bar{A}$ and $x \in \mathbf{z}$, which gives $x \in \bar{A}_{\mathbf{z}}$. Now we are going to prove that $\overline{A \setminus U}_{\mathbf{z}} = \bar{A}_{\mathbf{z}}$. The inclusion $\subseteq$ holds obviously. Let $x \in \bar{A}_{\mathbf{z}}$, i.e. $x \in \bar{A} \cap \mathbf{z}$. We have $x \notin U$ because $x \in \mathbf{z}$ and $\mathrm{Fig}\,(\mathbf{z}) = \mathbf{u} \setminus U$. Henceforth $x \in (A \setminus U) \cap \mathbf{z}$. But U is open so that $A \setminus U = \overline{A \setminus U}$. Hence $x \in \overline{A \setminus U}_{\mathbf{z}}$. All we have to prove is that $\mathrm{Fig}\,(\mathbf{a} \cap \mathbf{z}) = \mathrm{Fig}\,(A \setminus U)$. The inclusion $\subseteq$ is clear again. Let us put $\mathbf{B}(X) = \bar{X} \setminus \mathring{X}$. First of all we shall prove that $\mathbf{B}(A) \subseteq \mathbf{z}$. According to the assumption that $\overline{U} \subseteq \mathring{A}$, we have

$$\mathbf{B}(A) = \bar{A} \setminus \mathring{A} \subseteq \mathbf{u} \setminus \mathring{A} \subseteq \mathbf{u} \setminus \overline{U} \quad \text{ i.e. } \quad \mathbf{B}(A) \subseteq \overline{U}\,.$$

The fact that $\mathbf{z}$ is a $(\mathbf{u} \setminus U)$-full set could yield to $\mathbf{B}(A) \subseteq \mathbf{z}$ provided e.g. that $\mathbf{u} \setminus \overline{U} \subseteq \mathrm{Int}\,(\mathbf{u} \setminus U)$. But the last statement is satisfied because $\mathbf{u} \setminus \overline{U}$ is an open figure such that $\mathbf{u} \setminus \overline{U} \subseteq \mathbf{u} \setminus U$, hence $\mathbf{u} \setminus \overline{U} \subseteq \mathrm{Int}\,(\mathbf{u} \setminus U)$. We have also $\mathbf{B}(U) \subseteq \mathbf{a}$ because $\mathbf{B}(U) \subseteq \mathring{A} \subseteq \mathbf{a}$. Now we can return back to the proof of $\mathrm{Fig}\,(\mathbf{a} \cap \mathbf{z}) = \mathrm{Fig}\,(A \setminus U)$. Let $x \in A \setminus U = \mathrm{Fig}\,(A \setminus U)$. We shall consider three cases:
Let $x \in \mathbf{B}(A)$. As $\mathbf{a}$ is an A-full set, there is $y \in \mathbf{a}$ such that $x\mathbf{R}y$. But $y \in \mathbf{B}(A)$ therefore $y \in \mathbf{z}$. Hence $x \in \mathrm{Fig}\,(\mathbf{a} \cap \mathbf{z})$.
Let $x \in \mathring{A} \setminus \overline{U}$. Then $x \in \mathbf{a}$ and $x \in \mathbf{z}$. So that $x \in \mathbf{a} \cap \mathbf{z}$, hence $x \in \mathrm{Fig}\,(\mathbf{a} \cap \mathbf{z})$.
Finally let $x \in \mathbf{B}(U)$. Then there is $y \in \mathbf{z}$ such that $x\mathbf{R}y$ because $\mathbf{z}$ is a $(\mathbf{u} \setminus U)$-full set and $\mathbf{B}(U) \subseteq \mathbf{u} \setminus U = \mathbf{u} \setminus \mathring{U}$. But because $y \in \mathbf{B}(U) \subseteq \mathbf{a}$, we have $y \in \mathbf{a} \cap \mathbf{z}$. Hence $x \in \mathrm{Fig}\,(\mathbf{a} \cap \mathbf{z})$, which concludes the proof. □

Now let us put (to the previous assignments) $\mathbf{C}(\mathbf{u} \setminus U, A \setminus U) = \mathbf{C}(\mathbf{R} \cap \mathbf{z}^2, \mathbf{z}, \mathbf{a} \cap \mathbf{z})$. We shall prove that the inclusion $i : \mathbf{z} \to \mathbf{u}$ induces isomorphisms of homology groups $i_* : \mathbf{H}_n\,(\mathbf{R}, \mathbf{u}, \mathbf{a}) \to \mathbf{H}_n\left(\mathbf{R} \cap \mathbf{z}^2, \mathbf{z}, \mathbf{a} \cap \mathbf{z}\right)$.

Theorem 4.16. *For each $n \in \mathbf{FN}$ we have the following equality between classes*

$$\mathbf{C}_n \left(\mathbf{R} \cap \mathbf{z}^2, \mathbf{z}, \mathbf{a} \cap \mathbf{z}\right) = \mathbf{C}_n \left(\mathbf{R}, \mathbf{u}, \mathbf{a}\right).$$

PROOF: To simplify the proof let us put $\mathbf{a}' = \mathbf{a} \cap \mathbf{z}$. Let d be a simplex "in $\mathbf{C}_n \left(\mathbf{R} \cap \mathbf{z}^2, \mathbf{z}, \mathbf{a}'\right)$". It means that for each vertex $x \in d$ we have $x \in \mathbf{z}$, i.e. $x \in \mathbf{u}$. The assumption also says that there is $x \in d$ such that $x \notin \mathbf{a}'$, i.e. $x \notin \mathbf{a}$. Hence d is a simplex "in $\mathbf{C}_n \left(\mathbf{R}, \mathbf{u}, \mathbf{a}\right)$". From the opposite let d be a simplex "in $\mathbf{C}_n \left(\mathbf{R}, \mathbf{u}, \mathbf{a}\right)$". So that there is $x \in d$ such that $x \notin \mathbf{a}$. But because $\mathbf{a}$ is an A-full set, we have $x \notin \mathring{A}$. And because $\mathring{A}$ is a figure, none of the vertices of d belongs to $\mathring{A}$. And because $\mathbf{u} \setminus \mathring{A} \subseteq \mathbf{z}$ (we have proved it in the previous proof) we have $d \subseteq \mathbf{z}$. Hence d is "in $\mathbf{C}_n \left(\mathbf{R} \cap \mathbf{z}^2, \mathbf{z}, \mathbf{a}'\right)$". □

The mentioned isomorphisms between homology groups follow directly from this theorem.

Dimension axiom

is the very last homology axiom which is to be proved.

Theorem 4.17. *Let $\mathbf{u}$ be a one-element set. Then for each $n \geq 1$ the convenient absolute homology group is trivial, i.e. $\mathbf{H}_n \left(\mathbf{r} \left(\mathbf{R} \cap \mathbf{u}^2\right)\right) = \mathbf{0}$.*

PROOF: This is true because of the lack of elements to have at least one n-dimensional simplex for $n \geq 1$.

□

Theorem 4.18. *Let $\mathbf{u}$ be a set such that $\mathbf{u}^2 \subseteq \mathbf{R}$ for the given indiscernibility relation $\mathbf{R}$. Then for each $n \in \mathbf{FN}$ we have $\mathbf{H}_n \left(\mathbf{r} \left(\mathbf{R} \cap \mathbf{u}^2\right)\right) = \mathbf{0}$.*

PROOF: It follows from the assumption that for $x \in \mathbf{u}$ we have that $\mathrm{Fig}\,(\mathbf{u}) = \mathrm{Fig}\,(\{x\})$. Now the statement follows from the previous theorem and from Corollary 3.3.

□

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