Zero-dimensional Dugundji spaces admit profinite lattice structures

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Abstract. We prove what the title says. It then follows that zero-dimensional Dugundji space are supercompact. Moreover, their Boolean algebras of clopen subsets turn out to be semigroup algebras.

Keywords: Dugundji space, projective Boolean algebra, profinite lattice, supercompact
Classification: Primary 54B25; Secondary 54H12, 06E05

Introduction.

The aim of this note is to prove what its title asserts. Let us first explain the concepts involved. For the sake of convenience, we do not give the original definitions but some that are known to be equivalent. Let us agree that *all topological spaces ocurring in this paper are compact and zero-dimensional.* Then a Dugundji space is just a retract of a Cantor cube 2^τ (of arbitrary weight τ). A topological lattice is profinite iff each pair of distinct points can be separated by a continuous homomorphisms into a finite lattice. It has been proved by Numakura [N] that compact zero-dimensional *distributive* lattices are automatically profinite. As the lattices that we construct are not necessarily distributive, we need special care to guarantee that they become profinite.

Our theorem has two interesting corollaries. It implies that all Dugundji spaces are supercompact (definition below) and even possess binary subbases consisting of clopen sets.This was proved before by S. Koppelberg (unpublished) with other methods.

The second corollary concerns the algebraic structure of the Boolean algebras $\mathrm{Clop}(X)$ for Dugundji spaces, i.e. the so-called projective Boolean algebras. It turns out that they are semigroup algebras in the sense of representation theory (cf. [CP, 5.2] or [He] for the Boolean case). This answers a question in [He].

Here are some examples to contrasts our results. Theorem 4.8 of [S] says that for $\tau > \omega_1$, the dyadic space $\exp_3 2^\tau$ does not admit any continuous binary operation that is commutative and idempotent. An example of a non-supercompact dyadic space can be found in [B]. Let us also mention Example 2.2 of [BG]; a supercompact, zero-dimensional space that admits a semilattice structure but has no binary subbase consisting of clopen sets.

A construction of profinite lattices.

Until further notice, $\mathbf{X} = \langle X; \wedge, \vee\rangle$ denotes a topological lattice and A a clopen subset of X. We want to manufacture a new lattice to be denoted by $s(\mathbf{X}, A)$. Its underlying space will be $Y = (X \setminus A) \cup (A \times 2)$ and the operations are defined according to the following clauses in which $p : Y \to X$ denotes the projection sending $x \in X \setminus A$ to itself and (a, ε) to a. By A_ε we abbreviate $A \times \{\varepsilon\}$, $\varepsilon = 0, 1$.

For $y, z \in Y$ we put $t = p(y) \wedge p(z)$ and define

$$(\underset{\sim}{\wedge}) \qquad y \underset{\sim}{\wedge} z = \begin{cases} t, & \text{if } t \notin A \\ (t,0), & \text{if } t \in A \text{ and } y \in A_0 \text{ or } z \in A_0 \\ (t,1), & \text{otherwise.} \end{cases}$$

Analogously, we put $u = p(y) \vee p(z)$ and define

$$(\underset{\sim}{\vee}) \qquad y \underset{\sim}{\vee} z = \begin{cases} u, & \text{if } u \notin A \\ (u,0), & \text{if } u \in A \text{ and } y \notin A_1,\ z \notin A_1 \\ (u,1), & \text{otherwise.} \end{cases}$$

Having continuous restrictions to each part of a clopen decomposition of $Y \times Y$, both operations are continuous. So, $s(\mathbf{X}, A) = \langle Y; \underset{\sim}{\wedge}, \underset{\sim}{\vee} \rangle$ is a topological algebra.

Lemma 1. *If A is convex, i.e. $a \leq x \leq b$ and $a, b \in A$ imply $x \in A$, then $s(\mathbf{X}, A)$ is a lattice and $p : s(\mathbf{X}, A) \to \mathbf{X}$ is a continuous lattice homomorphism.*

PROOF: It follows immediately from the definitions of $\underset{\sim}{\wedge}$ and $\underset{\sim}{\vee}$ and the corresponding properties of $\wedge$ and $\vee$ that both operations are commutative and idempotent. It is equally easy to see that p is a continuous homomorphism.

To prove the associativity of $\underset{\sim}{\wedge}$ we consider three elements u, v, w of Y and denote $p(u) \wedge p(v) \wedge p(w)$ by t.

$$\text{CLAIM:} \qquad (u \underset{\sim}{\wedge} v) \underset{\sim}{\wedge} w = \begin{cases} t, & \text{if } t \notin A \\ (t,0), & \text{if } t \in A \text{ and } \{u, v, w\} \cap A_0 \neq \emptyset \\ (t,1), & \text{otherwise.} \end{cases}$$

To check this we denote $(u \underset{\sim}{\wedge} v) \underset{\sim}{\wedge} w$ by $\underset{\sim}{t}$ and notice that

$$(*) \qquad t = p(u \underset{\sim}{\wedge} v) \ \wedge \ p(w).$$

If $t \notin A$, then $\underset{\sim}{t} = t$ follows immediately from $(*)$ and $(\underset{\sim}{\wedge})$.

Assume that $t \in A$ and $u \in A_0$. From $t \leq p(u) \wedge p(v) \leq p(u)$ and $t, p(u) \in A$ we get $p(u) \wedge p(v) \in A$, by convexity. Consequently $u \underset{\sim}{\wedge} v \in A_0$, by $(\underset{\sim}{\wedge})$ and $\underset{\sim}{t} = (t, 0)$, by $(*)$ and $(\underset{\sim}{\wedge})$.

If $t \in A$ and $v \in A_0$, one argues in the same way.

If $t \in A$ and $w \in A_0$, then $\underset{\sim}{t} = (t, 0)$ follows immediately from $(*)$ and $(\underset{\sim}{\wedge})$.

In the "otherwise" case, $t \in A$ and none of u, v, w belongs to A_0. By the definition of $\underset{\sim}{\wedge}$, $u \underset{\sim}{\wedge} v$ does not belong to A_0 either. Hence $\underset{\sim}{t} = (t, 1)$, again by $(*)$ and $(\underset{\sim}{\wedge})$.

The claim is proved. It shows that $(u \underset{\sim}{\wedge} v) \underset{\sim}{\wedge} w$ is a symmetric function of all three arguments. Together with commutativity this gives $(u \underset{\sim}{\wedge} v) \underset{\sim}{\wedge} w = (v \underset{\sim}{\wedge} w) \underset{\sim}{\wedge} u = u \underset{\sim}{\wedge} (v \underset{\sim}{\wedge} w)$, the desired associativity.

The associativity of $\underset{\sim}{\vee}$ is proved in the same way.

The absorption laws $u \underset{\sim}{\wedge} (v \underset{\sim}{\vee} u) = u$ and $u \underset{\sim}{\vee} (v \underset{\sim}{\wedge} u) = u$ are even easier. Their proofs do not require the convexity of A. $\square$

Lemma 2. *Let $\varphi : \mathbf{X} \to \mathbf{L}$ be a continuous homomorphism into a finite lattice $\mathbf{L} = \langle L; \wedge, \vee\rangle$. Assume that $M \subseteq L$ is convex. If $\mathbf{X}$ is profinite, then $s(\mathbf{X}, \varphi^{-1}(M))$ is also profinite.*

Proof: Denote $\varphi^{-1}(M)$ by A and notice that A is clopen and convex. By Lemma 1, $s(\mathbf{X}, A)$ is a lattice, as is $s(\mathbf{L}, M)$. Denote by ψ the mapping $s(\mathbf{X}, A) \to s(\mathbf{L}, M)$ sending $x \in X \setminus A$ to $\varphi(x)$ and (a, ε) to $(\varphi(a), \varepsilon)$. A routine verification shows that ψ is a continuous lattice homomorphism. It separates any pair of the form $(a, 0)$, $(a, 1)$. If the pair $u \neq v$ is not of that form, then $\xi(p(u)) \neq \xi(p(v))$ for some homomorphism $\xi : \mathbf{X} \to \mathbf{K}$. It follows that $\xi \circ p : s(\mathbf{X}, A) \to \mathbf{K}$ separates u and v. □

The proof of our third lemma relies on the following well-known folklore.

Fact. If $\mathcal{A} \subseteq \mathrm{Clop}(Z)$ separates the points of Z, then $\mathcal{A}$ generates the Boolean algebra $\mathrm{Clop}(Z)$. (As before, Z is compact and zero-dimensional.)

Lemma 3. *If $\mathbf{Z} = \langle Z; \wedge, \vee\rangle$ is a profinite lattice, then for each $C \in \mathrm{Clop}(Z)$ there exist a continuous homomorphism $\varphi : Z \to \mathbf{L}$ into a finite lattice $\mathbf{L}$ such that $C = \varphi^{-1}(M)$ for a suitable subset $M \subseteq L$.*

Proof: Denote the collection of all $\varphi^{-1}(M)$ by $\mathcal{B}$. Clearly, $\mathcal{B}$ consists of clopen sets and, by profiniteness, separates the points of Z. So $\mathcal{B}$ generates $\mathrm{Clop}(Z)$. On the other hand, the equations

$$Z \setminus \varphi^{-1}(M) = \varphi^{-1}(L \setminus M),$$

and

$$\varphi_1^{-1}(M_1) \cap \varphi_2^{-1}(M_2) = (\varphi_1 \times \varphi_2)^{-1}(M_1 \times M_2)$$

show that $\mathcal{B}$ is a subalgebra of $\mathrm{Clop}(Z)$. □

Now we can sum up and prove

Proposition 1. *Let $\mathbf{Z}$ be a profinite lattice and $C \in \mathrm{Clop}(Z)$. Put $Y = (Z \setminus C) \cup (C \times 2)$ and let $r : Y \to Z$ be the projection. Then r is a lattice homomorphism with respect to some profinite lattice structure on Y.*

Proof: Use Lemma 3 to choose $\varphi : \mathbf{Z} \to \mathbf{L}$ and $M \subseteq L$ such that $C = \varphi^{-1}(M)$. If M happens to be convex, then $s(\mathbf{Z}, C)$ does the job, by Lemmas 1 and 2. The general case is reduced to the convex one by induction on $|M|$. If $|M| = 1$, then M is convex. Assume that $M = N \cup \{m\}$ with $m \notin N$ and put $\varphi^{-1}(N) = B$ and $\varphi^{-1}(m) = A$. By induction hypothesis, there is a profinite lattice structure $\mathbf{X}$ on $X = (Z \setminus B) \cup (B \times 2)$ such that the projection $q : X \to Z$ is a homomorphism.

As $A = (\varphi \circ q)^{-1}(m)$ is convex, $s(\mathbf{X}, A)$ is a profinite lattice with underlying space Y such that the projection $p : Y \to X$ is a lattice homomorphism. Clearly, $r = q \circ p$ is also a lattice homomorphism $s(\mathbf{X}, A) \to \mathbf{Z}$, and we are done. □

The results.

Theorem. *Every zero-dimensional Dugundji space admits a profinite lattice structure.*

Proof: According to Haydon [Ha] we can represent the space X in question as the inverse limit

$$X = \varprojlim (X_\alpha, p_\alpha^\beta)_{\alpha \leq \beta < \tau}$$

of a well-ordered inverse system, indexed by the ordinals less than some τ such that

(a) $|X_0| = 1$,

(b) $X_\gamma = \varprojlim (X_\alpha, p_\alpha^\beta)_{\alpha \leq \beta < \gamma}$, if γ is a limit ordinal,

(c) for all $\beta < \tau$, $p_\beta^{\beta+1}$ is an open mapping with a metrizable kernel.

From Koppelberg's analysis of projective Boolean algebras (cf. Theorem 2.7 in [K]), we know that in the zero-dimensional case condition (c) can be replaced by the stronger

(c^+) for all $\beta < \tau$ there is a clopen subset $A_\beta \subseteq X_\beta$ such that $X_{\beta+1} = (X_\beta \setminus A_\beta) \cup (A_\beta \times 2)$ and $p_\beta^{\beta+1}$ is the projection sending $x \notin A_\beta$ to itself and $(a, \varepsilon) \in A_\beta \times 2$ to a.

By induction on $\alpha < \tau$, we turn the spaces X_α into profinite lattices $\mathbf{X}_\alpha = \langle X_\alpha; \wedge_\alpha, \vee_\alpha \rangle$ in such a way that $(\mathbf{X}_\alpha, p_\alpha^\beta)_{\alpha \leq \beta < \tau}$ becomes an inverse system of profinite lattices. Its inverse limit will be a profinite lattice with underlying space X. As X_0 has only one point, there is no choice for $\wedge_0$ and $\vee_0$. For limit ordinals γ we take $\mathbf{X}_\gamma = \varprojlim (\mathbf{X}_\alpha, p_\alpha^\beta)_{\alpha \leq \beta < \gamma}$. As any pair of points of X_γ is separated by some p_α^γ, this procedure preserves profiniteness. The step from β to $\beta + 1$ is done according to Proposition 1. □

Remark. Let $\mathbf{L}$ be the four-element Boolean lattice with underlying set $\{0, a, -a, 1\}$. Then $s(\mathbf{L}, \{a\})$ is a pentagon. This shows that our construction does not yield modular, let alone distributive lattices. The author was not able to decide whether Dugundji spaces admit distributive lattice structures.

Corollary 1. *Zero-dimensional Dugundji spaces are supercompact.*

Before we prove that we recall the definitions, for the sake of completeness.

A collection of subsets of some set is called *linked* if any two of its members intersect. A collection is called *binary* if each linked subcollection has non-empty intersection.

A family $\mathscr{S}$ of closed subsets of a space X is called a *closed subbase* provided that each closed subset of X is an intersection of finite unions of sets in $\mathscr{S}$. X is called *supercompact* if it possesses a binary closed subbase.

The corollary follows immediately from the theorem and

Proposition 2. *The underlying space of a profinite lattice has a binary closed subbase that consists of clopen sets.*

PROOF: Let $\mathbf{X} = \langle X; \wedge, \vee\rangle$ be a profinite lattice. Denote by $\mathscr{S}$ the collection of all $\varphi^{-1}(m)$, where φ runs through all continuous homomorphisms of $\mathbf{X}$ into finite lattices $\mathbf{L}$ and $m \in L$. Then $\mathscr{S} \subseteq \mathrm{Clop}(X)$ and, by Lemma 3, $\mathscr{S}$ is a closed subbase of X. To see that $\mathscr{S}$ is binary, we have to show that $\bigcap \mathscr{T} \neq \emptyset$ for each linked subcollection $\mathscr{T} \subseteq \mathscr{S}$. By compactness, we can assume that $\mathscr{T}$ is finite, i.e.

$$\mathscr{T} = \{\varphi_1^{-1}(m_1), \dots, \varphi_n^{-1}(m_n)\}.$$

If $n \leq 2$, then $\bigcap \mathscr{T} \neq \emptyset$ is trivial. For $n \geq 3$ we find, by induction, elements x_1, x_2, x_3 such that $\varphi_i(x_j) = m_i$ for $j \leq 3$ and all $i \neq j$.

Using that the φ_i are lattice homomorphisms, it is then easy to check that $x = (x_1 \wedge x_2) \vee (x_1 \wedge x_3) \vee (x_2 \wedge x_3)$ satisfies $\varphi_i(x) = m_i$ for all $i = 1, \dots, n$, i.e. $x \in \bigcap \mathscr{T}$. $\square$

The second corollary of the theorem concerns the algebraic structure of projective Boolean algebras (cf. [K] for a survey on them). These are exactly the clopen algebras of Dugundji spaces.

To consider Boolean semigroup algebras (see [CP] and also [He] for the Boolean case), one has to conceive Boolean algebras as (linear) algebras over the two-element field $\mathbb{F}_2$, where the meet operation acts as multiplication and symmetric difference as addition.

Let again a profinite lattice $\langle X; \wedge, \vee\rangle$ be given and consider the set $\Sigma \subseteq \mathrm{Clop}(X)$ consisting of all non-empty clopen upper sets (i.e. $x \in S$ and $x \leq y$ imply $y \in S$) that are closed under $\wedge$. Notice that each $S \in \Sigma$ has a least element. Indeed, by Zorn's Lemma and compactness, S contains minimal elements, but being closed under $\wedge$, S can have only one of them. The same argument yields a top element for X, which, obviously, belongs to all $S \in \Sigma$.

It is easy to see that Σ is closed under set-theoretic intersection, i.e. multiplication in $\mathrm{Clop}(X)$.

Moreover, Σ separates the points of X. Indeed, for $x \neq y$ we find a continuous homomorphism $\varphi : \mathbf{X} \to \mathbf{L}$ into a finite lattice $\mathbf{L}$ such that $\varphi(x) \neq \varphi(y)$. Without loss, $\varphi(y)$ does not belong to $M = \{m \in L : m \geq \varphi(x)\}$. So, $\varphi^{-1}(M) \in \Sigma$ contains x but not y. It now follows from the fact quoted above that Σ generates $\mathrm{Clop}(X)$ as a Boolean algebra.

To see that Σ is linearly independent over $\mathbb{F}_2$, we use the test from [He]. Assuming that $S, S_1, S_2, \dots S_n \in \Sigma$ satisfy $S \subseteq S_1 \cup S_2 \cup \dots \cup S_n$, we have to find $i \leq n$ such that $S \subseteq S_i$. But this is easy, for the least element of S must belong to some S_i.

Summing up the above discussion, we can conclude

Corollary 2. *Clopen algebras of profinite lattices and, in particular, projective Boolean algebras are semigroup algebras.*

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(*Received* February 4, 1992)