Continuous actions of pseudocompact groups and axioms of topological group

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Abstract. In this paper, we show that it is possible to extend the Ellis theorem, establishing the relations between axioms of a topological group on a new class $\mathcal{N}$ of spaces containing all countably compact spaces in the case of Abelian group structure. We extend statements of the Ellis theorem concerning separate and joint continuity of group inverse on the class of spaces $\mathcal{N}$ that gives some new examples and statements for the C_p-theory and theory of topologically homogeneous spaces.

Keywords: m-topological group, semitopological group, paratopological group, topological group, topology of pointwise convergence, Eberlein compact, weak functional tightness

Classification: 22A05, 54B15, 54C35

1. Introduction.

The paper is a short English version of the paper [11] (which can be regarded as a preprint) and contains an essential part of the author's Ph.D. Thesis (Dissertation) (see [12]).

Many generalizations of the Ellis theorem [8, Theorem 2, p. 124] have been achieved since this theorem was proved in 1957 (see [7]). Here we present some new generalizations of the Ellis theorem which arose as a development of C_p-theory. Some theorems and notions, considered in C_p-theory, are used in the proofs of our generalizations, which, on the other hand, give examples for the C_p-theory and for theory of homogeneous spaces (see Corollary 2 and 3, Examples 1, 2 and 3 of the paper).

All spaces are assumed to be Tychonoff spaces. For every space X we denote the set of real-valued continuous functions on X and the space of continuous functions on X in the topology of pointwise convergence by $C(X)$ and $C_p(X)$ respectively.

Cardinality of a set A and the first infinite cardinal number are denoted by $|A|$ and ω.

A closure of a set A in a space X is denoted by $[A]_X$.

For every group G and for each element $g \in G$ we denote the multiplication, the inverse of G, the left and the right translations on the element g by $\mathfrak{M}(G)$, $\mathfrak{J}(G)$, l_g and r_g.

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A mapping $d : G \times X \to X$, where X is a space and G is a group, is called the action of the group G on the space X, iff d is an action of G on the set X and for every element $g \in G$ the mapping

$$d(g;\cdot) : X \to X : x \mapsto d(g;x)$$

is continuous. The action d is called transitive, iff $d(G \times \{x\}) = X$ for every $x \in X$.

A mapping $F : X \times Y \to Z$, where X, Y and Z are spaces, is separately continuous in a point $(\hat{x};\hat{y}) \in X \times Y$, iff the mappings

$$F(\hat{x};\cdot) : Y \to Z : y \mapsto F(\hat{x};y) \quad \text{and} \quad F(\cdot;\hat{y}) : X \to Z : x \mapsto F(x;\hat{y})$$

are continuous in the points $\hat{y}$ and $\hat{x}$ respectively. The separate continuity of the mapping F means that F is separately continuous in any point $(\hat{x};\hat{y}) \in X \times Y$. The mapping F is jointly continuous (in a point $(\hat{x};\hat{y}) \in X \times Y$) iff F is continuous mapping (in $(\hat{x};\hat{y})$) with respect to the topologies of the product $X \times Y$ and the space Z.

A pair $(X;\cdot)$ (where X is a space and $\cdot$ is a group structure on the set X) is called:

(a) a semitopological group, iff the multiplication $\mathfrak{M}(X;\cdot)$ is separately continuous and the inverse $\mathfrak{I}(X;\cdot)$ is continuous;
(b) a paratopological group, iff the multiplication $\mathfrak{M}(X;\cdot)$ is a jointly continuous mapping;
(c) a topological group, iff $(X;\cdot)$ is both a semitopological and a paratopological group (see [5]);
(d) an m-topological group, iff the multiplication $\mathfrak{M}(X;\cdot)$ is separately continuous.

We need the following classes of spaces:

$C = \{X : X$ is a countably compact space$\}$;
$\mathcal{P} = \{X : X$ is a pseudocompact space$\}$;
$\mathcal{P}_s = \{X : X$ is a separable pseudocompact space$\}$;
$\mathcal{P}_\omega = \{X : X^\omega \in \mathcal{P}\}$;
$\mathcal{Y} = \{X : X \in \mathcal{P} : [Y]_{C_p(X)}$ is compact for any pseudocompact subspace Y of the space $C_p(X)\}$;
$\mathcal{N} = \{X : [Y]_{C_p(X)}$ is compact if the subspace Y of the space $C_p(X)$ is a continuous image of the space $X\}$;
$C\mathcal{P} = \{X : X$ is a countably pracompact space$\}$;
$\mathcal{P}_{t_m} = \{X : X \in \mathcal{P} : t_m(X) \leq \omega\}$

(for definitions of weak functional tightness t_m and countable pracompactness, see [1] and [2]).

Well-known relations between the classes of spaces introduced above are formulated in

Proposition 1 (see [1]).

(a) $C \subset C\mathcal{P} \subset \mathcal{N}$;
(b) $\mathcal{P}_s \subset \mathcal{P}_{t_m} \subset \mathcal{Y}$;
(c) $\mathcal{Y} \subset \mathcal{N}$;
(d) $\mathcal{P}_\omega \setminus \mathcal{Y} \neq \emptyset$;
(e) $\mathcal{N} \subset \mathcal{P}$.

Proposition 1 describes the sphere of applications of Theorems 1, 2 and 3 and Corollaries 1 and 2 of the paper.

In the proofs of Theorem 1 of the paper we use the following two statements:

Proposition 2 (see [1, Theorem IV.5.3, p. 172]; [15]). *Let Y be a dense pseudocompact subspace of an Eberlein compact Z. Then $Y = Z$.*

Proposition 3. *Let $d : G \times X \to X$ be an action of a group G on a space X. Suppose a topology τ determined on G is such that all right translations are continuous with respect to τ. Let $f : X \to Y$ be a mapping from X in a space Y, let $D_f = f \circ d$.*

Then if there exists such an element $g_0 \in G$ that the mapping D_f is jointly continuous in every point of the set $\{g_0\} \times X$, then D_f is jointly continuous in every point of the space $(G;\tau) \times X$.

Proof: Let us note that for any element $g \in G$, the mapping

$$r_g \times d(g^{-1};\cdot) : (G;\tau) \times X \to (G;\tau) \times X : (h;x) \mapsto (h \cdot g; d(g^{-1};x))$$

is an autohomeomorphism of the space $(G;\tau) \times X$. Moreover,

$$D_f \circ (r_g \times d(g^{-1};\cdot))(h;x) = f(d(h \cdot g; d(g^{-1};x))) = f(d(h;x)) = D_f(h;x).$$

Thus, for any element $g \in G$, the mapping D_f is jointly continuous in every point of the set $\{g_0 \cdot g^{-1}\} \times X$, which completes the proof. □

We also need the following simple statement in order to prove Theorem 3 and Corollary 2 of the paper:

Proposition 4. *Let $\delta : H \times Y \to Y$ be a transitive action of an Abelian group $(H;\cdot)$ on a space Y, let $y_0 \in Y$. Then the operation $+$*

$$\delta(h_1;y_0) + \delta(h_2;y_0) = \delta(h_1 \cdot h_2;y_0)$$

(where $h_1 \in H$; $h_2 \in H$) is such a group operation on Y that the pair $(Y;+)$ is an m-topological Abelian group and the mapping

$$\varphi : G \to (Y;+) : g \mapsto \delta(g;y_0) \tag{0}$$

is an epimorphism of the groups (without the topology on Y).

2. Separate and joint continuity of actions on the classes $\mathcal{N}$ and $\mathcal{Y}$ spaces.

Theorem 1. *Let $d : C \times X \to X$ be a transitive action of a group G on a space X. Suppose a topology τ determined on a group G is such that $(G;\tau) \in \mathcal{N}$, all right translations of a group G are continuous with respect to τ and the mapping d is separately continuous with respect to τ and the topology of the space X. Then d is a jointly continuous mapping.*

Proof: Since $X \in T_{3\frac{1}{2}}$, in order to prove the joint continuity of the mapping d, it is enough to show that for every function $f \in C(X)$ the mapping $D_f = f \circ d$ is jointly continuous.

Let us choose a function $f \in C(X)$. For every point $x \in X$, we determine the function

$$\varphi_f(x) : G \to \mathbb{R} : g \mapsto f(d(g;x)) \tag{1}$$

which is continuous as the composition of the continuous mappings f and $d(\cdot;x) : G \to X : g \mapsto d(g;x)$.

Then the mapping

$$\varphi_f : X \to C_p(G) : x \mapsto \varphi_f(x) \tag{2}$$

is continuous, because pre-images of the elements of the natural subbase of the topology of the space $\varphi_f(X)$ are open in X (for any element $g \in G$ and for each open subset O of the real line, the set $\{x : x \in X : \varphi_f(x)(g) \in O\} = \{x : x \in X : d(g;x) \subset f^{-1}(O)\} = (d(g;\cdot))^{-1}(f^{-1}(O))$ is open that follows from the continuity of the mappings $d(g;\cdot)$ and f).

We consider:
the subspace $K = [\varphi_f(X)]_{C_p(G)}$ of the space $C_p(G)$,
the "reflexion" mapping (see [1])
$\psi_f : G \to C_p(K)$; $\psi_f(g)(k) = k(g)$,
the subspace $C = [\psi_f(G)]_{C_p(K)}$ of the space $C_p(K)$.

Since $K \subset C_p(G)$, the mapping ψ_f is continuous.

Let us show that

$$K \text{ and } C \text{ are Eberlein compacts} \tag{3}$$

and

$$K = \varphi_f(X);\ C = \psi_f(G). \tag{4}$$

Since the action d is transitive, the spaces X and $\varphi_f(X)$ are continuous images of the space $(G;\tau)$. Using this condition and the condition $(G;\tau) \in \mathcal{N}$ we get the compactness of the space K.

The continuity of the mapping ψ_f and the pseudocompactness of the space $(G;\tau)$ (see the condition (e) of Proposition 1) imply the pseudocompactness of the space $\psi_f(G)$.

The compactness of the space C follows now immediately from the Asanov–Veličko generalization of the Grothendieck theorem (see [1, Theorem III.4.1, p. 110]; [4]).

The space C is an Eberlein compact as a compact subspace of the space $C_p(K)$ (where K is a compact) (see [1, p. 115]; [3, Definition 2.2, p. 17]). On the other hand, since the set of functions C separates all points of the compact K, K is an Eberlein compact, too (see [1, Proposition IV.1.6., p. 144]; [3, Proposition 2.1, p. 16]).

The condition (4) follows directly from the condition (3) and Proposition 2.

Let us consider the mapping

$$F : C \times K : (c; k) \mapsto c(k).$$

Using the Namioka theorem (see [1, Theorem III.5.5, p. 132]; [14, Theorem 1.2, p. 517]) we can choose such an element $g_0 \in G$ that the mapping F is jointly continuous in any point of the set $\psi_f(g_0) \times K$. Taking into consideration the fact that $D_f(g; x) = F(\psi_f(g); \varphi_f(x))$ for any element $g \in G$ and for each point $x \in X$, we can infer the joint continuity of the mapping D_f in every point of the set $\{g_0\} \times X$ from the continuity of the mappings φ_f and ψ_f and the condition (4).

Proposition 3 now implies the (joint) continuity of the function D_f (and, hence, the continuity of the mapping d). □

Theorem 2. *Let $d : G \times X \to X$ be an action of a group G on a space X. Let $X \in \mathcal{Y}$, and τ be such a topology on G that $(G; \tau) \in \mathcal{P}$, all right translations of the group G are continuous with respect to τ and the action d is separately continuous with respect to τ and the topology of the space X. Then the mapping d is jointly continuous.*

Theorem 1 immediately implies the following statement:

Corollary 1. *Let $(X; \cdot)$ be an m-topological group, let $X \in \mathcal{N}$. Then $(X; \cdot)$ is a paratopological group.*

In the case of Abelian group structure it is possible to prove a stronger statement than Corollary 1:

Theorem 3. *Let $X \in \mathcal{N}$, $(X, \cdot)$ be an Abelian m-topological group. Then $(X; \cdot)$ is a topological group.*

Proof: The joint continuity of the multiplication of $(X; \cdot)$ is proved in Corollary 1, where the multiplication $\mathfrak{M}(X; \cdot)$ can be considered as the action $\bar{d}$ of the group $(X; \cdot)$ on itself by left translations ($\bar{d} = \mathfrak{M}(X; \cdot) : (x; y) \mapsto x \cdot y$). After that, Theorem 1 can be applied to the action $\bar{d}$.

Conditions (1) and (2) have the following meaning in this case:

$$(1') \qquad \varphi_f(x) : X \to \mathbb{R} : y \mapsto f(y \cdot x)$$

and

$$(2') \qquad \varphi_f : X \to C_p(X) : x \mapsto \varphi_f(x).$$

In order to prove the continuity of the inverse $\mathfrak{J}(X;\cdot)$ it is enough to show the continuity of the map $\mathfrak{J}_f = \varphi_f \circ \mathfrak{J}$ for every function $f \in C_p(X)$ and to take into consideration the fact that the family of the mappings $\{\varphi_f : f \in C_p(X)\}$ determines the topology of the space X (we have $\varphi_f(e)(x) = f(x)$ for every element $x \in X$).

Let us choose a function $f \in C_p(X)$, consider the compact K determined in the proof of Theorem 1 and the mapping

$$d : (X;\cdot) \times K \to K : (z;\varphi_f(x)) \mapsto \varphi_f(z \cdot x)$$

which is a transitive action of the group $(X;\cdot)$ (without the topology of X) on the space K.

Applying Proposition 4 to the action d and to the point $f \in K$, we get the Abelian compact m-topological group $(K;+)$ which according to the Ellis theorem (see [8, Theorem 2, p. 124]) is a topological group. Hence, the inverse $\mathfrak{J}(K;+)$ is continuous.

Let us note that the epimorphism of groups $\varphi : (X;\cdot) \to (K;+)$ (without the topologies of X and K) determined by Proposition 4 and the continuous mapping φ_f coincide. (It follows from the conditions $(1')$, $(2')$ and (0).)

Thus, we have:

$$\mathfrak{J}_f = \varphi_f \circ \mathfrak{J}(X;\cdot) = \varphi \circ \mathfrak{J}(X;\cdot) = \mathfrak{J}(K;+) \circ \varphi = \mathfrak{J}(K;+) \circ \varphi_f;$$

this shows the continuity of the mapping $\mathfrak{J}_f$ as the composition of the continuous mappings φ_f and $\mathfrak{J}(K;+)$. □

Theorem 3, the Ellis theorem and Proposition 4 imply some corollaries.

Corollary 2. *Let X be a locally compact space or $X \in \mathcal{N}$. Then the following conditions are equivalent:*

(a) *X is a space of some Abelian topological group;*
(b) *there exists a transitive action of some Abelian group on the space X.*

Corollary 2 and the Ivanovskiĭ–Kuz'minov theorem establishing dyadic compactness of any compact topological group (see [10], [13]) also imply the following statement:

Corollary 3. *Let X be a non-dyadic compact. Then any Abelian group cannot act transitively on the space X.*

Corollaries 2 and 3 give some examples of such spaces X that

> the space X is topologically homogeneous (i.e. there exists a group which acts transitively on the space X)

and

> any Abelian group cannot act transitively on the space X.

Example 1. "Double arrow space" (see [9]).

Example 2. The Alexandroff "long line" (see [9]).

3. Transitive actions of Abelian groups on pseudocompact spaces.

Corollary 2 and the following theorem illustrate differences between the classes $\mathcal{N}$ and $\mathcal{P}_\omega$.

Theorem 4. *For every space $X \in \mathcal{P}_\omega$, there exists such a continuous pre-image Y of the space X that $Y \in \mathcal{P}_\omega$ and some Abelian group G which acts transitively on the space Y.*

Sketch of the proof: Let us choose such an Abelian group G that $|G^\omega| = |G| \geq |X| \cdot \omega$. Let $\tau = |G|$.

We need the family $\mathcal{A} = \{A : A \subset G : |A| \subset \omega\}$. For any set $A \in \mathcal{A}$, we define the set of all mappings from A in X by means of $\mathcal{F}(A)$ and considering the set $\mathcal{F} = \bigcup\{\mathcal{F}(A) : A \in \mathcal{A}\}$. For every mapping $f \in \mathcal{F}$ we pick the set $A(f)$ satisfying the condition $f \in \mathcal{F}(A(f))$.

Since $\tau^\omega = \tau$, we have $|\mathcal{A}| = |\mathcal{F}| = \tau$. Let us choose a one-to-one numeration $\{f_\alpha : \alpha \in \tau\}$ of the set $\mathcal{F}$.

By means of transfinite induction we shall pick such a one-to-one numerated subset $\{g_\alpha : \alpha \in \tau\}$ of the group G that

$$\text{the family } \{g_\alpha \cdot A(f_\alpha) : \alpha \in \tau\} \text{ is disjoint.} \tag{5}$$

Suppose that for some ordinal $\beta \in \tau$ we chose such a subset $\{g_\alpha : \alpha < \beta\}$ of the group G that the family $\{g_\alpha \cdot A(f_\alpha) : \alpha < \beta\}$ is disjoint.

Let us find an element g_β in the set $G \setminus H$, where H is the minimal subgroup of the group G, containing the set

$$\{A(f_\alpha) : \alpha \leq \beta\} \cup \{g_\alpha : \alpha < \beta\}.$$

(Since $|H| \leq |\beta| \cdot \omega < \tau = |G|$, we have $G \setminus H \neq \emptyset$.)

It is not difficult to prove that the family

$$\{g_\alpha \cdot A(f_\alpha) : \alpha \leq \beta\}$$

is disjoint.

Continuing the inductive process we shall get the set $\{g_\alpha : \alpha \in \tau\}$ satisfying the condition (5)

Using the condition (5) we are able to pick the mapping $\hat{f} \in X^G$ satisfying the following condition:

$$\hat{f} \mid g_\alpha \cdot A(f_\alpha) = f_\alpha \circ l g_\alpha^{-1} \mid g_\alpha \cdot A(f_\alpha) \quad \text{for any} \quad \alpha \in \tau. \tag{6}$$

It is easy to show that the subspace

$$Y = \{\hat{f} \circ l_g : g \in G\}$$

of the space X^G is a continuous pre-image of the space X.

The mapping $d : G \times Y \to Y : (g; y) \mapsto y \circ l_g$ is a transitive action of G on the space Y.

In order to prove that $Y^\omega \in \mathcal{P}$, let us consider the space Y^ω as the subspace of the space $X^{G\times\omega}$. Using the condition (6) it is possible to show that for any set $A \in \mathcal{A}$, the projection $\pi_{A\times\omega} : X^{G\times\omega} \to X^{A\times\omega}$ maps the set Y^ω onto the space $X^{A\times\omega}$. Then the condition $Y \in \mathcal{P}_\omega$ follows from the fact that $X^\omega \in \mathcal{P}$. $\square$

Theorem 4 shows the impossibility to extend Theorem 3 on the class $\mathcal{P}_\omega$ and gives

Example 3. There exists a space $Y \in \mathcal{P}_\omega \setminus \mathcal{N}$.

In order to find the space $Y \in \mathcal{P}_\omega \setminus \mathcal{N}$, let us choose for some non-dyadic compact X a space Y as in Theorem 4. Using Proposition 4 we can pick such an Abelian group structure $+$ on the set Y that $(Y; +)$ is an m-topological group.

If $Y \in \mathcal{N}$, then $(Y; +)$ is a topological group which follows from Theorem 3. The Comfort–Ross theorem (see [6, Theorem 4.1, p. 494]) implies then that the Čech–Stone compactification βY of the space Y is a space of a topological group. According to the Ivanovskĭ–Kuz'minov theorem (see [10]; [13]), the compact βY is dyadic. Moreover, the compact X is also dyadic as a continuous image of βY that contradicts the choice of the compact X.

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