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Properties of the solution of evolution inclusions driven by time dependent subdifferentials

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Abstract. In this paper we consider evolution inclusions driven by a time-dependent subdifferential. First we prove a relaxation result and then we use it to show that if the solution set is closed in a space of continuous functions, then the orientor field is almost everywhere convex valued.

Keywords: subdifferential, monotonicity, relaxation, continuous selection, lower semicontinuous multifunction

Classification: 34G20

1. Introduction.

It is well known from functional analysis that convexity and the weak topology go together. Just recall the well known "Mazur's theorem" which says that a convex set in a Banach space is closed if and only if it is weakly closed. Also from nonlinear analysis we know that if the integral functional $x \to I_f(x) = \int_0^r f(t, x(t))\, dt$ is weakly lower semicontinuous on the Lebesgue-Bochner space $L^1(X)$, then for almost all $t \in T = [0, r]$, $f(t, \cdot)$ is convex (see Papageorgiou [8, Theorem 5.2]). Furthermore from set valued analysis we know that if $F : T \to 2^X \setminus \{\emptyset\}$ is a graph measurable multifunction with a nonempty set of integrable selectors which is w-closed in $L^1(X)$, then for almost all $t \in T$, $F(t)$ is convex (see [8, Theorem 4.1]). Recently Cellina-Ornelas [5] established the same kind of compatibility between convexity and the weak topology in the context of differential inclusions in $\mathbb{R}^n$. Namely they proved that if the solution set of such a multivalued Cauchy problem is closed in $AC(T, \mathbb{R}^n)_w$ (i.e. the space of absolutely continuous functions from T into $\mathbb{R}^n$ equipped with the weak topology), then for almost all $t \in T$, the orientor field $F(t, \cdot)$ is convex valued. Recall that the topology of uniform convergence on T, which $AC(T, \mathbb{R}^n)$ inherits from the space of all continuous functions on T with values in $\mathbb{R}^n$, is weaker than the weak topology.

The purpose of this note is to extend the above mentioned result of Cellina-Ornelas [5] to a much broader class of multivalued Cauchy problems, namely to evolution inclusions driven by time dependent subdifferentials and also use on the solution set, the $C(T, H)$-topology. Systems of this form model various classes of nonlinear partial differential equations and appear in several applied areas such as mechanics (study of problems with unilateral constraints), mathematical economics (variational differential inequalities) and free boundary problems. For further details on those applications we refer to the books by Aubin-Cellina [2] and Barbu [3].

To obtain our extension of the Cellina-Ornelas [5] result we first prove a new relaxation theorem for subdifferential evolution inclusions which is of independent interest and extends an earlier such result established by the author [10, Theorem 5.1].

2. Preliminaries.

Let $T = [0, r]$ and H be a separable Hilbert space. Throughout this paper, by $P_{f(c)}(H)$ we will denote the family of nonempty, closed (and convex) subsets of H. Recall that a multifunction $F : T \to P_f(H)$ is said to be measurable, if for all $z \in H$, $t \to d(z, F(t)) = \inf\{\|z - x\| : x \in F(t)\}$ is measurable. By S_F^p $(1 \le p \le \infty)$ we will denote the set of selectors of $F(\cdot)$ that belong to the Lebesgue-Bochner space $L^p(H)$; i.e. $S_F^p = \{f \in L^p(H) : f(t) \in F(t) \text{ a.e.}\}$. It is easy to check using Aumann's selection theorem (see Wagner [12, Theorem 5.10]) that for a measurable multifunction $F : \Omega \to P_f(H)$, the set S_F^p is nonempty if and only if $\omega \to \inf\{\|x\| : x \in F(\omega)\} \in L_+^p$.

Let $\phi : H \to \overline{\mathbb{R}} = \mathbb{R} \cup \{+\infty\}$. We say that the function $\phi(\cdot)$ is proper, if it is not identically $+\infty$. Assume that $\phi(\cdot)$ is proper, convex and l.s.c. (usually this family of $\overline{\mathbb{R}}$-valued functions is denoted by $\Gamma_0(H)$). The effective domain of $\phi(\cdot)$ is defined to be the set $\operatorname{dom}\phi = \{x \in H : \phi(x) < +\infty\}$. The subdifferential of $\phi(\cdot)$ at $x \in \operatorname{dom}\phi$ is defined to be the set $\partial\phi(x) = \{x^* \in H^* = H : (x^*, y - x) \le \phi(y) - \phi(x) \text{ for all } y \in \operatorname{dom}\phi\}$ (in this definition $(\cdot, \cdot)$ denotes the inner product of H). The subdifferential generalizes the concept of derivative to nondifferentiable convex functions and if $\phi(\cdot)$ is Gâteaux differentiable at $x \in H$, then $\partial\phi(x) = \{\nabla\phi(x)\}$. Also we say that $\phi(\cdot)$ is of compact type, if for every $\lambda \in \mathbb{R}$, the level set $\{x \in H : \phi(x) + \|x\|^2 \le \lambda\}$ is compact. For further details on those convex analytic concepts, we refer to Barbu [3].

A multifunction $G : H \to 2^H \setminus \{\emptyset\}$ is said to be lower semicontinuous (l.s.c.) if for every $U \subseteq H$ open, the set $G^-(U) = \{x \in H : G(x) \cap U \neq \emptyset\}$ is open. It turns out (see for example Klein-Thompson [7]) that this definition of lower semicontinuity is equivalent to saying that if $x_n \to x$, then $G(x_n) \subseteq s\text{-}\underline{\lim}\, G(x_n) = \{y \in H : \lim d(y, G(x_n)) = 0\} = \{y \in H : y = s\text{-}\lim y_n,\ y_n \in G(x_n),\ n \ge 1\}$. Here s-denotes the strong topology on H.

On $P_f(H)$ we can define a generalized metric known in the literature as Hausdorff metric, by setting

$$h(A, B) = \max[\sup_{a \in A} d(a, B), \sup_{b \in B} d(b, A)]$$

for all $A, B \in P_f(H)$. It is a well-known fact that $(P_f(H), h)$ is a complete metric space.

We will be studying the following evolution inclusion defined on $T \times H$:

$$(^*)_1 \qquad \left\{ \begin{array}{l} -\dot{x}(t) \in \partial\phi(t, x(t)) + F(t, x(t)) \text{ a.e.} \\ x(0) = x_0 \end{array} \right\}.$$

By a strong solution of $(^*)_1$, we mean a function $x(\cdot) \in C(T, H)$ s.t. $x(\cdot)$ is strongly absolutely continuous on $(0, b)$, $x(t) \in \operatorname{dom}\phi(t, \cdot)$ a.e. and satisfies

$-\dot{x}(t) \in \partial\phi(t, x(t)) + f(t)$ a.e., $x(0) = x_0$, with $f \in S^1_{F(\cdot,x(\cdot))}$. Recall that since H is a Hilbert space, an absolutely continuous function from T into H is almost everywhere strongly differentiable (see Diestel-Uhl [6]).

The following hypothesis concerning $\phi(t, x)$ will be in effect throughout the rest of the paper.

$\underline{H(\phi)}$: $\phi : T \times H \to \overline{\mathbb{R}} = \mathbb{R} \cup \{+\infty\}$ is a function s.t.

($\underline{1}$) for every $t \in T$, $\phi(t, \cdot)$ is proper, convex, l.s.c. (i.e. $\phi(t, \cdot) \in \Gamma_0(H)$) and of compact type,

($\underline{2}$) for every positive integer b, there exists a constant $K_b > 0$, an absolutely continuous function $g_b : T \to \mathbb{R}$ with $\dot{g}_b \in L^\beta(T)$ and a function of bounded variation $h_b : T \to \mathbb{R}$ s.t. if $t \in T$, $x \in \operatorname{dom}\phi(t, \cdot)$ with $\|x\| \le b$ and $s \in [t, r]$, then there exists $\hat{x} \in \operatorname{dom}\phi(s, \cdot)$ satisfying

$$\|\hat{x} - x\| \le |g_b(s) - g_b(t)|(\phi(t, x) + K_b)^\alpha$$

and $\varphi(s, \hat{x}) \le \phi(t, x) + |h_b(s) - h_b(t)|(\phi(t, x) + K_b)$, where $\alpha \in [0, 1]$, and $\beta = 2$ if $\alpha \in [0, 1/2]$ or $\beta = 1/1 - \alpha$ if $\alpha \in [1/2, 1]$.

This hypothesis on $\phi(t, x)$ was first introduced by Yotsutani [14] (hypothesis A) and is more general than the one used by Watanabe [13] in his pioneering work.

In [10], the author obtained existence theorems for $(^*)_1$ for both convex and nonconvex valued orientor field $F(t, x)$ (see Theorems 3.1 and 4.1 in [10]).

3. Relaxation theorem.

In addition to problem $(^*)_1$, we also consider the following convexified version of it:

$$(^*)_2 \qquad \left\{ \begin{array}{l} -\dot{x}(t) \in \partial\phi(t, x(t)) + \overline{conv}F(t, x(t)) \text{ a.e.} \\ x(0) = x_0 \end{array} \right\}.$$

We will denote the solution set of $(^*)_1$ by $P(x_0)$ and that of $(^*)_2$ by $P_c(x_0)$. In this section we prove that every trajectory in $P_c(x_0)$ can be approximated in the $C(T, H)$-norm, by a trajectory in $P(x_0)$. Such a result is known as "relaxation theorem", and can be useful in the study of infinite dimensional nonlinear control systems. We will need the following hypothesis on $F(t, x)$:

$\underline{H(F)}$: $F : T \times H \to P_f(H)$ is a multifunction s.t.

($\underline{1}$) $t \to F(t, x)$ is measurable,

($\underline{2}$) $h(F(t, x), F(t, y)) \le \theta(t)\|x - y\|$, with $\theta(\cdot) \in L^1_+$, $x, y \in H$,

($\underline{3}$) $|F(t, x)| = \sup\{\|y\| : y \in F(t, x)\} \le a(t) + b(t)\|x\|$ a.e. with $a, b \in L^2_+$.

Theorem 3.1. *$\underline{\text{If}}$ the hypotheses $H(\phi)$, $H(F)$ hold and $x_0 \in \operatorname{dom}\phi(0, \cdot)$, $\underline{\text{then}}$ $P_c(x_0) = \overline{P(x_0)}$, the closure taken in $C(T, H)$.*

Proof: Let $x(\cdot) \in P_c(x_0)$. Then by definition we have

$$\left\{ \begin{array}{l} -\dot{x}(t) \in \partial\phi(t, x(t)) + f(t) \text{ a.e.} \\ x(0) = x_0 \end{array} \right\}$$

with $f \in S^2_{F(\cdot,x(\cdot))}$. First we will establish an a priori bound for the elements in $P_c(x_0)$. So let $-\dot{u}(t) \in \partial\phi(t, u(t))$, $u(0) = x_0$. The existence of $u(\cdot) \in C(T, H)$ is guaranteed by the result of Yotsutani [14]. Exploiting the monotonicity of the subdifferential, we have:

$$(-\dot{x}(t) + \dot{u}(t), u(t) - x(t)) \le (f(t), u(t) - x(t)) \text{ a.e.},$$

where $(\cdot,\cdot)$ denotes the inner product of H. So we get:

$$\frac{1}{2}\frac{d}{dt}\|x(t) - u(t)\|^2 \le \|f(t)\| \cdot \|x(t) - u(t)\| \text{ a.e.}$$
$$\Longrightarrow \frac{1}{2}\|x(t) - u(t)\|^2 \le \int_0^t \|f(s)\| \cdot \|x(s) - u(s)\|\, ds.$$

Applying Lemma A.5, p. 157 of Brezis [4], we get

$$\|x(t) - u(t)\| \le \int_0^t \|f(s)\|\, ds \le \int_0^t (a(s) + b(s)\|x(s)\|)\, ds$$
$$\Longrightarrow \|x(t)\| \le \|u\|_{C(T,H)} + \int_0^t (a(s) + b(s)\|x(s)\|)\, ds.$$

Invoking Gronwall's inequality, we get that there exists $M_1 > 0$ s.t.

$$\|x(t)\| \le M_1$$

for all $t \in T$ and all $x(\cdot) \in P_c(x_0)$. Let $\psi(t) = a(t) + b(t)M_1$, $\psi(\cdot) \in L^2_+$ and set $V = \{h \in L^2(H) : \|h(t)\| \le \psi(t) \text{ a.e.}\}$. This set endowed with the relative weak $L^2(H)$-topology is compact metrizable. In what follows, this will be topology considered on V. Let $p : V \to C(T, H)$ be the solution map; i.e. for every $h \in V$, $p(h)(\cdot)$ is the unique solution of the Cauchy problem $-\dot{y}(t) \in \partial\phi(t, y(t)) + h(t)$ a.e., $y(0) = x_0$. From the proof of Theorem 3.1 in [10], we know that $p : V \to C(T, H)$ is continuous. So given $\varepsilon > 0$, we can find $\mathcal{U}$ a weak symmetric neighborhood of the origin in $L^2(H)$ s.t. $f - f_1 \in \mathcal{U} \cap V \Rightarrow \|x - z_1\|_{C(T,H)} < \varepsilon$, where $z_1 = p(f_1)$. From Proposition 4.1 of [8], we know that we can choose $f_1 \in S^2_{F(\cdot,x(\cdot))}$. An easy application of Aumann's selection theorem gives us $f_2 \in S^2_{F(\cdot,z_1(\cdot))}$ s.t.

$$\begin{aligned}\|f_1(t) - f_2(t)\| &= d(f_1(t), F(t, z_1(t))) \\ &\le h(F(t, x(t)), F(t, z_1(t))) \\ &\le \theta(t)\|x(t) - z_1(t)\| \le \theta(t)\varepsilon \text{ a.e.}\end{aligned}$$

Let $z_2 = p(f_2)$. Exploiting once again the monotonicity of the subdifferential operator, we have

$$(\dot{z}_2(t) - \dot{z}_1(t), z_2(t) - z_1(t)) \le (f_1(t) - f_2(t), z_2(t) - z_1(t)) \text{ a.e.}$$
$$\begin{aligned}\Longrightarrow \frac{1}{2}\|z_1(t) - z_2(t)\|^2 &\le \int_0^t \|f_1(s) - f_2(s)\| \cdot \|z_1(s) - z_2(s)\|\, ds \\ &\le \varepsilon \int_0^t \theta(s)\|z_1(s) - z_2(s)\|\, ds\end{aligned}$$
$$\Longrightarrow \|z_1(t) - z_2(t)\| \le \varepsilon \cdot \int_0^t \theta(s)\, ds,\ t \in T \text{ (see Brezis [4, Lemma A.5, p. 157])}.$$

Then we have

$$\|z_2(t)-x(t)\| \le \|z_2(t)-z_1(t)\|+\|z_1(t)-x(t)\| \le \varepsilon \int_0^t \theta(s)\,ds+\varepsilon = \varepsilon[\int_0^t \theta(s)\,ds+1].$$

Suppose we have picked $f_1,\dots,f_n \in L^2(H)$ s.t.

$$\text{(\underline{\underline{1}})} \qquad \|f_{k+1}(t)-f_k(t)\| \le \frac{\varepsilon\theta(t)}{(k-1)!}\Big[\int_0^t \theta(s)\,ds\Big]^{k-1}$$

and

$$\text{(\underline{\underline{2}})} \qquad f_{k+1}(t) \in F(t,z_k(t)) \text{ a.e., } z_k = p(f_k),\ k=1,2,\dots,n-1.$$

As before, through the monotonicity of the subdifferential, we get

(\underline{\underline{3}})

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|z_{k+1}(t)-z_k(t)\|^2 \le (f_k(t)-f_{k+1}(t), z_{k+1}(t)-z_k(t)) \text{ a.e.}\\
&\Longrightarrow \frac{1}{2}\|z_{k+1}(t)-z_k(t)\|^2 \le \int_0^t \|f_{k+1}(s)-f_k(s)\|\cdot\|z_{k+1}(s)-z_k(s)\|\,ds\\
&\qquad\qquad \le \int_0^t \frac{\varepsilon\theta(s)}{(k-1)!}\Big[\int_0^s \theta(\tau)\,d\tau\Big]^{k-1}\|z_{k+1}(s)-z_k(s)\|\,ds\\
&\Longrightarrow \|z_{k+1}(t)-z_k(t)\| \le \frac{\varepsilon}{(k-1)!}\int_0^t \theta(s)\Big[\int_0^s \theta(\tau)\,d\tau\Big]^{k-1}ds = \frac{\varepsilon}{k!}\Big[\int_0^t \theta(s)\,ds\Big]^k,\\
&\qquad\qquad\qquad k=1,2,\dots,n-1\\
&\Longrightarrow \|z_{k+1}(t)-x(t)\| \le \varepsilon\sum_{\ell=1}^{k+1}\frac{1}{\ell!}\Big[\int_0^t \theta(s)\,ds\Big]^\ell \le \varepsilon\exp\|\theta\|_1,\ k=1,2,\dots,n-1.
\end{aligned}$$

Then via Aumann's selection theorem we get $f_{n+1} \in S^2_{F(\cdot,x_n(\cdot))}$ s.t.

$$\begin{aligned}
\|f_{n+1}(t)-f_n(t)\| &\le h(F(t,z_n(t)),F(t,z_{n-1}(t)))\\
&\le \theta(t)\cdot\|z_n(t)-z_{n+1}(t)\|\\
&\le \frac{\varepsilon\theta(t)}{(n-1)!}\Big[\int_0^t \theta(s)\,ds\Big]^{n-1} \text{ a.e.}
\end{aligned}$$

So by induction we have a sequence $\{f_n\}_{n\ge1} \subseteq L^2(H)$ satisfying (\underline{\underline{1}}) and (\underline{\underline{2}}) above for every $k \ge 1$. Observe that

$$\|f_{n+1}(t)-f_n(t)\| \le \frac{\varepsilon\|\theta\|_1^{n-1}\vartheta(t)}{(n-1)!} \text{ a.e.}$$

$$\Longrightarrow \{f_n\}_{n\ge1} \subseteq L^2(H) \text{ is Cauchy.}$$

Therefore $f_n \xrightarrow{s} \hat f$ in $L^2(H)$, $\Longrightarrow p(f_n) = z_n \xrightarrow{s} p(\hat f) = \hat x$ in $C(T,H)$, and $\hat f(t) \in s\text{-}\lim F(t,z_n(t)) = F(t,\hat x(t))$ a.e. (the hypothesis $H(F)$ (\underline{2})). So $\hat x \in P(x_0)$ and from (\underline{3}) above we have $\|\hat x - x\|_{C(T,H)} \le \varepsilon\exp\|\theta\|_1$. Since $\varepsilon > 0$ was arbitrary and $P_c(x_0)$ is closed in $C(T,H)$ (see Theorem 4.1 in [10]), we conclude that $P_c(x_0) = \overline{P(x_0)}$ is the closure taken in $C(T,H)$. □

Remark. Our result here improves Theorem 5.2 of [10], where we assumed that $F(t,x) = g(t,x,U(t))$.

4. Convexity of the orientor field.

In this section we prove that the closedness of the solution of $(^*)_1$ in $C(T,H)$ implies that the orientor field $F(t,x)$ is convex valued for almost all $t \in T$. In this way, we generalize the recent result of Cellina-Ornelas [5], who examined differential inclusions in $\mathbb{R}^n$, with no subdifferential term present. In addition, Cellina-Ornelas [5] considered on the solution set the $AC(T,\mathbb{R}^n)_w$ topology, which is stronger than the $C(T,\mathbb{R}^n)$-topology. So even in the context of finite dimensional systems, our result is stronger than that of [5]. Furthermore our proof is simpler.

In what follows by $P(\hat{T},\hat{x}_0)$ (resp. $P_c(\hat{T},\hat{x}_0)$) we will denote the solution set of $(^*)_1$ (resp. of $(^*)_2$) defined on $\hat{T}$ and having initial state $\hat{x}_0$. Also as before $P(x_0)$ is the solution set of $(^*)_1$, $P(x_0)(t) = \{y \in H : y = x(t), x(\cdot) \in P(x_0)\}$ and $GrP(x_0)(\cdot) = \{(t,y) : y \in P(x_0)(t)\}$.

Theorem 4.1. *<u>If</u> the hypotheses $H(\phi)$, $H(F)$ hold, $x_0 \in \operatorname{dom}\phi(0,\cdot)$ and $P(x_0) \in P_f(C(T,H))$, <u>then</u> the orientor field $F(\cdot,\cdot)$ is convex valued on $W = \{(t,x) : t \in T \setminus N,\ \lambda(N) = 0,\ x \in P(x_0)(t)\}$ (here $\lambda(\cdot)$ denotes the Lebesgue measure on T).*

Proof: We will proceed by contradiction. Assuming that the claim of our theorem is false, we will establish a contradiction to the relaxation theorem (Theorem 3.1).

Since we have assumed that the conclusion of the theorem is not valid, we can find $T_1 \subseteq T$ measurable with $\lambda(T_1) > 0$ s.t. for any $t \in T_1$, there exists an $x^t \in H$ s.t. $(t,x^t) \in W$ and $F(t,x^t)$ is not a convex subset of H. Also because of the hypothesis $H(F)_1$ we can apply Theorem 2.1 of Artstein-Prikry [1] and get $T_2 \subseteq T$ nonempty, closed with $\lambda(T_2) > r - \lambda(T_1)$ s.t. $F\,|_{T_2\times H}$ is lower semicontinuous. Furthermore invoking Theorem 1 of Rzezuchowski [16], we can find $F_0 : T \times H \to P_f(H)$, a multifunction s.t. (i) $F_0(t,x) \subseteq F(t,x)$, (ii) if $\Delta \subseteq T$ is measurable and $x,y : \Delta \to H$ are measurable functions, then $y(t) \in F(t,x(t))$ a.e. implies that $y(t) \in F_0(t,x(t))$ a.e. and (iii) for every $\varepsilon > 0$, there is a closed set $B_\varepsilon \subseteq T$ s.t. $\lambda(T \setminus B_\varepsilon) < \varepsilon$ and $F_0\,|_{B_\varepsilon\times H}$ has closed graph. In particular we can assume that $F_0\,|_{T_2\times H}$ has closed graph. Note that the nonemptiness of the values of $F_0(t,x)$ follows easily from Theorem 3.1 of Jarnik-Kurzweil [15] and Theorem 3.1 of [10]. Now observe that $\lambda(T_1 \cap T_2) > 0$. Choose $t_0 \in T_1 \cap T_2$, $t_0 < r$ to be a Lebesgue point (point of density) of $T_1 \cap T_2$. Then we can find $x_0 \in H$ s.t. $F(t_0,x_0)$ is not convex. So there exists $y_0 \in \overline{\operatorname{conv}}F(t_0,x_0) \setminus F(t_0,x_0)$. Apply Michael's selection theorem to get $g : T_2 \times H \to H$ a continuous map s.t. $g(t,x) \in \overline{\operatorname{conv}}F(t,x)$ for all $(t,x) \in T_2 \times H$ and $g(t_0,x_0) = y_0$. Then define $\hat{F} : T \times H \to P_{fc}(H)$ by $\hat{F}(t,x) = \chi_{T_2}(t)g(t,x) + \chi_{T_2^c}\overline{\operatorname{conv}}F(t,x)$. Note that since $g(\cdot,\cdot)$ is a Carathéodory function, it is jointly measurable and similarly for $\overline{\operatorname{conv}}F(\cdot,\cdot)$ (see the hypotheses $H(F)$ (<u>1</u>) and (<u>2</u>) and Theorem 3.3 of [9]). Hence $(t,x) \to \hat{F}(t,x)$ is measurable. Also because $T_2 \subseteq T$ is closed, $\hat{F}(t,\cdot)$ is l.s.c. Now on $T_0 = [t_0,r]$ consider the following evolution inclusions:

$$(^*)_3 \qquad \left\{ \begin{array}{l} -\dot{x}(t) \in \partial\phi(t, x(t)) + \hat{F}(t, x(t)) \text{ a.e.} \\ x(0) = x_0 \end{array} \right\}.$$

From [10] we know that this has a solution $x(\cdot) \in W(T_0)$ and clearly $x(\cdot) \in P_c(x_0)$. On the other hand we claim that $x(\cdot) \notin P(x_0)$. Suppose the contrary. Then $-\dot{x}(t) \in \partial\phi(t, x(t)) + F(t, x(t))$ a.e., which by definition means that there exists $u \in L^2(H)$, $u(t) \in \partial\phi(t, x(t))$ a.e. s.t. $-\dot{x}(t) - u(t) \in F(t, x(t))$ a.e. Then from the properties of the multifunction $F_0(t, x)$, we deduce that $-\dot{x}(t) - u(t) \in F_0(t, x(t))$ a.e. Also note that for $\delta > 0$ small enough, $[t_0, t_0 + \delta] \cap T_2$ will have a positive measure. We also claim that for such small $\delta > 0$ and $t \in [t_0, t_0 + \delta] \cap T_2$, we will have $d(g(t, x(t)), F_0(t, x(t))) > 0$. Suppose not. Then we can get $t_n \in [t_0, t_0 + \delta] \cap T_2$, $t_n \to t_0$ s.t. $d(g(t_n, x(t_n)), F_0(t_n, x(t_n))) = 0 \Longrightarrow g(t_n, x_n(t_n)) \in F_0(t_n, x(t_n)) \Longrightarrow (t_n, x(t_n), g(t_n, x(t_n))) \in GrF_0$. But note that $(t_n, x(t_n), g(t_n, x(t_n))) \to (t_0, x(t_0) = x_0, g(t_0, x_0))$ in $([t_0, t_0 + \delta] \cap T_2) \times H \times H$ and since $F_0|_{T_2 \times H}$ has a closed graph, we get that $g(t_0, x_0) \in F_0(t_0, x_0)$ a contradiction. So indeed $x(\cdot) \notin P(x_0) = \overline{P(x_0)}^{C(T,H)}$ (by hypothesis) and $x(\cdot) \in P_c(x_0) = \overline{P(x_0)}^{C(T,H)} \subsetneq P_c(x_0)$ — a contradiction to Theorem 3.1 (relaxation theorem). $\square$

Remark. Let $K : T \to P_{fc}(\mathbb{R}^n)$ be an absolutely continuous multifunction with modulus $m(\cdot) \in L^1_+$; i.e. $|d(x, K(t)) - d(y, K(t'))| \leq \|x - y\| + \int_t^{t'} m(s)\, ds$. Set $\phi(t, x) = \delta_{K(t)}(x)$, where $\delta_{K(t)}(x) = 0$ if $x \in K(t)$ and $+\infty$ if $x \notin K(t)$ (indicator function of $K(t)$). It is easy to check that the hypothesis $H(\phi)$ is satisfied (in this case $\dot{g}_b = m$, $\beta = 1$, $h_b = 0$). Also recall that $\partial\phi(t, x) = \partial\delta_{K(t)}(x) = N_{K(t)}(x) =$ the normal cone to $K(t)$ at x. So the evolution inclusion $(^*)_1$ becomes

$$(^*)_4 \qquad \left\{ \begin{array}{l} -\dot{x}(t) \in N_{K(t)}(x(t)) + F(t, x(t)) \text{ a.e.} \\ x(0) = x_0 \end{array} \right\}.$$

Thus $(^*)_4$ is a special case of $(^*)_1$. Such evolution inclusions are important in mechanics and mathematical economics (different variational inequalities, see Aubin-Cellina [2]).

References

[1] Artstein Z., Prikry K., *Carathédory selections and the Scorza-Dragoni property*, J. Math. Anal. Appl. **127** (1987), 540–547.

[2] Aubin J.-P., Cellina A., *Differential Inclusions*, Springer, Berlin, 1984.

[3] Barbu V., *Nonlinear Semigroups and Differential Equations in Banach Spaces*, Noordhoff International Publishing, Leyden, The Netherlands, 1976.

[4] Brezis H., *Operateurs Maximaux Monotones*, North Holland, Amsterdam, 1973.

[5] Cellina A., Ornelas A., *Convexity and the closure of the solution set to differential inclusions*, SISSA, Report No. 136M, Trieste, Italy, 1988.

[6] Diestel J., Uhl J.J., *Vector Measures*, Math. Surveys, Vol. 15, AMS, Providence, R.I., 1977.

[7] Klein E., Thompson A., *Theory of Correspondences*, Wiley, New York, 1984.
[8] Papageorgiou N.S., *Measurable multifunctions and their applications to convex integral functionals*, Intern. J. Math. and Math. Sci. **12** (1989), 175–192.
[9] ———, *On measurable multifunctions with applications to random multivalued equations*, Math. Japonica **32** (1987), 437–464.
[10] ———, *On evolution inclusions associated with time dependent convex subdifferentials*, Comment. Math. Univ. Carolinae **31** (1990), 517–527.
[11] Tsukada M., *Convergence of best approximations in smooth Banach spaces*, J. Approx. Theory **40** (1984), 301–309.
[12] Wagner D., *Survey of measurable selection theorems*, SIAM J. Control and Optim. **15** (1977), 859–903.
[13] Watanabe J., *On certain nonlinear evolution equations*, J. Math. Soc. Japan **25** (1973), 446–463.
[14] Yotsutani S., *Evolution equations associated with subdifferentials*, J. Math. Soc. Japan **31** (1978), 623–646.
[15] Jarnik J., Kurzweil J., *On conditions on right hand sides of differential relations*, Časopis pro pěstování matematiky **102** (1977), 334–349.
[16] Rzezuchowski T., *Scorza-Dragoni type theorem for upper semicontinuous multivalued functions*, Bulletin Polish Academy of Sciences **28** (1980), 61–66.

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