Totally convex algebras

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Abstract. By definition a totally convex algebra A is a totally convex space $|A|$ equipped with an associative multiplication, i.e. a morphism $\mu : |A| \otimes |A| \longrightarrow |A|$ of totally convex spaces. In this paper we introduce, for such algebras, the notions of ideal, tensor product, unitization, inverses, weak inverses, quasi-inverses, weak quasi-inverses and the spectrum of an element and investigate them in detail. This leads to a considerable generalization of the corresponding notions and results in the theory of Banach spaces.

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0. Banach algebras and totally convex algebras.

In [6] totally convex spaces over $\mathbb{K}$, $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$, emerged as the Eilenberg-Moore algebras of the unit ball functor $\mathrm{O} : \mathbf{Ban}_1 \longrightarrow \mathbf{Set}$, where $\mathbf{Ban}_1$ is the category of Banach spaces over $\mathbb{K}$ and linear contractions. The step from totally convex spaces to totally convex algebras in (0.2) is quite natural; it corresponds completely to the step from abelian groups to rings. Moreover, totally convex algebras appear as the Eilenberg-Moore algebras of the unit ball functor from the category $\mathbf{Ban}_1$-$\mathbf{Alg}$ of Banach algebras over $\mathbb{K}$ and contractive homomorphisms to $\mathbf{Set}$. We will prove this in this section, because this result is important for the investigation of totally convex algebras. Moreover, this result means that the theory of totally convex algebras is the algebraic theory "generated" by the theory of Banach algebras. The unit ball functor from $\mathbf{Ban}_1$-$\mathbf{Alg}$ to $\mathbf{Set}$, assigning to each Banach algebra its closed unit ball, will be denoted by $\mathrm{O}^a : \mathbf{Ban}_1\text{-}\mathbf{Alg} \longrightarrow \mathbf{Set}$.

(0.1) Proposition. $\mathrm{O}^a : \mathbf{Ban}_1\text{-}\mathbf{Alg} \longrightarrow \mathbf{Set}$ *is pre-monadic but not monadic.*

Proof: First, we will show the existence of a left adjoint $\mathrm{l}_1^a : \mathbf{Set} \longrightarrow \mathbf{Ban}_1\text{-}\mathbf{Alg}$. If **Semi-Grp** is the category of semigroups, the unit ball $\mathrm{O}^a(A) := \{a \mid a \in A, \|a\| \leq 1\}$ of a Banach algebra carries a canonical semigroup structure and hence induces a canonical functor $\mathrm{O}^s : \mathbf{Ban}_1\text{-}\mathbf{Alg} \longrightarrow \mathbf{Semi\text{-}Grp}$. Let $\mathrm{V}^s : \mathbf{Semi\text{-}Grp} \longrightarrow \mathbf{Set}$ denote the usual forgetful functor, s. th. $\mathrm{O}^a = \mathrm{V}^s \circ \mathrm{O}^s$ holds. For a semigroup S define $\mathrm{l}_1^s(S) := \mathrm{l}_1(\mathrm{V}^s(S))$, where, for a set X, $\mathrm{l}_1(X)$ is the usual l_1-space generated by X. $\mathrm{l}_1^s(S)$ carries a canonical algebra structure, which makes it a Banach algebra. The multiplication is defined by putting

$$\delta_s * \delta_{s'} := \delta_{ss'},$$

where $\delta_s, s \in S$, are the Dirac symbols, which form a basis of $\mathrm{l}_1(\mathrm{V}^s(S))$. Using the well known fact that $\mathrm{l}_1 : \mathbf{Set} \longrightarrow \mathbf{Ban}_1$ is left adjoint to $\mathrm{O} : \mathbf{Ban}_1 \longrightarrow \mathbf{Set}$ with unit $\eta_X : X \longrightarrow \mathrm{O}(\mathrm{l}_1(X))$, $\eta_X(x) = \delta_x,\ x \in X$, it is elementary to verify that l_1^s is left adjoint to O^s . For $S \in$ **Semi-Grp** the unit is given by $\eta_S^s : S \longrightarrow \mathrm{O}^s(\mathrm{l}_1^s(S)), \eta_S^s(s) = \delta_s,\ s \in S$.

If F^s : **Set** $\longrightarrow$ **Semi-Grp** is left adjoint to V^s with unit $\hat{\eta}_X : X \longrightarrow \mathrm{V}^s \circ \mathrm{F}^s(X)$, $\mathrm{l}_1^a := \mathrm{l}_1^s \circ \mathrm{F}^s$ is left adjoint to O^a with the unit $\eta_X^a : X \longrightarrow \mathrm{O}^a \circ\ \mathrm{l}_1^a(X)$, $\eta_X^a = \mathrm{V}^s(\eta^s_{\mathrm{F}^s(X)})\hat{\eta}_X$. To show the premonadicity of O^a we use (10.1) in [15] and prove that the counit $\varepsilon^a : \mathrm{l}_1^a \circ \mathrm{O}^a \longrightarrow \mathbf{Ban}_1\text{-}\mathbf{Alg}$ is a coequalizer. Using (1.1) in [6], one sees that $\varepsilon_A^a : \mathrm{l}_1^a(\mathrm{O}^a(A)) \longrightarrow A$, $A \in \mathbf{Ban}_1\text{-}\mathbf{Alg}$, is given by $\varepsilon_A^a(\delta_{(a_1,\dots,a_n)}) = a_1 a_2 \dots a_n$, for a basis element $\delta_{(a_1,\dots,a_n)} \in \mathrm{l}_1^s(\mathrm{F}^s(\mathrm{O}(A)))$, $a_i \in \mathrm{O}^a(A)$, $1 \leq i \leq n$. The $\mathbf{Ban}_1$-morphism

$$\lambda : \mathrm{l}_1^a(\mathrm{O}^a(A))/_{ker\varepsilon_A^a} \longrightarrow A$$

in [6, (1.1)], is an isomorphism in $\mathbf{Ban}_1$. But it is also a multiplicative homomorphism, because ε_A^a has this property. Hence, ε_A^a is a coequalizer and O^a is pre-monadic. That O^a is not monadic will be established presently by using the Linton space as in [6]. From now on, we will often write simply $\mathrm{O}(A)$ instead of $\mathrm{O}^a(A)$, whenever the context is clear. □

As we know the Eilenberg-Moore algebras of $\mathrm{O} : \mathbf{Ban}_1 \longrightarrow \mathbf{Set}$, namely the totally convex spaces (cp. [6]), and as $\mathbf{Ban}_1$-**Alg** lies over $\mathbf{Ban}_1$ with the usual forgetful functor denoted by $|\Box| : \mathbf{Ban}_1\text{-}\mathbf{Alg} \longrightarrow \mathbf{Ban}_1$, it is reasonable to expect to get the Eilenberg-Moore algebras of $\mathbf{Ban}_1$-**Alg** out of the totally convex spaces by adding a (compatible) multiplication. The Eilenberg-Moore algebras of the category of C^*-algebras and the category of Jordan-Banach algebras, which are monadic over **Set** with respect to appropriate modifications of the unit ball functor, were recently investigated in [4] by J.W. Pelletier and J. Rosický.

(0.2) Definition. A *totally convex algebra* is a totally convex space A together with a morphism $\mu : A \otimes A \longrightarrow A$ in **TC** (cp. [6, (5.3)]), s. th.

$$\mu(x \otimes \mu(y \otimes z)) = \mu(\mu(x \otimes y) \otimes z)$$

holds for all $x, y, z \in A$. If $\mu(x \otimes y) = \mu(y \otimes x)$ for all $x, y \in A$, A is called *commutative*. μ is called *the multiplication in* A and will be denoted by

$$xy := \mu(x \otimes y)$$

throughout. A is called a *unital totally convex algebra*, if there is a (necessarily) unique $e \in A$, s. th. for any $x \in A$ $ex = xe = x$ holds. e is called the *unit element* of A.

Natural examples for totally convex algebras are given by the endomorphism sets End(C) := Hom(C,C) for C $\in$ **TC** (cp. [6, §5]). End(C) is even unital, i.e. it has a unit element (see (0.7)).

(0.3) Definition. A *morphism* $\varphi : A \longrightarrow B$ of *totally convex algebras* is a morphism of the underlying totally convex spaces preserving the product as well: $\varphi(xy) = \varphi(x)\varphi(y),\ x, y \in A$. The totally convex algebras together with their morphisms form a category **TC-Alg**. The canonical functor assigning to a totally convex algebra its underlying totally convex space is denoted by $|\Box| :$ **TC-Alg** $\longrightarrow$ **TC**. There will be no misunderstandings by using the same notation as for the forgetful functor from $\mathbf{Ban}_1$**-Alg** to $\mathbf{Ban}_1$. This connection between **TC-Alg** and **TC** explains, why we often will call a totally convex algebra a **TC**-*algebra* for short.

For $A \in \mathbf{Ban}_1\text{-}\mathbf{Alg}$, $\mathrm{O}(A)$ is in a canonical way a totally convex algebra, which we denote by $\hat{\mathrm{O}}^a(A)$ or simply by $\hat{\mathrm{O}}(A)$, if the context is clear. This induces a functor $\hat{\mathrm{O}}^a : \mathbf{Ban}_1\text{-}\mathbf{Alg} \longrightarrow \mathbf{TC}\text{-}\mathbf{Alg}$ and we have $\mathrm{O}^a = \mathrm{W} \circ \hat{\mathrm{O}}^a$ for the canonical forgetful functor $\mathrm{W} : \mathbf{TC}\text{-}\mathbf{Alg} \longrightarrow \mathbf{Set}$.

TC-Alg is a category of equationally defined universal algebras, hence the canonical forgetful functor $\mathrm{W} : \mathbf{TC}\text{-}\mathbf{Alg} \longrightarrow \mathbf{Set}$ has a left adjoint. An explicit construction of it is given in the

(0.4) Theorem. $\hat{\mathrm{O}}^a \circ \mathrm{l}_1^a : \mathbf{Set} \longrightarrow \mathbf{TC}\text{-}\mathbf{Alg}$ *is a left adjoint of* $\mathrm{W} : \mathbf{TC}\text{-}\mathbf{Alg} \longrightarrow \mathbf{Set}$. **TC-Alg** *is the* (*up to isomorphism unique*) *category of Eilenberg-Moore algebras of* $\mathrm{O}^a : \mathbf{Ban}_1\text{-}\mathbf{Alg} \longrightarrow \mathbf{Set}$ *and* $\hat{\mathrm{O}}^a : \mathbf{Ban}_1\text{-}\mathbf{Alg} \longrightarrow \mathbf{TC}\text{-}\mathbf{Alg}$ *is the canonical comparison functor.*

Proof: There is a canonical forgetful functor $|\Box|^s : \mathbf{TC}\text{-}\mathbf{Alg} \longrightarrow \mathbf{Semi}\text{-}\mathbf{Grp}$ with $\mathrm{V}^s \circ |\Box|^s = \mathrm{U} \circ |\Box|$. Moreover, we know that $\mathrm{W} \circ \hat{\mathrm{O}}^a = \mathrm{O}^a$, hence $\eta_X^a : X \longrightarrow \mathrm{W} \circ \hat{\mathrm{O}}^a \circ \mathrm{l}_1^a(X)$. For any $A \in \mathbf{TC}\text{-}\mathbf{Alg}$ and any $f : X \longrightarrow \mathrm{U}(|A|)$ we have, because of $\mathrm{U}(|A|) = \mathrm{V}^s(|A|^s)$, a unique morphism f_o: $\mathrm{F}^s(X) \longrightarrow |A|^s$ in **Semi-Grp** with $f = \mathrm{V}^s(f_o)\hat{\eta}_X$. $\mathrm{V}^s(f_o)$ induces a unique morphism $\varphi : \hat{\mathrm{O}} \circ \mathrm{l}_1(\mathrm{V}^s \circ \mathrm{F}^s(X)) \longrightarrow |A|$ in **TC** with $\mathrm{V}^s(f_o) = \mathrm{U}(\varphi)\eta_{\mathrm{V}^s \circ \mathrm{F}^s(X)}$ because of [6, (3.1)]. It is obvious that φ also preserves the multiplication, because f_o does, s. th. $\varphi : \hat{\mathrm{O}}^a \circ \mathrm{l}_1^a(X) \longrightarrow A$ is in **TC-Alg** and $\mathrm{W}(\varphi)\eta_X^a = f$. This equation determines φ uniquely, because $\eta_X^a(X)$ is a set of generators of the **TC**-algebra $\hat{\mathrm{O}}^a \circ \mathrm{l}_1^a(X)$.

One has $\mathrm{W} \circ \hat{\mathrm{O}}^a \circ \mathrm{l}_1^a = \mathrm{O}^a \circ \mathrm{l}_1^a$, the unit η_X^a is the same for both adjunctions and it is elementary to verify by looking at the co-units that both adjunctions induce the same monad (cp. [6, (3.5)]). As $\mathrm{W} : \mathbf{TC}\text{-}\mathbf{Alg} \longrightarrow \mathbf{Set}$ is monadic, it may be identified with its own category of Eilenberg-Moore algebras i.e. with the category of Eilenberg-Moore algebras of $\mathrm{O}^a : \mathbf{Ban}_1\text{-}\mathbf{Alg} \longrightarrow \mathbf{Set}$. [13, (2.9)], then immediately shows that $\hat{\mathrm{O}}^a$ is the comparison functor.

(0.5) Remark. It is now easy to see that $\mathrm{O} : \mathbf{Ban}_1\text{-}\mathbf{Alg} \longrightarrow \mathbf{Set}$ is not monadic. The Linton space $\mathrm{L}(\mathbb{K}) := \{z \mid z \in \mathbb{K} \text{ and } |z| = 1\} \cup \{0\}$ in[6, p. 985], does the job. It has a canonical structure of a **TC**-algebra and quite obviously is not the unit ball of a Banach algebra.

(0.6) Corollary. $\hat{\mathrm{O}}^a : \mathbf{Ban}_1\text{-}\mathbf{Alg} \longrightarrow \mathbf{TC}\text{-}\mathbf{Alg}$ *is full and faithful and has a left adjoint* $\mathrm{S}^a : \mathbf{TC}\text{-}\mathbf{Alg} \longrightarrow \mathbf{Ban}_1\text{-}\mathbf{Alg}$.

Proof: The first assertion follows from the fact that $\hat{\mathrm{O}}^a$ is the comparison functor to the Eilenberg-Moore algebras of the pre-monadic functor O^a. The second assertion holds, because $\mathbf{Ban}_1$**-Alg** has coequalizers (cp. [13, (3.7)]).

It will turn out to be useful later on, to know S^a in a more explicit resp. constructive form. This is not difficult, because S^a is a canonical lifting of the left adjoint $\mathrm{S} : \mathbf{TC} \longrightarrow \mathbf{Ban}_1$ of $\hat{\mathrm{O}} : \mathbf{Ban}_1 \longrightarrow \mathbf{TC}$ (cp. [6, (7.7)]) along the forgetful functors $|\Box| : \mathbf{TC\text{-}Alg} \longrightarrow \mathbf{TC}$ and $|\Box| : \mathbf{Ban}_1\text{-}\mathbf{Alg} \longrightarrow \mathbf{Ban}_1$.

For $C \in \mathbf{TC\text{-}Alg}$, we define a $\mathbb{K}$-algebra structure on $\mathrm{S}(|C|)$ by introducing a multiplication (for notation see [6, §7]) by

$$\overline{(\lambda_o, c_o)}\ \overline{(\lambda_1, c_1)} := \overline{(\lambda_o\lambda_1, c_oc_1)},$$

$\overline{(\lambda_i, c_i)} \in \mathrm{S}(|C|)$, $i = 0, 1$. This multiplication is well-defined and turns $\mathrm{S}(|C|)$ into a $\mathbb{K}$ -algebra, which we denote by $\mathrm{S}^a(C)$. The norm on the Banach space $\mathrm{S}(|C|)$ is given by

$$\|\overline{(\lambda, c)}\| = inf\{\ |\mu|\ \ |\ \overline{(\lambda, c)} = \overline{(\mu, d)}\},$$

hence one gets

$$\|\overline{(1, c_o)}\ \overline{(1, c_1)}\| = \|\overline{(1, c_oc_1)}\| \le \|\overline{(1, c_o)}\|\|\overline{(1, c_1)}\|,$$

which shows that $\mathrm{S}^a(C)$ is even a Banach algebra over $\mathbb{K}$.

The universal morphism $\sigma_{|C|} : |C| \longrightarrow \hat{\mathrm{O}}\circ\mathrm{S}(|C|)$ in [6, §7], preserves products, because $\sigma_{|C|}(c_oc_1) = \overline{(1, c_oc_1)} = \overline{(1, c_o)}\ \overline{(1, c_1)} = \sigma_{|C|}(c_o)\sigma_{|C|}(c_1)$, $c_i \in C$, $i = 0, 1$. It is easy to verify that the unique morphism $\varphi : \mathrm{S}(|C|) \longrightarrow |A|$ in **TC**, which exists for a morphism $f : C \longrightarrow \hat{\mathrm{O}}^a(A)$ in **TC-Alg**, $A \in \mathbf{Ban}_1$**-Alg**, on the level **TC**, with $|f| = \hat{\mathrm{O}}(\varphi)\sigma_{|C|}$, also preserves products, i.e. can be lifted to **TC-Alg**. Denoting by σ^a_C the **TC**-morphism $\sigma_{|C|}$ regarded as an element of **TC-Alg**, we have shown that S^a is left adjoint to $\hat{\mathrm{O}}^a$ with unit $\sigma^a_C : C \longrightarrow \hat{\mathrm{O}}^a\circ\ \mathrm{S}^a(C)$. This construction is the reason why we will drop the superscript "a" in σ^a and S^a, whenever the context is clear.

(0.7) Remark. Analogous results hold for subtypes of Banach algebras. If $\mathbf{Ban}^u_1$**-Alg** denotes the category of unital Banach algebras and contractive algebra homomorphisms preserving the unit element, the unit ball functor (denoted for simplicity by the same symbol) $\mathrm{O}^a : \mathbf{Ban}^u_1\text{-}\mathbf{Alg} \longrightarrow \mathbf{Set}$ is premonadic but *not* monadic and its left adjoint l^a_1 is constructed as in (0.1) substituting for **Semi-Grp** the category **Mon** of monoids. The zero algebra $\{0\}$ is a unital Banach algebra as well as a **TC**-algebra. The Eilenberg-Moore algebras of O^a are given by the category $\mathbf{TC}^u$**-Alg** of *unital totally convex algebras* and unit preserving **TC-Alg** morphisms. (0.4) remains true.

(0.8) Remark. Considering the category $\mathbf{Ban}^c_1$**-Alg** of commutative Banach algebras over $\mathbb{K}$, the unit ball functor $\mathrm{O}^a : \mathbf{Ban}^c_1\text{-}\mathbf{Alg} \longrightarrow \mathbf{Set}$ is pre-monadic but *not* monadic. Its left adjoint l^a_1 is constructed as in (0.1) substituting the category **Ab-Semi-Grp** of abelian semigroups for **Semi-Grp**. Its Eilenberg-Moore algebras are given by the category $\mathbf{TC}^c$**-Alg** of *commutative totally convex algebras*, where a totally convex algebra is called commutative, iff its multiplication is. (0.4) remains valid for $\hat{\mathrm{O}}^a$.

(0.9) Remark. For the category $\mathbf{Ban}_1^{cu}$**-Alg** of commutative, unital Banach algebras the same results hold, mutatis mutandis. The place of **Semi-Grp** is taken by the category **Ab-Mon** of abelian monoids and the category $\mathbf{TC}^{cu}$**-Alg** of *commutative, unital* **TC**-*algebras* and unit preserving **TC-Alg**-morphisms is the corresponding category of Eilenberg-Moore algebras.

(0.10) Remark. For a Banach algebra B define another Banach algebra B^{op} on the same underlying set by taking the same addition, defining a new multiplication with scalars by $\alpha \circ b := \overline{\alpha} b$, $\alpha \in \mathbb{K}$, $b \in B$, and by taking the opposite multiplication $b_o \times b_1 := b_1 b_o$. A *-Banach algebra is then defined as a Banach algebra B together with a morphism $\Box^* : B \longrightarrow B^{op}$ in $\mathbf{Ban}_1$, which is an involution as a set mapping. A *-Banach algebra is essentially a *-normed Banach algebra in the notation of [14]. The *-Banach algebras with the contractive homomorphisms commuting with the *-operation form the category $\mathbf{Ban}_1^*$**-Alg**. The unit ball functor O : $\mathbf{Ban}_1^*$**-Alg** $\longrightarrow$ **Set** is pre-monadic but not monadic. One gets its left adjoint by substituting the category **Semi-Grp*** of *-semigroups for **Semi-Grp** in the proof of (0.1). The category **TC*-Alg** of *-**TC**-algebras, i.e. **TC**-algebras with an involution, and involution preserving **TC**-algebra morphisms is the category of Eilenberg-Moore algebras of O.

(0.11) Definition. A *finitely totally convex algebra* A is a finitely totally convex space (cp. [6]) together with a morphism $\mu : A \otimes A \longrightarrow A$ in $\mathbf{TC}_{fin}$, s. th.

$$\mu(x \otimes \mu(y \otimes z)) = \mu(\mu(x \otimes y) \otimes z), x, y, z \in A.$$

Again, one writes for short

$$xy := \mu(x \otimes y),$$

$x, y \in A$. A morphism $\varphi : A \longrightarrow B$ between two finitely totally convex algebras is a $\mathbf{TC}_{fin}$-morphism, which also preserves this multiplication. With these morphisms the finitely totally convex algebras constitute the category $\mathbf{TC}_{fin}$**-Alg**. It is evident, how one has to define the category of commutative, resp. unital, resp. finitely totally convex $*$-algebra.

As the following theorem shows, $\mathbf{TC}_{fin}$**-Alg** appears, too, as a category of Eilenberg-Moore algebras.

(0.12) Theorem. *If* $\mathbf{Norm}_1$**-Alg** *is the category of normed* $\mathbb{K}$*-algebras and contractive* $\mathbb{K}$*-algebra homomorphisms, the unit ball functor* $\mathrm{O}^a : \mathbf{Norm}_1\textbf{-Alg} \longrightarrow \mathbf{Set}$ *is pre-monadic but not monadic.* $\mathbf{TC}_{fin}$**-Alg** *together with the canonical forgetful functor* $\mathrm{W} : \mathbf{TC}_{fin}\textbf{-Alg} \longrightarrow \mathbf{Set}$ *is the category of Eilenberg-Moore algebras of* O^a *and the canonical lifting of* O^a, $\hat{\mathrm{O}}^a : \mathbf{Norm}_1\textbf{-Alg} \longrightarrow \mathbf{TC}_{fin}\textbf{-Alg}$, *is the comparison functor.*

The proof will be omitted, because of its analogy to the proof of (0.4). We will give just a hint as to what has to be changed in the proof of (0.4) to yield (0.12). The left adjoint $\mathrm{l}_{1,fin} : \mathbf{Set} \longrightarrow \mathbf{Vec}_1$ of $\mathrm{O} : \mathbf{Vec}_1 \longrightarrow \mathbf{Set}$ in [6, (1.5)], can be lifted, as in (0.4), to a left adjoint $\mathrm{l}^s_{1,fin} : \mathbf{Semi\text{-}Grp} \longrightarrow \mathbf{Norm}_1\textbf{-Alg}$ of $\mathrm{O}^s : \mathbf{Norm}_1\textbf{-Alg} \longrightarrow \mathbf{Semi\text{-}Grp}$.

The explicit construction of the left adjoint of $\hat{\mathrm{O}}^a$: $\mathbf{Norm_1\text{-}Alg} \longrightarrow \mathbf{TC}_{fin}\mathbf{\text{-}Alg}$ is analogous, taking into account the slightly different construction of S_{fin} in [6, (7.10)]. The results (0.8) to (0.10) hold, of course, also for finitely totally convex algebras, mutatis mutandis.

1. Ideals.

As a category of equationally defined universal algebras **TC-Alg** is complete and cocomplete. Of course, the same statement holds for the category of unital, resp. commutative (and unital), resp. *-**TC**-algebras and for the corresponding categories of finitely totally convex algebras. As usual, limits are modelled on the corresponding limits of the underlying sets and are also limits on the level of totally convex spaces.

To obtain explicit knowledge of colimits is by far more complicated, as it is in the case of totally convex spaces (cp. [6, §4], [9]). At first we will investigate coequalizers or, equivalently, congruence relations. As in any algebraic theory, a congruence relation in a **TC**-algebra is an equivalence relation on the underlying set, compatible with the algebra operations. For many purposes it is convenient to consider for an equivalence relation "$\sim$" on a set X its graph, graph$(\sim) := \{(x_1, x_2) \mid x_1 \in X, x_1 \sim x_2\}$. Because graph $(\square)$ is a bijection between the equivalence relations on X and the subsets I of $X \times X$ that contain the diagonal $\triangle_X$ and are closed under the reflection $(x_1, x_2) \longmapsto (x_2, x_1)$ and under the transitivity operation $((x_1, x_2), (x_2, x_3) \in I$ implies $(x_1, x_3) \in I)$, we will often not distinguish between an equivalence relation and its graph.

For a **TC**-algebra A, a *left-A* **TC**-*space* C is given by a totally convex space C and an (external) left multiplication of elements of A with elements of C, $(a, c) \longmapsto ac$, which is distributive with respect to the totally convex operations on C, i.e. for $\alpha \in \Omega$ (cp. [6]) and $c_i \in C$, $i \in \mathbb{N}$, and $a \in A$

$$a(\sum_i \alpha_i c_i) = \sum_i \alpha_i (ac_i)$$

holds, such that the resulting map $\varphi : A \longrightarrow \mathrm{End}(C)$ is a **TC**-algebra morphism. *Right-A* **TC**-spaces are defined analogously.

(1.1) Definition. Let A be a (finitely) totally convex algebra. "$\sim$" is called a *left ideal* of A, if $\sim < |A| \times |A|$ is a congruence relation on the underlying totally convex space $|A|$ (cp. [10]) and a *left-A* **TC**-*space*, where the operation of A on $\sim$ is given by $a(x, y) := (ax, ay)$.

Analogously, a *right ideal* of A is a congruence relation on $|A|$, which is a *right-A* **TC**-*space* with $(x, y)a := (xa, ya)$ for $a \in A$. "$\sim$" is called an *ideal* of A, if it is both a left and a right ideal. It is easy to verify that a congruence relation "$\sim$" on $|A|$ is an ideal of A, iff $\sim < A \times A$, i.e. $\sim$ is a subalgebra of the **TC**-algebra $A \times A$.

If $\sim$ is an ideal of the **TC**-algebra A, then $A/\sim$ is canonically a **TC**-algebra and the canonical projection $\pi : A \longrightarrow A/\sim$ is a **TC-Alg** morphism. All the usual homomorphism and isomorphism theorems of algebra remain in force, as do the results on direct and subdirect products (see e.g. [3], [5]). Moreover, 2.10 in [8] remains true in **TC-Alg**:

(1.2) Proposition. *Let $\sim$ be an ideal of the* **TC***-algebra A and $\pi : A \longrightarrow A/\!\sim$ the canonical projection. Then*

(i) $\|\pi\| = 0$ *or* $\|\pi\| = 1$*;*

(ii) π *is a coequalizer in* **TC-Alg**.

Among the so called *representations* of a **TC**-algebra in the endomorphism algebra of a left-A **TC**-space C, i.e. **TC-Alg** morphisms $\varphi : A \longrightarrow \mathrm{End}(C)$, is, as in the classical case, the *left-regular representation* of A,

$$L : A \longrightarrow \mathrm{End}(|A|),$$

defined for $a, x \in A$ by $L_a(x) := ax$. L is obviously a morphism in **TC-Alg**. A is called *left-regular*, if L is injective. The *right-regular representation*, $R : A \longrightarrow \mathrm{End}(|A|)$, is defined dually. R is an anti-morphism. A is called *right-regular*, if R is injective. A is called *regular*, if A is left- and right-regular.

(1.3) Definition. If $\varphi : A \longrightarrow B$ is a morphism of **TC**-algebras, then, as usual, $\sim_\varphi$ defined by $a_o \sim_\varphi a_1$, if $\varphi(a_o) = \varphi(a_1)$, for $a_o, a_1 \in A$, is an ideal of A, called the *ideal associated with* φ.

Before looking at some congruences, which are of special interest, it should be noted that any **TC**-algebra A carries a “norm” $\|\Box\|$ on the underlying totally convex space $|A|$. It follows from [6], 6.3 and 6.4, that we have

(1.4) Proposition. *Let A be a totally convex algebra. Then, for any $a, b \in A$*

$$\|ab\| \leq \|a\|\|b\|.$$

In [6] and [7] several interesting types of totally convex spaces are investigated. Some of these properties also play a role in the theory of **TC**-algebras. Hence, we introduce the following notation.

(1.5) Definition. If P is a property of (finitely) totally convex spaces, we say that a *totally convex algebra A has property P*, iff the underlying totally convex space $|A|$ has property P. For instance the notion of a spherical or separated totally convex algebra (cp. [7]), is well-defined.

The interior $|\mathring{A}|$ of the totally convex space $|A|$ underlying the **TC**-algebra A is canonically a **TC**-algebra, called the *interior* of A and denoted by $\mathring{A}$ (cp. [7, (10.1)]). Several of the types of totally convex spaces discussed in [7] induce canonical congruence relations on any totally convex space. These congruence relations induce ideals in totally convex algebras, as is shown in the following

(1.6) Proposition. *The full subcategory* $\mathbf{TC}^{sep}$**-Alg** *of separated totally convex algebras is a reg-epi-reflective subcategory of* **TC-Alg**.

Proof: We look at the reflection $\mathbf{TC} \longrightarrow \mathbf{TC}^{sep}$ on the level of totally convex spaces as described in [8, (2.11)]: For $A \in$ **TC-Alg**, $x, y \in A$, put $x \sim y$, if $\frac{1}{2}x = \frac{1}{2}y$. Obviously $\sim$ is an ideal and the canonical projection $\pi_A : A \longrightarrow A/\!\sim$ is the desired reflection. $\Box$

(1.7) Proposition. *The full subcategory* $\mathbf{TC}^{sph}$**-Alg** *of spherical totally convex algebras is a mono-coreflective and a reg-epi-reflective subcategory of* **TC-Alg**.

PROOF: The first assertion follows, because, for $A \in$ **TC-Alg**, the subspace $\mathrm{T}_s(A)$ of spherical elements of A (cp. [7, 12.5]) is evidently even a subalgebra.

As for the second assertion, one considers the congruence relation defining the reg-epi-reflection on the level of **TC**, i.e. for $A \in$ **TC-Alg** and $x, y \in A$ one puts $x \sim y$, if $x = y$ or $\|x\|, \|y\| < 1$ (cp. [7, (14.3)]). (1.4) shows that $\sim$ is even an ideal of A, which proves the last assertion. □

For the investigation of ideals of **TC**-algebras the following result plays an important rôle. It corresponds to Proposition (1.2) in [10], which was crucial for the investigation of congruences in **TC**.

(1.8) Proposition. *Let* $\sim$ *be an ideal of the totally convex algebra* A *and denote by* $\mathrm{I}(\sim)$ *the subvector space of* $\mathrm{S}(A)$ *generated by the set* $\{\sigma_A(x) - \sigma_A(y) \mid x, y \in A$ *and* $x \sim y\}$. *Then* $\mathrm{I}(\sim)$ *is a closed ideal of the Banach algebra* $\mathrm{S}(A)$ *and the quotient map* $\pi : A \longrightarrow A/\sim$ *induces an isomorphism*

$$\mathrm{S}(A)/\mathrm{I}(\sim) \cong \mathrm{S}(A/\sim).$$

Moreover, $\mathrm{I}(\sim)$ *is generated by* $\sigma_A(\mathring{A} \cap \ \ker(\sim))$, *where* $\ker(\sim) := \{x \mid x \in A$ *and* $x \sim 0\}$.

The proof is completely analogous to that of (1.2) in[10], because Lemma (1.1) in [10] carries over to totally convex algebras. A statement analogous to (1.3) in [10] holds, too. (1.8) obviously does not hold for unital **TC**-algebras, but is nevertheless useful in this case, too. If one just forgets the identity of a unital **TC**-algebra and applies (1.8), it is obvious that $\mathrm{I}(\sim)$ is also an ideal in the unital Banach algebra $\mathrm{S}(A)$. $\ker(\sim)$, for an ideal $\sim$ of A, has some interesting properties (cp. [6, (4.3)]):

(a) $\ker(\sim)$ is a **TC**-subalgebra of A.
(b) $\ker(\sim)A \subseteq \ker(\sim)$ and $A\ker(\sim) \subseteq \ker(\sim)$.

In the classical case, i.e. in Banach algebras, these properties characterize ideals, whereas, in the theory of **TC**-algebras, ideals are certain subalgebras of $A \times A$. Nevertheless, subalgebras satisfying properties (a) and (b) play an important role, too, in the theory of **TC**-algebras. Generalizing the considerations of congruence relations in **TC** in [8] one defines:

(1.9) Definition. (cp. [8, 2.4]): Let A be a **TC**-algebra and K a subalgebra of A. Then, for any $x, y \in A$ one puts $x \sim_K y$, if $\frac{1}{2}x - \frac{1}{2}y \in K$. "$\sim_K$", is called the *relation induced by* K, which is obviously reflexive and symmetric.

In [8, 2.5], it was shown that this relation is also a congruence relation of totally convex spaces, if K is r-closed. Condition (b) implies the compatibility with multiplication and is also necessary for this property, provided K is r-closed (cp. [8, 1.5]). 2.5, [8], implies the

(1.10) Proposition. *Let K be a subalgebra of the* **TC**-*algebra A. Then the following are equivalent.*

(i) $\sim_K$ *is a separated ideal,*
(ii) K *is r-closed and satisfies $KA \subseteq K$ and $AK \subseteq K$.*

The other results of [8] carry over mutatis mutandis. The above results also hold, mutatis mutandis, for finitely totally convex algebras. For instance, to obtain the analogue of (1.10) for finitely totally convex algebras one has to use 4.6 in [8].

Finally, we present the following interesting example. For a semigroup S put

$$U := \hat{\mathrm{O}}(\mathrm{l}_1^s(S)) \setminus \{\ \varepsilon\delta_s \mid s \in S,\ \varepsilon \in \mathbb{K},\ |\varepsilon| = 1\}$$

for the **TC**-algebra $\hat{\mathrm{O}}(\mathrm{l}_1^s(S))$, (cp. (0.1)). The partition

$$\hat{\mathrm{O}}(\mathrm{l}_1^s(S)) = U \cup \bigcup \{\{\ \varepsilon\delta_s\} \mid s \in S,\ \varepsilon \in \mathbb{K},\ |\varepsilon| = 1\}$$

induces an equivalence relation $\sim$ on $\hat{\mathrm{O}}(\mathrm{l}_1^s(S))$. $\sim$ is even a congruence relation on $\hat{\mathrm{O}}(\mathrm{l}_1^s(S))$ regarded as a totally convex space. The proof for this rests on the fact that the $\varepsilon\delta_s, s \in S$, $|\varepsilon| = 1$, are exactly the extremal points of the totally convex space $\hat{\mathrm{O}}(\mathrm{l}_1^s(S))$ (cp. [10]). The following lemma gives a sufficient condition for $\sim$ to be an ideal.

(1.11) Lemma. *If S is a regular semigroup, then $\sim$ is an ideal.*

Proof: A semigroup is called *regular*, if, for any $a \in S$, $ax = ay$ implies $x = y$ and $xa = ya$ implies $x = y$.

We have to show that, for $u, v, w \in \hat{\mathrm{O}}(\mathrm{l}_1^s(S))$, $u \sim v$ implies $wu \sim wv$ and $uw \sim vw$. The implication is trivial for $u = v$, hence we assume $u \neq v$, i.e. $u, v \in U$.

First, we assume $w \notin U$, i.e. $w = \varepsilon_o\delta_{s_o}$, $|\varepsilon_o| = 1$, $s_o \in S$. One has a representation $u = \sum\limits_{i=1}^{\infty} \alpha_i\delta_{s_i}$ with $s_i \neq s_j$ for $i \neq j$. $wu \notin U$ would imply

$$\sum_{i=1}^{\infty} \alpha_i\varepsilon_o\delta_{s_o s_i} = \varepsilon\delta_t$$

with suitable $t \in S$, $\varepsilon \in \mathbb{K}$, $|\varepsilon| = 1$, which implies $\varepsilon_o\delta_{s_o s_i} = \varepsilon\delta_t$ for $i \in \mathbb{N}$ with $\alpha_i \neq o$. Hence $s_o s_i = s_o s_j$ for $i, j \in \mathbb{N}$ with $\alpha_i\alpha_j \neq o$, i.e. $s_i = s_j$, because S is regular. Hence, there is exactly one $i \in \mathbb{N}$ with $\alpha_i \neq o$ and we get $u = \alpha_i\delta_{s_i}$, which leads to the contradiction $u \notin U$, because $\|u\| = 1$ holds. Analogously one proves that $u \in U$ implies $uw \in U$ for every $w \notin U$.

Now, take $w \in U$ and assume, for the sake of simplicity, that $\alpha_i \neq o$ for every i. If $wu \notin U$, then, as above,

$$\sum_i \alpha_i w\delta_{s_i} = \varepsilon\delta_t$$

with suitable ε and $t \in S$, $|\varepsilon| = 1$, which implies $\rho_i \omega \delta_{s_i} = \varepsilon \delta_t \notin U$ for at least one $i, \rho_i := \alpha_i |\alpha_i|^{-1}$. But this is a contradiction to our first result and we get $wu \in U$. $uw \in U$ is proved analogously.

Thus we have proved that $\hat{\mathrm{O}}(\mathrm{l}_1^s(S))_{/\sim}$ is a **TC**-algebra. The same proof works, mutatis mutandis, for commutative semigroups and (commutative) monoids. If $\pi : \hat{\mathrm{O}}(\mathrm{l}_1^s(S)) \longrightarrow \hat{\mathrm{O}}(\mathrm{l}_1^s(S))_{/\sim}$ is the canonical projection, then the **TC**-algebra $\hat{\mathrm{O}}(\mathrm{l}_1^s/S)_{/\sim}$ has as elements 0 and the elements $\varepsilon\pi(\delta_s)$, $\varepsilon \in \mathbb{K}$, $|\varepsilon| = 1$, $s \in S$. And an equation

$$\sum_i \alpha_i \pi(x_i) = \varepsilon \pi(\delta_s),$$

$(\alpha_i \mid i \in I) \in \Omega$, holds, if for every i with $\alpha_i \neq 0$ $\pi(x_i) = \varepsilon_i \pi(\delta_s)$ and $\sum_i \alpha_i \varepsilon_i = \varepsilon$. Moreover, the product is given by

$$\varepsilon_1 \pi(\delta_{s_1}) \varepsilon_1 \pi(\delta_{s_2}) = \varepsilon_1 \varepsilon_2 \pi(\delta_{s_1 s_2})$$

and all other products are 0. □

If one applies this, for a set $X \neq \phi$, to the free semigroup $\mathrm{F}^s(X)$ generated by X, the quotient $\hat{\mathrm{O}}(\mathrm{l}_1^a(X))_{/\sim}$ is called the *totally convex algebra of monomials* in X and is denoted by $\mathbb{K}\mathbf{M}\{X\}$. The analogous construction for abelian semigroups yields the commutative **TC**-algebra $\mathbb{K}\mathbf{M}[X]$ *of commuting monomials* in X. If one considers the case of monoids resp. abelian monoids one gets the *unital* **TC**-algebra $\mathbb{K}\mathbf{M}_u\{X\}$ resp. the *commutative, unital* **TC**-algebra $\mathbb{K}\mathbf{M}_u[X]$ *of monomials* in X. For a group the above construction yields a **TC**-field.

The above results remain valid for finitely totally convex algebras, mutatis mutandis.

2. The tensor product.

The tensor product of totally convex spaces, which is the canonical generalization of the projective tensor product of Banach spaces, was introduced in [6, (5.3)]. As in the classical case this tensor product in **TC** induces a tensor product of **TC**-algebras.

(2.1) Proposition. *Let A, B be* **TC***-algebras. Then:*

(i) *There is a unique multiplication in $A \otimes B$, which makes $A \otimes B$ a* **TC***-algebra, s.th. $(a_0 \otimes b_0)(a_1 \otimes b_1) = (a_0 a_1) \otimes (b_0 b_1)$, $a_i \in A$, $b_i \in B$, $i = 0, 1$. This* **TC***-algebra is again denoted by $A \otimes B$ and called the tensor product of A and B.*

(ii) *If A and B are commutative, resp. unital resp. *-***TC***-algebras, then so is $A \otimes B$.*

(iii) *If A and B are unital with unit elements e_A resp. e_B, then the mappings $i_A : A \longrightarrow A \otimes B$, $i_B : B \longrightarrow A \otimes B$, defined by $i_A(a) := a \otimes e_B$, $i_B(b) := e_A \otimes b$, $a \in A$, $b \in B$, are unital* **TC***-algebra morphisms and the subalgebras $i_A(A)$ and $i_B(B)$ of $A \otimes B$ commute elementwise.*

Proof: (i): To simplify the notation, we will often denote the underlying **TC**-space $|A|$ of a **TC**-algebra also by A. With the canonical isomorphism $\lambda_{A,B,A,B} : (A \otimes B) \otimes (A \otimes B) \longrightarrow (A \otimes A) \otimes (B \otimes B)$ we put $\mu := (\mu_A \otimes \mu_B)\lambda_{A,B,A,B}$. $\mu_A : A \otimes A \longrightarrow A$ resp. $\mu_B : B \otimes B \longrightarrow B$ is the multiplication of A resp. B. It is easily verified that μ fulfills (i).

(ii): In the commutative resp. unital case the assertion follows immediately from the definition of μ. For *-**TC**-algebras A, B, the easiest way to turn the **TC**-algebra $A \otimes B$ into a *-**TC**-algebra is as follows (oral communication by R. Börger). For a **TC**-algebra A define another **TC**-algebra A^{op} on the same underlying set by putting

$$(*) \qquad \sum_{i=1}^{\infty}{}' \alpha_i a_i := \sum_{i=1}^{\infty} \bar{\alpha_i} a_i,$$

for $a_i \in A$, $i \in \mathbb{N}$, and $(\alpha_i \mid i \in \mathbb{N}) \in \Omega$ (cp. [6, §2]). The left side of $(*)$ denotes the effect of $(\alpha_i \mid i \in \mathbb{N})$ on the "new" totally convex space $|A^{op}|$, the right side is the operation of $(\bar{\alpha_i} \mid i \in \mathbb{N})$ on A, where $\bar{\alpha_i}$ is the complex conjugate of $\alpha_i \in \mathbb{K}$, $i \in \mathbb{N}$. $|A^{op}|$ becomes a **TC**-algebra A^{op}, if one defines $\mu_{A^{op}} : |A^{op}| \otimes |A^{op}| \longrightarrow |A^{op}|$ by $\mu_{A^{op}}(a \otimes b) := \mu_A(b \otimes a)$.

A *-**TC**-algebra A is then a **TC**-algebra A together with a morphism $s_A : A \longrightarrow A^{op}$ in **TC**-Alg, which is an involution. It is customary to write $a^* := s_A(a)$ for $a \in A$. One may identify $(A \otimes B)^{op} = A^{op} \otimes B^{op}$ for **TC**-algebras A, B. Now, it is routine to check that, for *-**TC**-algebras A, B, $s_A \otimes s_B : A \otimes B \longrightarrow A^{op} \otimes B^{op}$ is again a *-**TC**-algebra.

(iii) is obvious. □

(2.2) Proposition. *The tensor product of two unital, commutative* **TC***-algebras* A_0, A_1, *resp. of two unital, commutative* *-**TC***-algebras is the coproduct in the category* $\mathbf{TC}^{cu}$**-Alg** *resp. in the category* $\mathbf{TC}^{cu*}$**-Alg** *of unital, commutative* *-**TC***-algebras. The injections* $i_\nu : A_\nu \longrightarrow A_0 \otimes A_1$, $\nu = 0, 1$, *of the coproduct are given by* $i_0(a_0) := a_0 \otimes e_1$, $i_1(a_1) := e_0 \otimes a_1$, *with the unit element* $e_\nu \in A_\nu$ *and* $a_\nu \in A_\nu$, $\nu = 0, 1$.

The proof is carried out in complete analogy to the proof in the classical case of real or complex algebras. (2.2) gives a description of finite coproducts with the tensor product in the two subcategories. The universal property expressed in (2.2) remains valid for not necessarily commutative, unital **TC**- and ***TC**-algebras relative to morphisms $f_\nu : A_\nu \longrightarrow X$, $\nu = 0, 1$, satisfying $f_1(a_1)f_2(a_2) = f_2(a_2)f_1(a_1)$ for any $a_\nu \in A_\nu$, $\nu = 0, 1$.

The *infinite tensor product* of unital **TC**-algebras can be described, as in the classical case, as the inductive limit of the finite partial tensor products. This infinite tensor product has the usual properties and yields the infinite coproduct in the category of unital, commutative **TC**-algebras resp. *-**TC**-algebras (cp. [2]). The investigation of the structure of coproducts of non-commutative or non-unital **TC**-algebras might meet considerable difficulties in view of the difficulties one has with the structure of the coproduct of general $\mathbb{K}$-algebras. In $\mathbf{TC}_{fin}$-**Alg** all the results of this section hold, mutatis mutandis.

As $\mathrm{S} : \mathbf{TC} \longrightarrow \mathbf{Ban_1}$ is left adjoint to $\hat{\mathrm{O}} : \mathbf{Ban_1} \longrightarrow \mathbf{TC}$ ([6]), it preserves the tensor product of totally convex spaces. In [10, (1.1)], it was shown that the (lifted) open unit ball functor $\tilde{\mathrm{O}} : \mathbf{Ban_1} \longrightarrow \mathbf{TC}$ is left adjoint to S, hence $\tilde{\mathrm{O}}$ preserves the (projective) tensor product in $\mathbf{Ban_1}$. Both statements remain true for **TC**-algebras. The proof, however, is different, because the tensor product in $\mathbf{Ban_1}$-**Alg** as well as in **TC-Alg** is no longer a left adjoint functor.

(2.3) Proposition. $\mathrm{S} : \mathbf{TC}\text{-}\mathbf{Alg} \longrightarrow \mathbf{Ban_1}\text{-}\mathbf{Alg}$ *and* $\tilde{\mathrm{O}} : \mathbf{Ban_1}\text{-}\mathbf{Alg} \longrightarrow \mathbf{TC}\text{-}\mathbf{Alg}$, *the open unit ball functor, preserve finite tensor products.*

Proof: For $C, D \in \mathbf{TC}\text{-}\mathbf{Alg}$ define the morphism $\lambda : \mathrm{S}(C)\otimes\mathrm{S}(D) \longrightarrow \mathrm{S}(C \otimes D)$ in $\mathbf{Ban_1}$-**Alg** by $\lambda(\sigma_C(c) \otimes \sigma_D(d)) : = \sigma_{C\otimes D}(c \otimes d)$. Conversely, the bi-morphism $\beta : C \times D \longrightarrow \hat{\mathrm{O}}(\mathrm{S}(C)\otimes\mathrm{S}(D))$, given by $\beta(c, d) := \sigma_C(c) \otimes \sigma_D(d)$, induces a **TC**-morphism $\kappa : C \otimes D \longrightarrow \hat{\mathrm{O}}(\mathrm{S}(C)\otimes\mathrm{S}(D))$ and this, in turn, a $\mathbf{Ban_1}$-morphism $\varphi : \mathrm{S}(C \otimes D) \longrightarrow \mathrm{S}(C)\otimes\mathrm{S}(D)$. A routine computation shows φ to be the set-theoretical inverse of λ, i.e. λ is an isomorphism in $\mathbf{Ban_1}$-**Alg**.

For $A \in \mathbf{Ban_1}\text{-}\mathbf{Alg}$, $A \cong \mathrm{S}(\tilde{\mathrm{O}}(A))$ holds in $\mathbf{Ban_1}$-**Alg**. Hence, for $A, B \in \mathbf{Ban_1}$-**Alg**, we get with the first assertion

$$A \otimes B \cong \mathrm{S}(\tilde{\mathrm{O}}(A)) \otimes \mathrm{S}(\tilde{\mathrm{O}}(B)) \cong \mathrm{S}(\tilde{\mathrm{O}}(A) \otimes \tilde{\mathrm{O}}(B)),$$

or, with the analogue of [10, (1.1)],

$$\tilde{\mathrm{O}}(A \otimes B) \cong \tilde{\mathrm{O}}(A) \otimes \tilde{\mathrm{O}}(B),$$

i.e. the second assertion. □

As isomorphisms in $\mathbf{Ban_1}$-**Alg** automatically preserve unit elements, S also preserves the tensor product of unital **TC**-algebras. Moreover, an analysis of the above proof shows that S preserves finite tensor products of *-**TC**-algebras and $\tilde{\mathrm{O}}$ finite tensor products of $\mathbf{Ban_1^*}$-**Alg**. As left adjoint functors, S and $\tilde{\mathrm{O}}$ preserve infinite tensor products in all cases, in which an infinite tensor product is an inductive limit of finite tensor products.

3. Strongly aspherical algebras.

In [6, §6], a norm was defined for all elements of a totally convex space X. It satisfies the inequality $\|\alpha x\| \leq |\alpha|\|x\|$ for all $\alpha \in \mathrm{O}(\mathbb{K})$, $x \in X$, with the equality attained for $|\alpha| = 1$ or $\|x\| < 1$. There are examples of totally convex spaces, in which the above inequality is strict for some elements x and all α with $|\alpha| < 1$ (cp. [10, §5]). Totally convex spaces, for which $\|\alpha x\| = |\alpha|\|x\|$ for all $\alpha \in \mathrm{O}(\mathbb{K})$ and all $x \in X$ are called *normed* (cp. [7, §13]). We will show here that the above inequality can be strengthened to an equality for arbitrary totally convex spaces.

(3.1) Proposition. *Let X be a totally convex space. For $x \in X$, $x \neq 0$, put*

$$s(x) := \frac{\|\sigma_X(x)\|}{\|x\|}.$$

Then $s(x)$ *is the unique real number,* $0 \leq s(x) \leq 1$*, such that for any* α *with* $0 \leq |\alpha| < 1$ *and* $x \in X$*,* $x \neq 0$*,*

$$\|\alpha x\| = |\alpha| s(x) \|x\|.$$

Proof: Let $x \in X$, $x \neq 0$ and $\|x\| < 1$. Then, because σ_X restricted to the interior $\mathring{X}$ is an isometry, we have $s(x) = 1$ and $\|\alpha x\| = |\alpha| \|x\|$. For the same reason, for any $x \in X$ and $0 \leq |\alpha| < 1$,

$$\|\alpha x\| = \|\sigma_X(\alpha x)\| = |\alpha| \|\sigma_X(x)\|,$$

hence, for $x \neq 0$,

$$\|\alpha x\| = |\alpha| s(x) \|x\|.$$

To show uniqueness, let $t : X \backslash \{0\} \longrightarrow [0,1]$ be a mapping with $\|\alpha x\| = |\alpha| t(x) \|x\|$ for $0 \leq |\alpha| < 1$, $x \in X$, $x \neq 0$. Then, for any $x \neq 0$,

$$t(x) = \frac{\|\frac{1}{2}x\|}{\frac{1}{2}\|x\|} = s(x).$$

□

In order to have s defined on all of X, we put $s(0) := 1$. For a totally convex space X, the subspace $\mathrm{T}_s(X) < X$ of all spherical elements of X was introduced in [7, 12.5]. If $x \in \mathrm{T}_s(X)$, $x \neq 0$, then $\frac{1}{2}x = 0$ and

$$s(x) = \frac{\|\sigma_X(x)\|}{\|x\|} = \frac{2 \cdot 2^{-1} \|\sigma_X(x)\|}{\|x\|} = 2 \frac{\|\sigma_X(2^{-1}x)\|}{\|x\|} = 0.$$

Conversely, $s(x) = 0$ implies $\|\frac{1}{2}x\| = 0$, i.e. $\frac{1}{2}x = 0$ resp. $x \in \mathrm{T}_s(X)$. Hence,

$$\mathrm{T}_s(X) = s^{-1}(\{0\}) \cup \{0\} \tag{3.2}$$

holds.

(3.3) Corollary. *Let* A *be a totally convex algebra. Then, for* $a, b \in A$ *with* $ab \neq 0$

$$s(ab)\|ab\| \leq s(a)s(b)\|a\|\|b\|$$

holds. In particular, $\mathrm{T}_s(A)$ *is a subalgebra, even a left- and right-*A **TC***-space.*

Proof: One has

$$\begin{aligned} s(ab)\|ab\| &= \frac{\|\sigma_A(ab)\|}{\|ab\|}\|ab\| = \|\sigma_A(a)\sigma_A(b)\| \\ &\leq \|\sigma_A(a)\|\|\sigma_A(b)\| = s(a)s(b)\|a\|\|b\|. \end{aligned}$$

□

(3.2) shows that a totally convex space X is aspherical, if $s(X)$ does not contain 0 or, equivalently, $\sigma_X(\partial(X))$ does not contain 0 for the boundary $\partial(X) = \{x \mid \|x\| = 1\}$. This leads to the following stronger notion (cp. [12, 3.5]).

(3.4) Definition. For a totally convex space X

$$\eta_X := inf\{s(x) \mid x \in X\}$$

is called the *norm factor* of X, $0 \leq \eta_X \leq 1$. X is called *strongly aspherical*, if $\eta_X > 0$. Obviously X is normed, if and only if $\eta_X = 1$.

This notion is a natural generalization of the norm quotient in [12, 3.5], and a special case of the following concept.

(3.5) Definition. For a morphism $f : X \longrightarrow Y$ of totally convex spaces we define the *norm factor of* f

$$\eta(f) := inf\left\{\frac{\|f(x)\|}{\|x\|} \mid x \in X,\ x \neq 0\right\}.$$

f is called *homometric*, if $\eta(f) > 0$ holds.

Obviously, a totally convex space X is strongly aspherical, iff $\sigma_X : X \longrightarrow \hat{\mathrm{O}}(\mathrm{S}(X))$ is homometric and, in this case, $\eta_X = \eta(\sigma_X)$. Moreover, because $s(x) = 1$ for $\|x\| < 1$, it suffices to verify $\|\sigma_X(x)\| \geq \eta > 0$ for some $\eta > 0$ and all $x \in X$ with $\|x\| = 1$ in order to see that X is strongly aspherical. For $|\alpha| < 1$ one defines $\hat{\alpha} : X \longrightarrow X$ by $\hat{\alpha}(x) = \alpha x$. Then $\hat{\alpha}$ is a **TC**-morphism and $\eta(\hat{\alpha}) = |\alpha|\eta_X$. And a subspace Y of X is norm-equivalent in the sense of [12, 3.5], iff the inclusion in: $Y \longrightarrow X$ is homometric.

(3.6) Proposition. *For a totally convex space X the following implications hold:*

(i) *If X is normed, then it is strongly aspherical.*
(ii) *If X is strongly aspherical, then it is aspherical.*

PROOF: Obvious. □

(3.7) Proposition. *Let A be a strongly aspherical* **TC***-algebra. If $a, b \in A$ satisfy $s(a)s(b) < \eta_A$ then $\|ab\| < \eta_A$.*

PROOF: For $\|ab\| = 0$ the assertion is trivial. Hence, assume $\|ab\| \neq 0$. Then (3.3) implies

$$s(ab)\|ab\| \leq s(a)s(b)\|a\|\|b\| \leq s(a)s(b) < \eta_A.$$

$\|ab\| = 1$ would lead to the contradiction $\eta_A \leq s(ab) < \eta_A$, hence we get $\|ab\| < 1$, i.e. $s(ab) = 1$ and $\|ab\| < \eta_A$. □

For finitely totally convex spaces (3.1) is valid, too, but one has to give a different proof, because, for a finitely totally convex space X, σ_X restricted to the interior $\mathring{X}$ is, in general, not an isometry. For regular finitely totally convex X (i.e. for any $x \in X$ $\|x\| = 0$ implies $x = 0$, cp. [7, (13.5)]), however, all proofs remain valid verbatim, because $\sigma_X/\mathring{X}$ is an isometry, i.e. 1.5 in [9] holds.

(3.8) Proposition. *Let X be a finitely totally convex space. Then, for every $x \in X$ with $\|x\| \neq 0$ there is a unique $s(x)$, $0 \leq s(x) \leq 1$, s.th. for any α with $0 \leq |\alpha| < 1$*

$$\|\alpha x\| = |\alpha| s(x) \|x\|$$

holds.

PROOF: The uniqueness statement is shown as in (3.1). If, for $x \in X$, $\|x\| \neq 0$, $\|\alpha x\| = |\alpha| \|x\|$ for any α with $|\alpha| \leq 1$, then put $s(x) := 1$. If $\|\alpha x\| < |\alpha| \|x\|$ for some α, for every λ, with $\|\alpha x\| < \lambda < |\alpha|$, there is a $y \in X$ with $\alpha x = \lambda y = \alpha(\lambda \alpha^{-1} y)$. Due to [6, (4.1)], this implies $\gamma x = \gamma(\lambda \alpha^{-1} y)$ for all γ with $0 \leq |\gamma| < 1$, or

$$\|\gamma x\| \leq |\gamma| \frac{\lambda}{|\alpha|}$$

and thus

$$\|\gamma x\| \leq |\gamma| \frac{\|\alpha x\|}{|\alpha|} < |\gamma| \|x\|,$$

for $\gamma \neq 0$. This means that we may interchange the roles of α and γ and get

$$|\alpha| \|\gamma x\| = |\gamma| \|\alpha x\|.$$

Therefore

$$s(x) := \frac{\|\alpha x\|}{|\alpha| \|x\|}$$

satisfies (3.8). □

Again one defines $s(0) := 1$. The other results of this section carry over to the finitely totally convex case, mutatis mutandis. For example, the characterization of $\mathrm{T}_s(X)$ for finitely totally convex X is

$$\mathrm{T}_s(X) = \mathrm{N}(X) \cup s^{-1}(\{0\})$$

(cp. [7, (14.10)]) and identical to (3.2), if X is regular. And the characterization of the interior $\mathring{X}$ by s is given in the

(3.9) Corollary. *Let X be a finitely totally convex space. Then, for $x \in X$ with $\|x\| < 1$, $s(x) = 1$ holds.*

PROOF: For β with $\|x\| < \beta < 1$, we have $x = \beta y$ with some $y \in X$. As $\|y\| \neq 0$, we get for any α with $0 < |\alpha| < 1$,

$$\|\alpha x\| = \|\alpha \beta y\| = |\alpha| \|\beta y\| = |\alpha| \|x\|.$$

hence $s(x) = 1$. □

4. Unitization.

The unit element e of a unital **TC**-algebra A is unique. If $A \neq \{0\}$, then $\|e\| = 1$ holds. For, assume $\|e\| < 1$, then, for any $n \in \mathbb{N}$, $\|e\| = \|e^n\| \leq \|e\|^n$ follows, which implies $\|e\| = 0$, i.e. $e = 0$ (cp. [6, (6.9)]) and hence the contradiction $A = \{0\}$. The unit element of a **TC**-algebra may be spherical, as it is shown by the example of the Linton algebra $\mathrm{L}(\mathbb{K})$ in (0.5). Obviously, e is spherical, iff A is spherical (see also (5.2)).

(4.1) Proposition. *Let A be a unital, not spherical* **TC**-*algebra, then $A \neq \{0\}$. Moreover, $u \in A$ satisfies $\alpha u = \alpha e$ for some $\alpha \in \mathbb{K}$ with $0 < |\alpha| < 1$, if and only if u acts as a unit element on $\mathring{A}$. If $\mathring{A} \neq \{0\}$, then such an element u satisfies $\|u\| = 1$.*

Proof: For $a \in \mathring{A}$ there are $b \in A$ and $\|a\| < \beta < 1$ with $a = \beta b$. Moreover, $\alpha u = \alpha e$ implies $\gamma u = \gamma e$ for all γ with $|\gamma| < 1$ ([6, (4.1)]). Hence,

$$a = \beta b = (\beta e)b = (\beta u)b = u(\beta b) = ua.$$

Similarly, we obtain $au = a$. The last assertion follows from (1.4). □

$\mathbf{TC}^u$**-Alg** is a subcategory of **TC-Alg**, which is not full (cp. (0.7)). The embedding functor will be denoted by $\mathrm{E} : \mathbf{TC}^u\text{-}\mathbf{Alg} \hookrightarrow \mathbf{TC}\text{-}\mathbf{Alg}$.

(4.2) Theorem. *E has a left adjoint* $\mathrm{U} : \mathbf{TC}\text{-}\mathbf{Alg} \longrightarrow \mathbf{TC}^u\text{-}\mathbf{Alg}$, *called the unitization functor.*

Proof: One possible proof consists of a simple application of the Adjoint Functor Theorem. However, we wish to give a more explicit description of U, hence we construct $\mathrm{U}(A), A \in \mathbf{TC}\text{-}\mathbf{Alg}$, with the coproduct of totally convex spaces.

For $A \in \mathbf{TC}\text{-}\mathbf{Alg}$, let $j_{\mathbb{K}} : |\hat{\mathrm{O}}(\mathbb{K})| \longrightarrow |\hat{\mathrm{O}}(\mathbb{K})| \amalg |A|$ and $j_A : |A| \longrightarrow |\hat{\mathrm{O}}(\mathbb{K})| \amalg |A|$ denote the canonical injections into the coproduct $|\hat{\mathrm{O}}(\mathbb{K})| \amalg |A|$ of the underlying totally convex spaces (cp. (0.3) and [9]). Given $a \in A$, we define a morphism in **TC**

$$g(a) : |\hat{\mathrm{O}}(\mathbb{K})| \amalg |A| \longrightarrow |\hat{\mathrm{O}}(\mathbb{K})| \amalg |A|$$

by the equations $g(a)(j_{\mathbb{K}}(\rho)) := j_A(\rho a),\ g(a)(j_A(b)) := j_A(ab)$ for $\rho \in \hat{\mathrm{O}}(\mathbb{K}),\ b \in A$. A routine computation (cp. [6, §5]) shows that this defines a **TC** morphism

$$g : |A| \longrightarrow \mathrm{Hom}(|\hat{\mathrm{O}}(\mathbb{K})| \amalg |A|, |\hat{\mathrm{O}}(\mathbb{K})| \amalg |A|).$$

Moreover,

$$f : |\hat{\mathrm{O}}(\mathbb{K})| \longrightarrow \mathrm{Hom}(|\hat{\mathrm{O}}(\mathbb{K})| \amalg |A|, |\hat{\mathrm{O}}(\mathbb{K})| \amalg |A|),$$

defined by $f(\lambda)(x) := \lambda x$ is trivially a morphism in **TC**. The equations $\hat{\mu} j_{\mathbb{K}} = f,\ \hat{\mu} j_A = g$ define a morphism

$$\hat{\mu} : |\hat{\mathrm{O}}(\mathbb{K})| \amalg |A| \longrightarrow \mathrm{Hom}(|\hat{\mathrm{O}}(\mathbb{K})| \amalg |A|, |\hat{\mathrm{O}}(\mathbb{K})| \amalg |A|)$$

in **TC**. Via the canonical adjunction isomorphism between tensor product and internal hom-functor (cp. [6, §5]), we get a morphism

$$\mu : (|\hat{\mathrm{O}}(\mathbb{K})| \amalg |A|) \otimes (|\hat{\mathrm{O}}(\mathbb{K})| \amalg |A|) \longrightarrow |\hat{\mathrm{O}}(\mathbb{K})| \amalg |A|$$

in **TC**, which makes $|\hat{\mathrm{O}}(\mathbb{K})| \amalg |A|$ a **TC**-algebra $\mathrm{U}(A)$. For $x, y \in |\hat{\mathrm{O}}(\mathbb{K})| \amalg |A|$ one has explicitly $\mu(x \otimes y) = \hat{\mu}(x)(y)$. As any element x of $|\hat{\mathrm{O}}(\mathbb{K})| \amalg |A|$ can be written in the form $x = \alpha j_{\mathbb{K}}(\lambda) + \beta j_A(a)$, where $\alpha\lambda$ and βa are uniquely determined by x (cp. [9]), we get for the product of two elements, which we denote as usual by juxtaposition:

$$
\begin{aligned}
&(\alpha j_{\mathbb{K}}(\lambda) + \beta j_A(a))(\alpha' j_{\mathbb{K}}(\rho) + \beta' j_A(b)) \\
=&\mu((\alpha j_{\mathbb{K}}(\lambda) + \beta j_A(a)) \otimes (\alpha' j_{\mathbb{K}}(\rho) + \beta' j_A(b))) \\
=&\hat{\mu}((\alpha j_{\mathbb{K}}(\lambda) + \beta j_A(a))(\alpha' j_{\mathbb{K}}(\rho) + \beta' j_A(b))) \\
=&\alpha\hat{\mu}(j_{\mathbb{K}}(\lambda))(\alpha' j_{\mathbb{K}}(\rho) + \beta' j_A(b)) + \beta\hat{\mu}(j_A(a))(\alpha' j_{\mathbb{K}}(\rho) + \beta' j_A(b)) \\
=&\alpha f(\lambda)(\alpha' j_{\mathbb{K}}(\rho) + \beta' j_A(b)) + \beta g(a)(\alpha' j_{\mathbb{K}}(\rho) + \beta' j_A(b)) \\
=&\alpha\alpha'\lambda j_{\mathbb{K}}(\rho) + \alpha\beta'\lambda j_A(b) + \alpha'\beta j_A(\rho a) + \beta\beta' j_A(ab).
\end{aligned}
\tag{4.3}
$$

This shows that $\mathrm{U}(A)$ is a unital **TC**-algebra with unit element $e := j_{\mathbb{K}}(1)$ and that $j_A : A \longrightarrow \mathrm{U}(A)$ is a morphism of **TC**-algebras. In order to see that $\mathrm{U}(A)$ induces a left adjoint of E resp. a reflection of **TC-Alg** into **TC^u-Alg**, we prove that j_A is a reflection morphism. Let ψ be a **TC-Alg** morphism and $C \in$ **TC^u-Alg** in the diagram $(*)$. The **TC** morphism $\hat{\psi} : |\hat{\mathrm{O}}(\mathbb{K})| \amalg |A| \longrightarrow |C|$ is defined by the equations $\hat{\psi} j_{\mathbb{K}} = \tau_C, \hat{\psi} j_A = \psi$, where $\tau_C : |\hat{\mathrm{O}}(\mathbb{K})| \longrightarrow |C|$ is given by $\tau_C(\lambda) := \lambda e_C, e_C$ the unit element of C. We get $\hat{\psi}(e) = \hat{\psi}(j_{\mathbb{K}}(1)) = \tau_C(1) = e_C$, i.e. $\hat{\psi}$ preserves the unit element. As ψ preserves the product, (4.3) immediately implies that the same holds for $\hat{\psi}$, i.e. $\hat{\psi}$ is a **TC^u-Alg** morphism making $(*)$ commutative. Obviously $\hat{\psi}$ is uniquely determined by ψ and $\hat{\psi} j_A = \psi$, because any **TC^u-Alg** morphism must satisfy $\hat{\psi} j_{\mathbb{K}} = \tau_C$. Hence, (4.2) is proved. □

$(*)$

$$
\begin{tikzcd}
A \arrow[r, "j_A"] \arrow[dr, "\psi"'] & \mathrm{E}(\mathrm{U}(A)) \arrow[d, dotted, "\mathrm{E}(\hat{\psi})"] \\
& \mathrm{E}(C)
\end{tikzcd}
$$

U describes the universal method of adjoining a unit element to a **TC**-algebra. Its construction is analogous to the one used in classical algebra theory.

(4.4) Corollary. *For any* **TC**-*algebra the following hold:*

(i) $\mathrm{U}(A)$ *is not spherical.*

(ii) $j_A : A \longrightarrow \mathrm{E}(\mathrm{U}(A))$ *is an injective isometry.*

(iii) *Every element of* $\mathrm{U}(A)$ *can be written as* $\lambda e + \alpha j_A(a)$*, for some* $a \in A$, $\lambda \in \mathrm{O}(\mathbb{K})$ *and* $|\alpha| + |\lambda| \leq 1$. *Both* λ *and* αa *are uniquely determined by* $\lambda e + \alpha j_A(a)$.

Proof: (i): $\alpha e = 0$ implies $\alpha j_{\mathbb{K}}(1) = j_{\mathbb{K}}(\alpha) = 0$, i.e. $\alpha = 0$. (ii) is proved in [9, (2.1), (ii)], and (iii) follows from [9, (2.1), (i)], by applying the canonical projections of the coproduct. □

For separated totally convex spaces the direct sum is the coproduct ([9, (2.7)]). Hence, for a separated **TC**-algebra A we have, up to isomorphism,

$$\mathrm{U}(A) = \{(\lambda, \alpha a) \mid |\lambda| + |\alpha| \leq 1, a \in A\}$$

with the product

$$(\lambda_1, \alpha_1 a_1)(\lambda_2, \alpha_2 a_2) = (\lambda_1\lambda_2, \lambda_1\alpha_2 a_2 + \alpha_1\lambda_2 a_1 + \alpha_1\alpha_2 a_1 a_2).$$

In particular, for a Banach algebra B, we get

$$\mathrm{U}(\hat{\mathrm{O}}(B)) = \{(\lambda, b) \mid (\lambda, b) \in \mathbb{K} \times B, |\lambda| + \|b\| \leq 1\}.$$

The construction of U implies that, for a morphism $f : A \longrightarrow B$ in **TC-Alg**, $\mathrm{U}(f) = |\hat{\mathrm{O}}(\mathbb{K})| \amalg f$ holds. Hence, we get the

(4.5) Corollary. U *preserves monomorphisms. In particular, for a separated* **TC**-*algebra* A, $\mathrm{U}(\sigma_A) : \mathrm{U}(A) \longrightarrow \mathrm{U}(\hat{\mathrm{O}} \circ \mathrm{S}(A))$ *is injective.*

Proof: Let $m : A \longrightarrow B$ be a monomorphism in **TC-Alg**, then $\mathrm{U}(m)(\lambda e + \alpha j_A(a)) = \lambda e + \alpha j_B(m(a))$. Hence $\mathrm{U}(m)(\lambda e + j_A(a)) = 0$ implies $\lambda e = 0$ and $\alpha j_B(m(a)) = j_B(m(\alpha a)) = 0$, i.e. $\lambda = 0$ and $\alpha a = 0$, resp. $\lambda e + \alpha j_A(a) = 0$. □

As has been mentioned before, the unitization functor U is the analogue of the unitization functor in classical algebra theory. We are now going to give an exact description of the connection between both functors. The unitization functor of Banach algebras, which we denote by $\mathrm{U}_0 : \mathbf{Ban}_1\text{-}\mathbf{Alg} \longrightarrow \mathbf{Ban}_1^u\text{-}\mathbf{Alg}$ is left adjoint to the canonical embedding $\mathrm{E}_0 : \mathbf{Ban}_1^u\text{-}\mathbf{Alg} \longrightarrow \mathbf{Ban}_1\text{-}\mathbf{Alg}$. For a Banach algebra B, one has $|\mathrm{U}_0(B)| = \mathbb{K} \oplus |B|$, where $\oplus$ denotes the direct sum resp. l_1-sum of Banach spaces (the coproduct in $\mathbf{Ban}_1$). $e_0 = \mu_{\mathbb{K}}(1)$ will turn out to be the unit element of $\mathrm{U}_0(B)$ and every element of $|\mathrm{U}_0(B)|$ has a unique representation $\lambda e_0 + \alpha\mu_B(b)$, $\lambda, \alpha \in \mathbb{K}$, $b \in B$, where $\mu_{\mathbb{K}} : \mathbb{K} \longrightarrow |\mathrm{U}_0(B)|$, $\mu_B : |B| \longrightarrow |\mathrm{U}_0(B)|$ are the canonical injections. $|\mathrm{U}_0(B)|$ becomes a unital Banach algebra $\mathrm{U}_0(B)$ with the product

$$\begin{aligned}
&(\lambda_1 e_0 + \alpha_1\mu_B(b_1))(\lambda_2 e_0 + \alpha_2\mu_B(b_2)) \\
&= \lambda_1\lambda_2 e_0 + \lambda_1\alpha_2\mu_B(b_2) + \alpha_1\lambda_2\mu_B(b_1) + \alpha_1\alpha_2\mu_B(b_1 b_2) \\
&= \lambda_1\lambda_2 e_0 + \mu_B(\lambda_1\alpha_2 b_2 + \alpha_1\lambda_2 b_1 + \alpha_1\alpha_2 b_1 b_2).
\end{aligned}$$

$\mu_B : B \longrightarrow \mathrm{E}_0(\mathrm{U}_0(B))$ is the unit of the adjunction and an isometric injection.

For the moment, let us denote the comparison functor for unital Banach algebras by $\hat{\mathrm{O}}^u : \mathbf{Ban}_1^u\text{-}\mathbf{Alg} \longrightarrow \mathbf{TC}^u\text{-}\mathbf{Alg}$ (cp. (0.7)), its left adjoint by $\mathrm{S}^u : \mathbf{TC}^u\text{-}\mathbf{Alg} \longrightarrow \mathbf{Ban}_1^u\text{-}\mathbf{Alg}$ and the unit by $\sigma_C^u : C \longrightarrow \hat{\mathrm{O}}^u(\mathrm{S}^u(C))$, $C \in \mathbf{TC}^u\text{-}\mathbf{Alg}$.

(4.6) Theorem. *There is a natural isomorphism* $\mathrm{S}^u \circ \mathrm{U} \cong \mathrm{U}_0 \circ \mathrm{S}$.

PROOF: Obviously $\hat{\mathrm{O}} \circ \mathrm{E}_0 = \mathrm{E} \circ \hat{\mathrm{O}}^u$ holds. $\mathrm{S}^u \circ \mathrm{U}$ and $\mathrm{U}_0 \circ \mathrm{S}$ are both left adjoints of $\hat{\mathrm{O}} \circ \mathrm{E}_0$ hence must be naturally isomorphic. □

(4.7) Corollary. *For an element* $\lambda e + \alpha j_A(a) \in U(A), A$ *a* **TC**-*algebra, the following hold:*

(i) $\|\lambda e + \alpha j_A(a)\| < 1$ *implies* $\|\lambda e + \alpha j_A(a)\| = |\lambda| + \|\alpha a\|$;

(ii) $\|\lambda e + \alpha j_A(a)\| = 1$ *implies* $\|\lambda e + \alpha j_A(a)\| = |\lambda| + |\alpha|$ *and* $\|a\| = 1$, *provided* $\alpha \neq 0$.

PROOF: If $\tau : \mathrm{S}^u \circ \mathrm{U} \longrightarrow \mathrm{U}_0 \circ \mathrm{S}$ denotes the natural isomorphism of (4.6), the complete statement of (4.6) actually is

$$\hat{\mathrm{O}}(\mu_{S(A)})\sigma_A = (\hat{\mathrm{O}} \circ \mathrm{E}_0)(\tau_A)\mathrm{E}(\sigma^u_{U(A)})j_A.$$

(i): If $\lambda = 0$, then $\|\alpha a\| < 1$; if $\lambda \neq 0$, then, because of $|\lambda| + |\alpha| \leq 1,\ |\alpha| < 1$, i.e. also $\|\alpha a\| < 1$. Hence,

$$\begin{aligned} \|\lambda e + \alpha j_A(a)\| \quad\quad &= \|\sigma^u_{U(A)}(\lambda e + \alpha j_A(a))\| = \\ \|\lambda e_0 + \alpha\sigma^u_{U(A)} j_A(a)\| &= \|\lambda e_0 + \sigma^u_{U(A)} j_A(\alpha a)\| = \\ |\lambda| + \|\sigma^u_{U(A)} j_A(\alpha a)\| \ &= |\lambda| + \|\alpha a\|. \end{aligned}$$

(ii): $1 = \|\lambda e + \alpha j_A(a)\| \leq |\lambda| + |\alpha|\|\alpha\| \leq |\lambda| + |\alpha| \leq 1$, i.e. $\|\lambda e + \alpha j_A(a)\| = |\lambda| + |\alpha|$ and $\|a\| = 1$, if $\alpha \neq 0$.

5. Inverses.

As we are going to investigate inverses, all **TC**-algebras in this section will be assumed to be unital and the unit element of a unital **TC**-algebra A will be denoted by e_A or simply by e. For technical reasons all unital **TC**-algebras A in this section are assumed to be not trivial, i.e. $A \neq \{0\}$.

(5.1) Definition. Let A be a unital **TC**-algebra. $a \in A$ is called *invertible*, if there is a $b \in A$ with $ab = ba = e$; b is called an *inverse* of a. $a \in A$ is called *weakly invertible*, if there is a $b \in A$ and a $\rho \in \mathrm{O}(\mathbb{K})$, s.th. $ab = ba = \rho e \neq 0$; b is called a *weak inverse* of a. Similarly one defines *left* resp. *right* (*weakly*) *invertible elements* and *left* resp. *right* (*weak*) *inverses*.

As for the unit element e of a **TC**-algebra $A \neq \{0\}$ $\|e\| = 1$ holds, a left resp. right invertible element a of A clearly satisfies $\|a\| = 1$. Also, an element that is both left and right (weakly) invertible is (weakly) invertible. Finally the inverse of an element a is uniquely determined and as usual denoted by a^{-1}. Weak inverses of a are not uniquely determined in general: If b is a weak inverse of a, so is αb for $0 < |\alpha| \leq 1$.

The set of invertible (weakly invertible) elements of A forms a group (monoid) under the multiplication of A; it is denoted by $\mathrm{IN}(A)$ ($\mathrm{WIN}(A)$). Obviously $A \longmapsto \mathrm{IN}(A)$ ($A \longmapsto \mathrm{WIN}(A)$) is the object function of a functor $\mathrm{IN} : \mathbf{TC}^u\text{-}\mathbf{Alg} \longrightarrow \mathbf{Grp}$($\mathrm{WIN} : \mathbf{TC}^u\text{-}\mathbf{Alg} \longrightarrow \mathbf{Mon}$). For any A, $\mathrm{IN}(A) \subset \mathrm{WIN}(A)$ holds.

(5.2) Proposition. *Let A be a unital* **TC**-*algebra. Then the following are equivalent:*

(a) *A is spherical,*
(b) *e is spherical,*
(c) $\mathrm{IN}(A) = \mathrm{WIN}(A)$.

Proof: (a) $\iff$ (b) is trivial (cp. [7, (12.5)]). If (b) holds, and $a \in \mathrm{WIN}(A)$, then there is a $b \in A$ and $\rho \in \mathrm{O}(\mathbb{K})$ with $ab = ba = \rho e \neq 0$, which implies $|\rho| = 1$, i.e. $a \in \mathrm{IN}(A)$. (c) yields $\frac{1}{2}e = 0$, i.e. e spherical. □

(5.3) Proposition. *Let A be a unital, not spherical* **TC**-*algebra. If $a \in \mathrm{WIN}(A)$ and $0 < |\rho| \leq 1$, then also $\rho a \in \mathrm{WIN}(A)$. In other words, $\mathrm{WIN}(A)$ is a cone (without vertex) in A.*

Proof: Obvious. □

(5.4) Proposition. *Suppose that A is a unital not spherical* **TC**-*algebra. Then $a \in A$ is weakly invertible, if and only if $\sigma_A(a) \in \mathrm{S}(A)$ is invertible.*

Proof: As A is not spherical, $\mathrm{S}(A) \neq \{0\}$ holds, s.th. $ba = ab = \rho e \neq 0$ in A implies $\sigma_A(a)(\rho^{-1}\sigma_A(b)) = (\rho^{-1}\sigma_A(b))\sigma_A(a) = 1 \neq 0$ in $\mathrm{S}(A)$. Conversely, if $\sigma_A(a)x = x\sigma_A(a) = 1$ in $\mathrm{S}(A)$, then $y := \frac{1}{2\|x\|}x$ satisfies $\|y\| = \frac{1}{2}$ and $\sigma_A(a)y = y\sigma_A(a) = \frac{1}{2\|x\|} \cdot 1$. As σ_A maps $\mathring{A}$ isomorphically onto $\tilde{\mathrm{O}}(\mathrm{S}(A))$ (cp. [9, (1.5)]), there is a unique $b \in \mathring{A}$ with $\sigma_A(b) = y$. Thus $\sigma_A(ab) = \sigma_A(ba) = \sigma_A\left(\frac{1}{2\|x\|}e\right)$ and therefore $ab = ba = \frac{1}{2\|x\|}e$. □

In [8] a distance function was introduced for totally convex spaces X by putting $d(x,y) := \|\frac{1}{2}x - \frac{1}{2}y\|$, $x, y \in X$.

(5.5) Lemma. *Let A be a totally convex algebra. Then, for all $a, b, x, y \in A$*

$$d(xy, ab) \leq M(2d(x,a)d(y,b) + \|a\|d(y,b) + \|b\|d(x,a)),$$

provided that one of the following conditions is satisfied:

(i) *at least one of the four points $a, b, x, y \in A$ has norm < 1, in which case $M = 1$.*
(ii) *A is strongly aspherical, in which case $M = \eta_A^{-1}$.*

Proof:

$$\frac{1}{2}(\frac{1}{2}x - \frac{1}{2}a)(\frac{1}{2}y - \frac{1}{2}b) + \frac{1}{4}a(\frac{1}{2}y - \frac{1}{2}b) + \frac{1}{4}(\frac{1}{2}x - \frac{1}{2}a)b =$$
$$\frac{1}{2}(\frac{1}{4}xy - \frac{1}{4}xb - \frac{1}{4}ay + \frac{1}{4}ab) + \frac{1}{4}(\frac{1}{2}ay - \frac{1}{2}ab) + \frac{1}{4}(\frac{1}{2}xb - \frac{1}{2}ab) =$$
$$\frac{1}{4}(\frac{1}{2}xy - \frac{1}{2}ab)$$

(cp. [6, (2.4), (ii)]). In case (i)

$$\|\frac{1}{2}xy - \frac{1}{2}ab\| \leq \frac{1}{2}\|x\|\|y\| + \frac{1}{2}\|a\|\|b\| < 1,$$

whence

$$\|\frac{1}{4}(\frac{1}{2}xy - \frac{1}{2}ab)\| = \frac{1}{4}d(xy, ab) \leq \frac{1}{2}d(x, a)d(y, b) + \frac{1}{4}\|a\|d(y, b) + \frac{1}{4}\|b\|d(x, a)$$

resulting in $M = 1$.

In case (ii),

$$\|\frac{1}{4}(\frac{1}{2}xy - \frac{1}{2}ab)\| \geq \frac{1}{4}\eta_A d(xy, ab),$$

due to (3.1) and (3.4), whence

$$\frac{1}{4}\eta_A d(xy, ab) \leq \frac{1}{2}d(x, a)d(y, b) + \frac{1}{4}\|a\|d(y, b) + \frac{1}{4}\|b\|d(x, a),$$

i.e. $M = \eta_A^{-1}$. □

(5.6) Lemma. *Let $a, b \in \mathrm{IN}(A)$, then $d(a^{-1}, b^{-1}) = d(a, b)$.*

Proof: $d(a^{-1}, b^{-1}) = \|\frac{1}{2}a^{-1} - \frac{1}{2}b^{-1}\| = \|a^{-1}(\frac{1}{2}b - \frac{1}{2}a)b^{-1}\| \leq \|a^{-1}\|d(a, b)\|b^{-1}\| = d(a, b)$. □

This distance function d induces a topology on A. In the following, if we refer to the notion "topology", we will always mean this topology.

(5.7) Proposition. *Let A be a unital* **TC**-*algebra. Then:*

(i) *If A is not spherical, then $\mathring{A} \cap \mathrm{WIN}(A)$ is a regular topological semigroup.*
(ii) *If A is strongly aspherical, then $\mathrm{WIN}(A)$ is a regular topological monoid and $\mathrm{IN}(A)$ is a topological group.*

Proof: (i): $\mathring{A} \cap \mathrm{WIN}(A)$ is a topological semigroup because of (5.5), (i). To show that it is regular, assume $ab = ac$ with $a, b, c \in \mathring{A} \cap \mathrm{WIN}(A)$. As $da = ad = \rho e \neq 0$ with suitable $d \in \mathrm{WIN}(A)$, $\rho \in \mathrm{O}(\mathbb{K})$, $\rho b = \rho c$ follows and hence $b = c$ (cp. [7, (11.6)]). This shows $\mathring{A} \cap \mathrm{WIN}(A)$ to be a left regular semigroup, right regularity is proved analogously. One may omit the assumption that A is not spherical, because for spherical A (5.2) implies $\mathring{A} \cap \mathrm{WIN}(A) = \emptyset$.

(ii): (5.5), (ii), and (5.6) imply the second assertion, while the first one follows from (5.5), (ii), alone.

(iii): As "separated" implies "strongly aspherical", the assertions follow from (ii) with the exception of regularity, which is proved as in (i). □

(5.8) Lemma. *Let A be a unital, not spherical* **TC**-*algebra. Then, for any $a \in A$, any $\lambda, \mu \in \mathrm{O}(\mathbb{K})$ with $|\mu| + |\lambda| \leq 1$ and $|\mu| < |\lambda|$, $\lambda e - \mu a$ is weakly invertible.*

Proof: With $\rho := 1 - |\mu\lambda^{-1}|$, $\sum\limits_{i=0}^{\infty} |\rho(\frac{\mu}{\lambda})^i| \leq 1$ holds and a routine computation using [6, (2.4), (ii)], shows

$$(\lambda e - \mu a)\sum_{i=0}^{\infty} \rho(\frac{\mu}{\lambda})^i a^i = \lambda\rho e \neq 0.$$

□

(5.9) Proposition. *For a unital, not spherical* **TC***-algebra A,* WIN(A) *is an open cone of A.*

Proof: Let $a \in \mathrm{WIN}(A)$ and $ab = ba = \rho e \neq 0$ with $0 < \rho \leq 1$. Put $\varepsilon := \frac{\rho}{4}$. Then we claim that $\{x \mid x \in A,\ d(x,a) < \varepsilon\} \subseteq \mathrm{WIN}(A)$ holds. For an element x with $d(x,a) < \varepsilon$ we have, due to [6, (6.1)], that

$$\frac{1}{2}a - \frac{1}{2}x = \varepsilon y$$

for some $y \in A$. Hence

$$\frac{1}{4}x = \frac{1}{4}a - \frac{1}{2}(\frac{1}{2}a - \frac{1}{2}x) = \frac{1}{4}a - \frac{\varepsilon}{2}y$$

and therefore

$$\frac{1}{4}bx = \frac{\rho}{4}e - \frac{\rho}{8}by = \frac{3}{8}\rho(\frac{2}{3}e - \frac{1}{3}by).$$

By (5.8) there is a $c \in A$, such that $c(\frac{2}{3}e - \frac{1}{3}by) = \frac{1}{3}e$, hence

$$(\frac{1}{4}cb)x = \frac{\rho}{8}e.$$

Similarly there is a $c' \in A$, such that

$$x(\frac{1}{4}bc') = \frac{\rho}{8}e.$$

Thus x is weakly invertible. □

(5.10) Definition. Let A be a **TC**-algebra, then $a \in A$ is called a *left* (resp. *right*) *topological zero divisor*, if $inf\{\ \|ax\| \mid \frac{1}{2} < \|x\| < 1\} = 0$ (resp. $inf\{\ \|xa\| \mid \frac{1}{2} < \|x\| < 1\} = 0$). a is said to be a *topological zero divisor*, if $inf\{\ \|ax\| + \|xa\| \mid \frac{1}{2} < \|x\| < 1\} = 0$.

(5.11) Proposition.

(i) *A spherical* **TC***-algebra does not have left* (*resp. right*) *topological zero divisors.*

(ii) *For a not spherical* **TC***-algebra every spherical element is a topological zero divisor.*

Proof: (i): Clearly, A is spherical, iff there is no $x \in A$ with $\frac{1}{2} < \|x\| < 1$. Hence, the infima in (5.10) are $+\infty$.

(ii): Let a be a spherical element and $\frac{1}{2} < \|x\| < 1$. Then $x = \rho y$, where $\|x\| < \rho < 1$ and $y \in A$ is suitably chosen. Hence

$$ax = a(\rho y) = (\rho a)y = 0y = 0.$$

□

(5.12) Theorem. *Let A be a unital, not spherical* **TC***-algebra and let $a \in \partial_{\mathring{A}}(\mathring{A} \cap \mathrm{WIN}(A))$. Then a is a topological zero divisor.*

Proof: (5.4) and [9, (1.5)], imply that σ_A maps $\mathring{A} \cap \mathrm{WIN}(A)$ homeomorphically onto $\mathrm{IN}(\mathrm{S}(A)) \cap \mathring{\mathrm{O}}(\mathrm{S}(A))$. Hence [1, Theorem 14], leads to our assertion. □

A unital **TC**-algebra B is called an *extension* of the unital **TC**-algebra A, if A is a unital subalgebra of B and a norm-equivalent subspace of B (cp. [12, (3.5)]).

(5.13) Definition. Let A be a unital **TC**-algebra. Then $a \in A$ is called *singular* (*strongly singular*), if it is not invertible (weakly invertible) in A. $a \in A$ is called *permanently singular*, if a is strongly singular in any extension of A.

Obviously, every strongly singular element is singular. The set of singular elements of A is denoted by $\mathrm{SING}(A)$, the set of strongly singular elements by $\mathrm{SSING}(A)$.

(5.14) Proposition. *Let A be a unital* **TC***-algebra. Then every left (resp. right) topological zero divisor is permanently singular. In particular, every element of $\partial_{\mathring{A}}(\mathring{A} \cap \mathrm{WIN}(A))$ is permanently singular.*

Proof: Let B be an extension of A with norm quotient $\eta > 0$ ([12, (3.5)]) and let $a \in A$ be a left topological zero divisor. For $x \in A$ let $\|x\|_B$ denote the norm in B. Then there is a sequence $x_n \in A$, $n \in \mathbb{N}$, with $\frac{1}{2} < \|x_n\| < 1$ and $\lim_{n\to\infty} \|ax_n\| = 0$. Suppose furthermore that a has a weak inverse $b \in B$, i.e. $ab = ba = \rho e \neq 0$. As $\|x_n\| < 1$, we have $\|\rho x_n\| = |\rho|\|x_n\|$ and hence

$$\eta|\rho|\|x_n\| = \eta\|\rho x_n\| \leq \|\rho x_n\|_B = \|bax_n\|_B \leq \|b\|_B\|ax_n\|$$

contradicting the assumption $\frac{1}{2} < \|x_n\|$. The assertion for right topological zero divisors follows analogously and the last assertion is implied by (5.12). □

(5.15) Corollary. *Let B be an extension of A. Then*

(i) $\mathring{A} \cap \mathrm{WIN}(A) \subseteq \mathring{B} \cap \mathrm{WIN}(B)$,
(ii) $\partial_{\mathring{A}}(\mathring{A} \cap \mathrm{WIN}(A)) \subseteq \partial_{\mathring{B}}(\mathring{B} \cap \mathrm{WIN}(B))$.

Proof: (i) is obvious. As for (ii), (5.14) implies $\partial_{\mathring{A}}(\mathring{A} \cap \mathrm{WIN}(A)) \subseteq \mathrm{SSING}(B)$. On the other hand, for $a \in \partial_{\mathring{A}}(\mathring{A} \cap \mathrm{WIN}(A))$ and every $\varepsilon > 0$ there is an element $a' \in \mathring{A} \cap \mathrm{WIN}(A) \subseteq \mathring{B} \cap \mathrm{WIN}(B)$, s.th. $d(a, a') = \|\frac{1}{2}a - \frac{1}{2}a'\| < \varepsilon$. Hence, $d_B(a, a') \leq d(a, a') < \varepsilon$ implies (ii), where d_B is the distance function of B. □

As the left regular representation $L : A \longrightarrow \mathrm{End}(|A|)$ (cp. Section 1) is an isometry, $\mathrm{End}(|A|)$ is, via L, an extension of A for A a unital **TC**-algebra.

(5.16) Corollary. *Let A be a unital* **TC***-algebra. Then*

(i) *$a \in \mathrm{SING}(A)$, if and only if $L_a \in \mathrm{SING}(\mathrm{End}(|A|))$.*
(ii) *If A is commutative and not spherical, then*

$$a \in \mathrm{SSING}(A)\ ,\ \textit{if and only if}\ \ L_a \in \mathrm{SSING}(\mathrm{End}(|A|)).$$

Proof: (i) is proved as in [1, p. 15].

(ii): "$\Longleftarrow$" is obvious. If $a \in \mathrm{SSING}(A)$ and L_a were weakly invertible, there would be a $\varphi \in \mathrm{End}(|A|)$, s.th. $L_a \circ \varphi = \varphi \circ L_a = \rho id_A \neq 0$. This would imply $a \in \mathrm{WIN}(A)$, i.e. a contradiction. □

6. Quasi-inverses.

(6.1) Definition. Let A be a **TC**-algebra and $x, y \in A$. One defines

$$x \circ y := \frac{1}{3}x + \frac{1}{3}y - \frac{1}{3}xy.$$

x is called a *left quasi-inverse* of y and y a *right quasi-inverse* of x, if $x \circ y = 0$ holds. For $\alpha \in \mathbb{K}$ with $|\alpha| < \frac{1}{3}$ one defines

$$x \circ_\alpha y := \alpha x + \alpha y - \alpha xy.$$

x is called a *weak left quasi-inverse* of y and y a *weak right quasi-inverse* of x, if $x \circ_\alpha y = 0$ holds for every α with $|\alpha| < \frac{1}{3}$. A (weak) quasi-inverse of x is an element, that is both a (weak) left and a (weak) right quasi-inverse of x.

Obviously the operation "$\circ_\alpha$" is trivial for $\alpha = 0$. Moreover, to see that y is a weak right quasi-inverse of x it is enough to verify $x \circ_\alpha y = 0$ for one $\alpha \neq 0$:

(6.2) Lemma. *Let A be a* **TC**-*algebra and $x \in A$. Then any right quasi-inverse y of x is a weak right quasi-inverse of x. If there is an α with $|\alpha| < \frac{1}{3}$ and $x \circ_\alpha y = 0$, then y is a weak right quasi-inverse of x.*

Proof: For $\alpha, \beta \in \mathbb{K}$ let $0 < |\beta| \leq |\alpha| \leq \frac{1}{3}$, then

$$\frac{\beta}{\alpha}(x \circ_\alpha y) = \frac{\beta}{\alpha}(\alpha x + \alpha y - \alpha xy) = x \circ_\beta y,$$

i.e.

$$\beta(x \circ_\alpha y) = \alpha(x \circ_\beta y).$$

Hence, for $\alpha = \frac{1}{3}, |\beta| < \frac{1}{3}, x \circ y = 0$ implies $\frac{1}{3}(x \circ_\beta y) = 0$, i.e. $x \circ_\beta y = 0$, because $\|x \circ_\beta y\| < 1$ holds. Similarly, if $|\alpha|, |\beta| < \frac{1}{3}$ and $x \circ_\alpha y = 0$, the above equation implies $\alpha(x \circ_\beta y) = 0$ and again $x \circ_\beta y = 0$ follows. Obviously the analogue of (6.2) for (weak) left quasi-inverses holds, too. $\square$

(6.3) Corollary. *If A is an aspherical* **TC**-*algebra, y is a left (right) quasi-inverse of x, if and only if y is a weak left (right) quasi-inverse of x.*

Proof: Let y be a weak left quasi-inverse of x, i.e. $y \circ_\alpha x = 0$ for (any) α with $0 < |\alpha| < \frac{1}{3}$. The equation in the proof of (6.2) then implies $\alpha(y \circ x) = 0$, i.e. $y \circ x = 0$, because A is aspherical. $\square$

(6.4) Proposition. *Let A be a* **TC**-*algebra. Suppose that $x \in A$ has a (weak) left quasi-inverse z and a (weak) right quasi-inverse y, then $\gamma y = \gamma z$, for all γ with $|\gamma| < 1$. In particular, if A is separated, then x is (weakly) quasi-invertible.*

Proof: For $\alpha, \beta \in \mathbb{K}$ with $|\alpha| \leq \frac{1}{3}, |\beta| \leq \frac{1}{3}$ the following formulae are easily verified:

$$x \circ_\alpha 0 = 0 \circ_\alpha x = \alpha x,$$

$$z \circ_\alpha (x \circ_\beta y) = \alpha z + \alpha\beta x + \alpha\beta y - \alpha\beta xy - \alpha\beta zx - \alpha\beta zy + \alpha\beta zxy,$$

$$(z \circ_\alpha x) \circ_\beta y = \alpha\beta z + \alpha\beta x - \alpha\beta zx + \beta y - \alpha\beta zy - \alpha\beta xy + \alpha\beta zxy.$$

A straightforward computation now leads to our assertion. $\square$

(6.5) Proposition. *Let A be a unital, not spherical* **TC***-algebra. If $x \in A$ has y as weak left quasi-inverse, then, for any $0 < |\alpha| < \frac{1}{2}, \alpha e - \alpha y$ is a weak left inverse of $\frac{1}{2}e - \frac{1}{2}x$.*

PROOF: For any β with $|\beta| < \frac{2}{3}$ one has

$$\beta^2(\frac{1}{2}e - \frac{1}{2}y)(\frac{1}{2}e - \frac{1}{2}x) = (\frac{\beta}{2})^2 e - \frac{\beta}{2}(y \circ_{\frac{\beta}{2}} x) = \beta^2(\frac{1}{4}e).$$

Hence, [6, (4.1)], leads to

$$(\frac{\lambda}{2}e - \frac{\lambda}{2}y)(\frac{1}{2}e - \frac{1}{2}x) = \frac{\lambda}{4}e$$

for all $|\lambda| < 1$, whence our claim follows. □

(6.6) Proposition. *Let A be a* **TC***-algebra and let $x, y \in A$. Then y is a weak left quasi-inverse of x, if and only if, in $\mathrm{U}(A)$, $\beta e - \beta j_A(y)$ is a weak left inverse of $\beta e - \beta j_A(x)$ for all $0 < |\beta| < \frac{1}{2}$. If, in addition, A is aspherical, then y is a left quasi-inverse of x, if and only if, in $\mathrm{U}(A)$, $\frac{1}{2}e - \frac{1}{2}j_A(y)$ is a weak left inverse of $\frac{1}{2}e - \frac{1}{2}j_A(x)$.*

PROOF: Similarly to the proof of (6.5) we obtain, for $0 < |\beta| \leq \frac{1}{2}$ and $\alpha := \frac{\beta^2}{1-\beta^2}$,

$$(\beta e - \beta j_A(y))(\beta e - \beta j_A(x)) = \beta^2 e - (1 - \beta^2) j_A(y \circ_\alpha x).$$

Hence, if y is a (weak) left quasi-inverse of x, the weak left invertibility follows from (4.4), (i). Conversely, if $\beta e - \beta j_A(y)$ is a weak left inverse of $\beta e - \beta j_A(x)$, then, for some $0 < |\rho| \leq 1$,

$$\beta^2 e - (1 - \beta^2) j_A(y \circ_\alpha x) = \rho e$$

holds. Hence, (4.4), (iii), implies that $y \circ_\alpha x$ is spherical. But, for $|\beta| < \frac{1}{2}$, we have $\|y \circ_\alpha x\| < 1$ and therefore $y \circ_\alpha x = 0$. For $\beta = \frac{1}{2}$ $\alpha = \frac{1}{3}$ holds, i.e. $y \circ x = 0$, if A is aspherical. □

Obviously, the assertions of (6.6) remain true, if one replaces "left" by "right".

(6.7) Corollary. *Let A be a* **TC***-algebra and let $x \in A$ with $\|x\| < \frac{1}{2}$. Then x is weakly quasi-invertible. If, in addition, A is aspherical, then x is quasi-invertible.*

PROOF: Since $\|x\| < \frac{1}{2}$ we have $x = \frac{1}{2}y$ for some $y \in A$. Hence

$$\frac{1}{2}e - \frac{1}{2}j_A(x) = \frac{1}{2}e - \frac{1}{4}j_A(y).$$

By (5.8) this has as a weak inverse the element

$$\sum_{i=0}^{\infty} \frac{1}{2}(\frac{1}{2})^i j_A(y^i) = \frac{1}{2}e - \frac{1}{2}\sum_{i=1}^{\infty}(\frac{1}{2})^i j_A(y^i).$$

Therefore the assertions follow from (6.6). □

The set of (weakly) quasi-invertible elements of a **TC**-algebra A is denoted by $(\mathrm{WQIN}(A))$ $\mathrm{QIN}(A)$. Obviously, $\mathrm{QIN}(A) \subseteq \mathrm{WQIN}(A)$.

7. The spectrum of an element.

(7.1) Definition. Let A be a complex unital $\mathbf{TC}_{fin}$-algebra and let $a \in A$. Then $\mathrm{SSp}_A(a)$ (resp. $\mathrm{Sp}_A(a)$) is the set of all $\lambda \in \mathbb{C}$ such that $\frac{\lambda}{1+|\lambda|}e - \frac{1}{1+|\lambda|}a$ is in $\mathrm{SING}(A)$ (resp. $\mathrm{SSING}(A)$). $\mathrm{Sp}_A(a)$ is called the *spectrum* of a in A, $\mathrm{SSp}_A(a)$ is called the *strong spectrum* of a in A. Whenever the context is clear we drop the subscript "A".

(7.2) Lemma. *For all $a \in A$, $\mathrm{Sp}(a) \subset \mathrm{SSp}(a)$. Moreover, A is spherical if and only if $\mathrm{SSp}(a) = \mathrm{Sp}(a)$, for all $a \in A$.*

Proof: Straightforward. □

(7.3) Lemma. *Let A be a complex, unital, not spherical $\mathbf{TC}_{fin}$-algebra. Then, for all $a, b \in A$, $\mathrm{Sp}_A(ab)\backslash\{0\} = \mathrm{Sp}_A(ba)\backslash\{0\}$.*

Proof: Suppose $\frac{\lambda}{1+|\lambda|}e - \frac{1}{1+|\lambda|}ab$ has x as a weak inverse, i.e. let

$$x(\frac{\lambda}{1+|\lambda|}e - \frac{1}{1+|\lambda|}ab) = (\frac{\lambda}{1+|\lambda|}e - \frac{1}{1+|\lambda|}ab)x = \rho e \neq 0.$$

Put

$$\alpha := \frac{1+|\lambda|}{1+|\lambda|+|\rho|^{-1}} \quad \text{and} \quad \beta := \frac{\rho^{-1}}{1+|\lambda|+|\rho|^{-1}}.$$

Then a simple computation shows

$$\begin{aligned}(\alpha e + \beta bxa)(\frac{\lambda}{1+|\lambda|}e - \frac{1}{a+|\lambda|}ba) &= (\frac{\lambda}{1+|\lambda|}e - \frac{1}{1+|\lambda|}ba)(\alpha e + \beta bxa) \\ &= \frac{\lambda}{1+|\lambda|+|\rho|^{-1}}e,\end{aligned}$$

which proves our assertion. □

(7.4) Lemma. *Let $f : A \longrightarrow B$ be a unital homomorphism of complex, unital $\mathbf{TC}_{fin}$-algebras. Then, for all $a \in A$,*

(i) $\mathrm{SSp}_B(f(a)) \subseteq \mathrm{SSp}_A(a)$,
(ii) $\mathrm{SSp}_{\mathrm{End}(|A|)}(L_a) \subseteq \mathrm{SSp}_A(a)$.

Proof: Obvious. □

(7.5) Lemma. *Let $f : A \longrightarrow B$ be a unital homomorphism of complex, unital, not spherical $\mathbf{TC}_{fin}$-algebras. Then for all $a \in A$,*

(i) $\mathrm{Sp}_B(f(a)) \subseteq \mathrm{Sp}_A(a)$,
(ii) $\mathrm{Sp}_{\mathrm{End}(|A|)}(L_a) \subseteq \mathrm{Sp}_A(a)$.

Proof: Obvious. □

(7.6) Theorem. *Let A be a complex, unital, not spherical* **TC***-algebra. Then for all $a \in A$,*

(i) $\mathrm{Sp}_A(a) \subset \{z \mid z \in \mathbb{C}, |z| \leq \|a\| \ \} \subset \mathrm{O}(\mathbb{C})$,
(ii) $\mathrm{Sp}_A(a) = \mathrm{Sp}_{\mathrm{S}(A)}(\sigma_A(a))$.

Proof: (i): Let $\|a\| < \rho \leq 1$. Then

$$\frac{\lambda}{1+|\lambda|}e - \frac{1}{1+|\lambda|}a = \frac{\lambda}{1+|\lambda|}e - \frac{1}{1+|\lambda|}\rho b,$$

for some $b \in A$. By (5.8) this element has a weak inverse provided $|\frac{\rho}{\lambda}| < 1$, that is, if $|\rho| < |\lambda|$. Hence our assertion follows in case $\|a\| < 1$. In case $\|a\| = 1$, choose $\rho = 1$ and $b = a$ in this argument.

(ii): Suppose that $0 \notin \mathrm{Sp}(a)$ and a is spherical. Then a is weakly invertible. Hence, for some $0 < |\rho| \leq 1$ and $b \in A$, $ab = \rho e$. However,

$$0 \neq \frac{1}{2}\rho e = \frac{1}{2}(ab) = (\frac{1}{2}a)b = 0,$$

which is a contradiction. Thus $0 \in \mathrm{Sp}(a)$ for spherical a. Now let $0 \neq \lambda \in \mathbb{C}$. Then

$$\frac{1}{2}e(\frac{\lambda}{1+|\lambda|}e - \frac{1}{1+|\lambda|}a) = \frac{\lambda}{2(1+|\lambda|)}e - \frac{1}{1+|\lambda|}(\frac{1}{2}a) = \frac{\lambda}{2(1+|\lambda|)}e,$$

whence $\frac{\lambda}{1+|\lambda|}e - \frac{1}{1+|\lambda|}a$ is weakly invertible, and thus $\lambda \notin \mathrm{Sp}(a)$. Therefore, for a spherical, $\mathrm{Sp}(a) = \{0\} = \mathrm{Sp}(\sigma_A(a))$.

Now suppose that a is not spherical and $\lambda \notin \mathrm{Sp}(a)$. Then there is a $0 < |\rho| \leq 1$ and $b \in A$ such that

$$(\frac{\lambda}{1+|\lambda|}e_A - \frac{1}{1+|\lambda|}a)b = b(\frac{\lambda}{1+|\lambda|}e - \frac{1}{1+|\lambda|}a) = \rho e_A \neq 0.$$

Hence

$$\sigma_A(\frac{\lambda}{1+|\lambda|}e_A - \frac{1}{1+|\lambda|}a)\sigma_A(b) = \sigma_A(b)\sigma_A(\frac{\lambda}{1+|\lambda|}e_A - \frac{1}{1+|\lambda|}a) = \rho e_{\mathrm{S}(A)} \neq 0.$$

This means that

$$\sigma_A(\frac{\lambda}{1+|\lambda|}e_A - \frac{1}{1+|\lambda|}a) = \frac{\lambda}{1+|\lambda|}e_{\mathrm{S}(A)} - \frac{1}{1+|\lambda|}\sigma_A(a)$$

is invertible and therefore $\lambda \notin \mathrm{Sp}_{\mathrm{S}(A)}(\sigma_A(a))$. Conversely, assume that $\lambda \notin \mathrm{Sp}_{\mathrm{S}(A)}(\sigma_A(a))$. Then there exists an $x \in \mathrm{S}(A)$ with

$$(\frac{\lambda}{1+|\lambda|}e_{\mathrm{S}(A)} - \frac{1}{1+|\lambda|}\sigma_A(a))x = x(\frac{\lambda}{1+|\lambda|}e_{\mathrm{S}(A)} - \frac{1}{1+|\lambda|}\sigma_A(a)) = e_{\mathrm{S}(A)}.$$

Since $y := \frac{x}{2\|x\|}$ satisfies $\|y\| = \frac{1}{2}$, there is a $b \in A$ such that $\sigma_A(b) = y$ and $\|b\| = \frac{1}{2}$. The last equation renders

$$\sigma_A(\frac{\lambda}{1+|\lambda|}e_A - \frac{1}{1+|\lambda|}a)\sigma_A(b) = \sigma_A(b)\sigma_A(\frac{\lambda}{1+|\lambda|}e_A - \frac{1}{1+|\lambda|}a) = \frac{1}{2\|x\|}e_{\mathrm{S}(A)}$$

or,

$$\sigma_A((\frac{\lambda}{1+|\lambda|}e_A - \frac{1}{1+|\lambda|}a)b) = \sigma_A(b(\frac{\lambda}{1+|\lambda|}e_A - \frac{1}{1+|\lambda|}a)) = \sigma_A(\frac{1}{2\|x\|}e_A).$$

Since σ_A maps $\mathring{A}$ injectively and since

$$\|(\frac{\lambda}{1+|\lambda|}e_A - \frac{1}{1+|\lambda|}a)b\| \leq \frac{1}{2} \quad \text{and} \quad \|b(\frac{\lambda}{1+|\lambda|}e_A - \frac{1}{1+|\lambda|}a)\| \leq \frac{1}{2},$$

we see that $\frac{\lambda}{1+|\lambda|}e_A - \frac{1}{1+|\lambda|}a$ is weakly invertible and hence $\lambda \notin \mathrm{Sp}_A(a)$. □

Let A be a **TC**-algebra and let $a \in A$. Then we define the *spectral radius* $r(a)$ of a by

$$r(a) := r(\sigma_A(a)),$$

where $r(\sigma_A(a))$ is the usual spectral radius (see [1, 2.7]).

(7.7) Corollary. *Let A be a complex, unital, not spherical* **TC**-*algebra. Then, for all $a \in A$, $\mathrm{Sp}_A(a)$ is a non-empty compact subset of* $\mathrm{O}(\mathbb{C})$ *and*

$$r(a) = \max\{|\lambda| \mid \lambda \in \mathrm{Sp}_A(a)\}.$$

PROOF: (7.6) and [1, Theorem 5.8]. □

(7.8) Proposition. *Let A be a complex. unital, not spherical* **TC**-*algebra. Let furthermore $p(z) = \sum_{i=0}^{n} \alpha_i z^i$ be a polynomial with $\sum_{i=0}^{n} |\alpha_i| \leq 1$. Then, for all $a \in A$*

$$\mathrm{Sp}_A(p(a)) = \{p(\lambda) \mid \lambda \in \mathrm{Sp}_A(a)\}.$$

PROOF: Same as for [1, Proposition 5.5], keeping in mind that for such a polynomial $p(z)$ with $\alpha_n \neq 0$, having the roots $\lambda_1, \dots, \lambda_n$, the equation

$$p(z) = \beta \prod_{i=1}^{n} (\frac{\lambda_i}{1+|\lambda_i|} - \frac{1}{1+|\lambda_i|}z)$$

holds, where

$$\beta := (-1)^n \alpha_n \prod_{i=1}^{n} (1+|\lambda_i|)$$

and $|\beta| \geq 1$. □

(7.9) Theorem. *Let A be a complex, unital, not spherical* $\mathbf{TC}_{fin}$*-algebra. Then*

(i) $\|a\| < 1$ *implies* $\mathrm{SSp}(a) = \mathbb{C}$,

(ii) $\|a\| = 1$ *and* $\|\sigma_A(a)\| < 1$ *imply* $\mathrm{SSp}(a) = \mathbb{C}$,

(iii) $\|a\| = 1$ *and* $\|\sigma_A(a)\| = 1$ *imply* $\mathrm{SSp}(a) \supseteq \mathbb{C}\backslash\mathbb{R}_0^+ e^{i\varphi}$*, for some* φ*, and either* $\mathrm{SSp}(a) \supseteq \mathbb{C}\backslash\{0\}$ *or* $\|\frac{te^{i\varphi}}{1+t}e_A - \frac{1}{1+t}a\| = 1$ *and* $\|\frac{te^{i\varphi}}{1+t}e_{\mathrm{S}(A)} - \frac{1}{1+t}\sigma_A(a)\| = 1$*, for all* $t \in \mathbb{R}_0^+$.

Proof: (i): Since

$$\|\frac{\lambda}{1+|\lambda|}e - \frac{1}{1+|\lambda|}a\| \leq \frac{|\lambda|}{1+|\lambda|} + \frac{\|a\|}{1+\|\lambda\|} < 1,$$

for all $\lambda \in \mathbb{C}$, $\frac{\lambda}{1+|\lambda|}e - \frac{1}{1+|\lambda|}a$ fails to be invertible for all $\lambda \in \mathbb{C}$.

(ii): Suppose $\lambda \notin \mathrm{SSp}(a)$. Then there is a $b \in A$ with

$$(\frac{\lambda}{1+|\lambda|}e_A - \frac{1}{1+|\lambda|}a)b = b(\frac{\lambda}{1+|\lambda|}e_A - \frac{1}{1+|\lambda|}a) = e_A.$$

Hence

$$\begin{aligned}(\frac{\lambda}{1+|\lambda|}e_{\mathrm{S}(A)} - \frac{1}{1+|\lambda|}\sigma_A(a))\sigma_A(b) &= \sigma_A(b)(\frac{\lambda}{1+|\lambda|}e_{\mathrm{S}(A)} - \frac{1}{1+|\lambda|}\sigma_A(a))\\ &= e_{\mathrm{S}(A)}\end{aligned}$$

and

$$\begin{aligned}1 = \|e_{\mathrm{S}(A)}\| \leq \|\frac{\lambda}{1+|\lambda|}e_{\mathrm{S}(A)} - \frac{1}{1+|\lambda|}\sigma_A(a)\|\|\sigma_A(b)\| \leq\\ \leq \frac{|\lambda|}{1+|\lambda|} + \frac{\|\sigma_A(a)\|}{1+|\lambda|} < 1,\end{aligned}$$

which is a contradiction.

(iii): For $\lambda \neq 0$ and any $\mu \in \mathbb{C}$ we have

$$\begin{aligned}(\frac{\lambda}{1+|\lambda|}e - \frac{1}{1+|\lambda|}a)(\frac{\mu}{1+|\mu|}e - \frac{1}{1+|\mu|}a) =\\ = \frac{\lambda\mu}{(1+|\lambda|)(1+|\mu|)}e - \frac{\lambda+\mu}{(1+|\lambda|)(1+|\mu|)}a + \frac{1}{(1+|\lambda|)(1+|\mu|)}a^2\end{aligned}$$

and hence

$$\|(\frac{\lambda}{1+|\lambda|}e - \frac{1}{1+|\lambda|}a)(\frac{\mu}{1+|\mu|}e - \frac{1}{1+|\mu|}a)\| \leq \frac{1+|\lambda+\mu|+|\lambda||\mu|}{(1+|\lambda|)(1+|\mu|)}.$$

However, the latter is < 1 precisely, if $\mu \notin \mathbb{R}_0^+\lambda$, in which case not both $\frac{\lambda}{1+|\lambda|}e - \frac{1}{1+|\lambda|}a$ and $\frac{\mu}{1+|\mu|}e - \frac{1}{1+|\mu|}a$ are invertible. This proves the first claim in (iii). [10, (2.7)], shows that either

$$\|\frac{te^{i\varphi}}{1+t}e_A - \frac{1}{1+t}a\| = 1 \quad \text{and} \quad \|\frac{te^{i\varphi}}{1+t}e_{\mathrm{S}(A)} - \frac{1}{1+t}\sigma_A(a)\| = 1, \text{ for all } \ t \in \mathbb{R}_0^+,$$

or

$$\|\frac{te^{i\varphi}}{1+t}e_A - \frac{1}{1+t}a\| = 1 \quad \text{and} \quad \|\frac{te^{i\varphi}}{1+t}e_{\mathrm{S}(A)} - \frac{1}{1+t}\sigma_A(a)\| < 1, \text{ for all } \ t \in \mathbb{R}_0^+.$$

In the second case we have $\mathbb{R}_0^+ e^{i\varphi} \subseteq \mathrm{SSp}(a)$, whence our assertion holds. □

(7.10) Theorem. *Let A be a complex, unital, spherical $\mathbf{TC}_{fin}$-algebra. Then*

(i) $\mathrm{SSp}(0) = \mathbb{C}$ *and* $\mathrm{SSp}(te) = \mathbb{C}\backslash\{-t\}\mathbb{R}_0^+$, *for* $t \in \sum(\mathbb{C})$.

(ii) $\|a\| = 1$ *implies* $\mathrm{SSp}(a) \supseteq \mathbb{C}\backslash\mathbb{R}_0^+ e^{i\varphi}$, *for some* φ, *and*

$$\textit{either} \qquad \|\frac{te^{i\varphi}}{1+t}e - \frac{1}{1+t}a\| = 1, \ \textit{for all} \ \ t \in \mathbb{R}_0^+$$

$$\textit{or} \qquad \frac{te^{i\varphi}}{1+t}e - \frac{1}{1+t}a = 0, \ \textit{for all} \ \ t \in \mathbb{R}^+.$$

Proof: (i): Obvious.

(ii): Similar to the proof of (7.9), (iii). □

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