Convergence theorems for the Perron integral and Sklyarenko's condition

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Abstract. It is shown that a uniform version of Sklyarenko's integrability condition for Perron integrals together with pointwise convergence of a sequence of integrable functions are sufficient for a convergence theorem for Perron integrals.

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A finite sequence of numbers

$$D = \{\alpha_0, \tau_1, \alpha_1, \dots, \alpha_{k-1}, \tau_k, \alpha_k\}$$

is called a partition (or division) of the interval $[a, b]$ if

$$a = \alpha_0 < \alpha_1 < \dots < \alpha_{k-1} < \alpha_k = b$$

and

$$\alpha_{j-1} \le \tau_j \le \alpha_j, \quad j = 1, 2, \dots, k.$$

Given an arbitrary positive function $\delta : [a, b] \mapsto (0, +\infty)$, called a *gauge* on $[a, b]$, the partition D is said to be δ-fine if

$$[\alpha_{j-1}, \alpha_j] \subset [\tau_j - \delta(\tau_j), \tau_j + \delta(\tau_j)]$$

for $j = 1, 2, \dots, k$.

Definition 1 (Kurzweil, Henstock)**.** A function $f : [a, b] \mapsto \mathbb{R}$ is said to be *integrable to the value* $I \in \mathbb{R}$ *on* $[a, b]$ if for every $\varepsilon > 0$ there is a gauge δ on $[a, b]$ such that

$$|\sum_{j=1}^{k} f(\tau_j)(\alpha_j - \alpha_{j-1}) - I| < \varepsilon$$

for every δ-fine partition of $[a, b]$. The value I will be denoted by $\int_a^b f(s)\, ds$ or shortly also $\int_a^b f$.

For a given $f : [a, b] \mapsto \mathbb{R}$ and a partition D of $[a, b]$ we use the notation

$$S(f, D) = \sum_{j=1}^{k} f(\tau_j)(\alpha_j - \alpha_{j-1})$$

for the corresponding Riemann type integral sum.

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Remark. Definition 1 belongs to J. Kurzweil (1957) and R. Henstock (1961). It is well known that this definition of an integral is equivalent to the classical concept of the Perron integral described e.g. in [8]. There is a still growing series of monographs and textbooks on this type of integration (see e.g. [1], [2], [3], [4], [5], [6]).

Theorem 1. *A function $f : [a,b] \mapsto \mathbb{R}$ is integrable over $[a,b]$ if and only if there exists $F : [a,b] \mapsto \mathbb{R}$ and for every $\varepsilon > 0$ there is a $\delta : [a,b] \mapsto (0,+\infty)$ and a nondecreasing $\mu : [a,b] \mapsto \mathbb{R}$ with $\mu(a) = 0$, $\mu(b) \le \varepsilon$ such that*

$$|F(x+h) - F(x) - h.f(x)| \le |\mu(x+h) - \mu(x)| \tag{1}$$

provided $x, x+h \in [a,b]$ and $0 \le |h| < \delta(x)$.

Remark. Theorem 1 gives a Sklyarenko-type necessary and sufficient condition for integrability of a function. The original result of Sklyarenko was given and proved in [9] for the Denjoy definition of the integral. The function μ in the original Sklyarenko's theorem is assumed to be absolutely continuous in addition to our requirements. A proof of the result based on the sum definition of the integral in the form given in Definition 1 can be also given. We postpone the proof of this result, because the result follows from a more general statement which will be stated later.

It is easy to show that if the function $f : [a,b] \mapsto \mathbb{R}$ is integrable then for every α, β, $a \le \alpha \le \beta \le b$ we have

$$\int_\alpha^\beta f(s)\,ds = F(\beta) - F(\alpha)$$

where $F : [a,b] \mapsto \mathbb{R}$ is the function given by Theorem 2. This means in fact that the function F is an indefinite integral corresponding to the integrable function f.

Definition 2. A system $\mathcal{G}$ of functions $f : [a,b] \mapsto \mathbb{R}$ is called *equi-integrable* if

(a) every $f \in \mathcal{G}$ is integrable (in the sense of Definition 1)
(b) to every $\varepsilon > 0$ there exists a gauge $\delta : [a,b] \mapsto (0,+\infty)$ such that for every δ-fine partition D and any $f \in \mathcal{G}$ the inequality

$$|S(f,D) - \int_a^b f(s)\,ds| = |\sum_{j=1}^k f(\tau_j)(\alpha_j - \alpha_{j-1}) - \int_a^b f(s)\,ds| < \varepsilon$$

holds.

Theorem 2. *Let $f_m : [a,b] \mapsto \mathbb{R}$, $m = 1,2,\dots$ be a sequence of functions such that*

$$\lim_{m\to\infty} f_m(s) = f(s) \ \text{ for } s \in [a,b]. \tag{2}$$

If the sequence (f_m) is an equi-integrable system then the function f is integrable and

$$\lim_{m\to\infty} \int_a^b f_m(s)\,ds = \int_a^b f(s)\,ds.$$

Proof: Let $\varepsilon > 0$ be given. By Definition 2 there is a gauge δ on $[a,b]$ such that for every δ-fine partition

$$D = \{\alpha_0, \tau_1, \alpha_1, \dots, \alpha_{k-1}, \tau_k, \alpha_k\}$$

of $[a,b]$ we have

$$|S(f_m, D) - \int_a^b f_m| < \frac{\varepsilon}{2}$$

for $m = 1, 2, \dots$. By (2) for every fixed partition D of $[a,b]$ there exists a positive integer m_0 such that for $m > m_0$ the inequality

$$|S(f_m, D) - S(f, D)| =$$

$$= |\sum_{j=1}^{k} \left[f_m(\tau_j)(\alpha_j - \alpha_{j-1}) - f(\tau_j)(\alpha_j - \alpha_{j-1}) \right] | < \frac{\varepsilon}{2}$$

holds and this means that

$$\lim_{m\to\infty} S(f_m, D) = S(f, D).$$

Therefore for any δ-fine partition D of $[a,b]$ there is a positive integer m_0 such that for $m > m_0$ we have

$$|S(f, D) - \int_a^b f_m| < \varepsilon. \tag{3}$$

First we get from (3) that for all positive integers $m, l > m_0$ the inequality

$$|\int_a^b f_m - \int_a^b f_l| < 2\varepsilon$$

holds. This means that $(\int_a^b f_m)_{m=1}^{\infty}$ is a Cauchy sequence in $\mathbb{R}$ and it has therefore a limit

$$\lim_{m\to\infty} \int_a^b f_m = H \in \mathbb{R}. \tag{4}$$

The second consequence of (3) is the inequality

$$|S(f, D) - H| \le |S(f, D) - \int_a^b f_m| + |\int_a^b f_m - H| < \varepsilon + |\int_a^b f_m - H|.$$

By (4) we obtain from this inequality immediately that for every δ-fine partition D of $[a,b]$ we have

$$|S(f, D) - H| < \varepsilon$$

and this means that the integral $\int_a^b f$ exists and that (4) is satisfied. □

Remark. Theorem 2 plays a central role in the theory of the Perron integral for the setting given by Kurzweil's sum Definition 1. Definition 1 makes transparent that the integral is in a certain sense the result of a "limit process". Therefore the problem of the convergence theorem becomes the problem of interchanging of two "limiting" processes. It is commonly accepted and known even from elementary calculus courses that two limits can be interchanged if one of the limits is uniform. The first possibility when the limit in (2) is uniform leads to a convergence result which is well known even for the Riemann integral and is not of sufficient power for applications. The other possibility is given by the theorem above in which the concept of equi-integrability of the sequence (f_m) expresses the uniformity with respect to m of the "limiting" process of integration. Consequently the proof is also easy because in fact it follows the classical lines of proving the interchangeability of two limits when one of them is uniform, see also [6], [7] and [10].

Theorem 3. *Let $\mathcal{F}$ be a system of functions $f : [a,b] \mapsto \mathbb{R}$. The following two conditions are equivalent.*

(a) *$\mathcal{F}$ is equi-integrable*

(b) *For every $f \in \mathcal{F}$ there is a $F_f : [a,b] \mapsto \mathbb{R}$ and for any $\varepsilon > 0$ there is a gauge $\delta : [a,b] \mapsto (0,+\infty)$ (independent of f) and a nondecreasing $\mu_f : [a,b] \mapsto \mathbb{R}$ such that $\mu_f(a) = 0$, $\mu_f(b) \le \varepsilon$ and*

$$|F_f(x+h) - F_f(x) - h.f(x)| \le |\mu_f(x+h) - \mu_f(x)| \tag{5}$$

provided $x, x+h \in [a,b]$, $0 \le |h| \le \delta(x)$.

Proof: First let us assume that the condition (b) is satisfied. Let $\varepsilon > 0$ be given and assume that

$$D = \{\alpha_0, \tau_1, \alpha_1, \dots, \alpha_{k-1}, \tau_k, \alpha_k\}$$

is an arbitrary δ-fine partition of $[a,b]$ where the gauge δ corresponds to ε by the condition (b) and is independent of $f \in \mathcal{F}$. Then using the inequality from (b) we obtain

$$\Big|\sum_{i=1}^{k} f(\tau_i)(\alpha_i - \alpha_{i-1}) - (F_f(b) - F_f(a))\Big| =$$

$$= \Big|\sum_{i=1}^{k} f(\tau_i)(\alpha_i - \alpha_{i-1}) - [F_f(\alpha_i) - F_f(\alpha_{i-1})]\Big| =$$

$$= \Big|\sum_{i=1}^{k} f(\tau_i)(\alpha_i - \tau_i) + f(\tau_i)(\tau_i - \alpha_{i-1}) - [F_f(\alpha_i) - F_f(\tau_i) + F_f(\tau_i) - F_f(\alpha_{i-1})]\Big| \le$$

$$\le \sum_{i=1}^{k} \{|f(\tau_i)(\alpha_i - \tau_i) - [F_f(\alpha_i) - F_f(\tau_i)]| +$$

$$+ |f(\tau_i)(\tau_i - \alpha_{i-1}) + [F_f(\tau_i) - F_f(\alpha_{i-1})]|\} \le$$

$$\le \sum_{i=1}^{k} [\mu_f(\alpha_i) - \mu_f(\tau_i) + \mu_f(\tau_i) - \mu_f(\alpha_{i-1})] = \sum_{i=1}^{k} [\mu_f(\alpha_i) - \mu_f(\alpha_{i-1})] = \mu_f(b) \le \varepsilon.$$

Hence by definition $\int_a^b f(s)\,ds$ exists, its value is $F_f(b) - F_f(a)$ and because the gauge δ does not depend on the choice of f from the system $\mathcal{F}$, the system is equi-integrable by Definition 2.

Assume that the condition (a) is satisfied ($\mathcal{F}$ is equi-integrable in the sense of Definition 2), i.e. for every $\varepsilon > 0$ there is a gauge δ on $[a, b]$ such that

$$|S(f, D) - \int_a^b f(s)\,ds| < \frac{\varepsilon}{2} \tag{6}$$

for every δ-fine partition

$$D = \{\alpha_0, \tau_1, \alpha_1, \dots, \alpha_{k-1}, \tau_k, \alpha_k\}$$

of $[a, b]$ where $S(f, D) = \sum_{i=1}^k f(\tau_i).(\alpha_i - \alpha_{i-1})$ and $f \in \mathcal{F}$ is arbitrary. Denote

$$F_f(x) = \int_a^x f(s)\,ds \text{ for } x \in [a, b] \text{ and } f \in \mathcal{F}.$$

If $a \le x < y \le b$ we set

$$\mathcal{F}_{[x,y]} = \{g : [x, y] \mapsto \mathbb{R};\quad g(s) = f(s),\ s \in [x, y],\ f \in \mathcal{F}\}.$$

It is easy to observe that the system $\mathcal{F}_{[x,y]}$ is equi-integrable (with the same gauge δ corresponding to $\varepsilon > 0$ by Definition 2). Hence for every $\eta > 0$ there is a $\delta_1 : [x, y] \mapsto (0, +\infty)$, $\delta_1(\xi) \le \delta(\xi)$ for $\xi \in [x, y]$ such that for every δ_1-fine partition $D^1_{[x,y]}$ of $[x, y]$ we have

$$|S(f, D^1_{[x,y]}) - \int_x^y f(s)\,ds| < \eta \text{ for any } f \in \mathcal{F} \tag{7}$$

where $S(f, D^1_{[x,y]})$ is the integral sum corresponding to the partition $D^1_{[x,y]}$. For $z \in [a, b]$ denote by $D_{[a,z]}$ an arbitrary δ-fine partition of $[a, z]$.

Define

$$M_f(a) = 0,\ M_f(x) = \sup_{D_{[a,x]}} S(f, D_{[a,x]})$$

and

$$m_f(a) = 0,\ m_f(x) = \inf_{D_{[a,x]}} S(f, D_{[a,x]})$$

for $x \in (a, b]$ where the supremum and the infimum are taken over all δ-fine partitions of $[a, x]$. Then

$$D_{[a,x]} \circ D^1_{[x,y]} = D_{[a,y]}$$

is a δ-fine partition of $[a,y]$ where by $\circ$ the union of partitions is denoted and

$$S(f, D_{[a,x]}) + S(f, D^1_{[x,y]}) = S(f, D_{[a,y]}).$$

By the definition of M_f, m_f respectively we get

$$m_f(y) \le S(f, D_{[a,x]}) + S(f, D^1_{[x,y]}) = S(f, D_{[a,y]}) \le M_f(y)$$

and also

$$m_f(y) \le m_f(x) + S(f, D^1_{[x,y]}) \le M_f(x) + S(f, D^1_{[x,y]}) \le M_f(y).$$

Taking (7) into account we conclude that for every $\eta > 0$ the inequalities

$$m_f(y) - m_f(x) - \eta < \int_x^y f(s)\,ds < M_f(y) - M_f(x) + \eta \tag{8}$$

hold and therefore for every $a \le x \le y \le b$ we have

$$m_f(y) - m_f(x) \le \int_x^y f(s)\,ds \le M_f(y) - M_f(x) \tag{9}$$

because $\eta > 0$ in (8) can be chosen arbitrarily.

Assume now that $x \in [a,b]$ and $0 \le h < \delta(x)$. Then

$$S(f, D_{[a,x]}) + h.f(x) = S(f, D_{[a,x+h]})$$

where $D_{[a,x+h]} = D_{[a,x]} \circ \{x, x, x+h\}$ and consequently also

$$M_f(x) + f(x).h \le M_f(x+h).$$

Similarly we obtain also $m_f(x+h) \le m_f(x) + f(x).h$, i.e.

$$m_f(x+h) - m_f(x) \le f(x).h \le M_f(x+h) - M_f(x); \tag{10}$$

for the case $-\delta(x) < h \le 0$ we get analogous inequalities.

Using (9) for $y = x + h$ and (10) we get

$$\begin{aligned} m_f(x+h) - m_f(x) - M_f(x+h) + M_f(x) \le \int_x^{x+h} f(s)\,ds - h.f(x) \le \\ \le M_f(x+h) - M_f(x) - m_f(x+h) + m_f(x), \end{aligned}$$

i.e.

$$\begin{aligned} |\int_x^{x+h} f(s)\,ds - h.f(x)| \le \\ \le |M_f(x+h) - M_f(x) - m_f(x+h) + m_f(x)| = |\mu_f(x+h) - \mu_f(x)| \end{aligned}$$

for $x, x+h \in [a,b]$, $0 \le |h| < \delta(x)$ where $\mu_f(x) = M_f(x) - m_f(x)$ for $x \in [a,b]$. Hence

$$|F_f(x+h) - F_f(x) - h.f(x)| \le |\mu_f(x+h) - \mu_f(x)|$$

provided $x, x+h \in [a,b]$, and $0 \le |h| < \delta(x)$, i.e. (1) is satisfied. It remains to show that the function μ_f given by

$$\mu_f(x) = M_f(x) - m_f(x)$$

for $x \in [a,b]$ satisfies all the properties stated in the theorem. We have evidently $\mu_f(a) = 0$. Using the definition of M_f and m_f we obtain easily $f(x).(y_2 - y_1) + M_f(y_1) \le M_f(y_2)$, i.e.

$$f(x) \le \frac{M_f(y_2) - M_f(y_1)}{y_2 - y_1}$$

provided $a \le y_1 \le x \le y_2 \le b$, $|y_i - x| < \delta(x)$, $i = 1, 2$ and similarly also

$$\frac{m_f(y_2) - m_f(y_1)}{y_2 - y_1} \le f(x)$$

for such x, y_1, y_2. Hence

$$\frac{\mu_f(y_2) - \mu_f(y_1)}{y_2 - y_1} = \frac{M_f(y_2) - M_f(y_1)}{y_2 - y_1} - \frac{m_f(y_2) - m_f(y_1)}{y_2 - y_1} \ge f(x) - f(x) = 0$$

for $a \le y_1 \le x \le y_2 \le b$, $|y_i - x| < \delta(x)$, $i = 1, 2$ and

$$\frac{\mu_f(x+h) - \mu_f(x)}{h} \ge 0$$

for $0 < |h| < \delta(x)$. Hence

$$\underline{D}\mu_f(x) = \liminf_{h \to 0} \frac{\mu_f(x+h) - \mu_f(x)}{h} \ge 0$$

for every $x \in [a,b]$. Consequently the function μ_f is nondecreasing on $[a,b]$. By (6) we get

$$\int_a^b f(s)\,ds - \frac{\varepsilon}{2} < S(f, D_{[a,b]}) < \int_a^b f(s)\,ds + \frac{\varepsilon}{2}.$$

Consequently we have

$$\int_a^b f(s)\,ds - \frac{\varepsilon}{2} \le m_f(b) \le M_f(b) \le \int_a^b f(s)\,ds + \frac{\varepsilon}{2}.$$

and

$$0 \le \mu_f(b) = M_f(b) - m_f(b) \le \varepsilon.$$

Since δ does not depend on the choice of $f \in \mathcal{F}$ by the assumption of equi-integrability, it is easy to see that the condition (b) is satisfied.

□

Remark. The condition (b) represents a uniform version of Sklyarenko's condition given in Theorem 1 for a system $\mathcal{F}$ of functions. Theorem 3 shows the equivalence of the uniform Sklyarenko condition and the equi-integrability. Hence the requirement of equi-integrability of the pointwise convergent sequence $(f_m)_{m=1}^{\infty}$ in Theorem 2 can be replaced by the condition (b) for the sequence $(f_m)_{m=1}^{\infty}$. Moreover we are now in the position to give a simple proof of the Sklyarenko-type Theorem 1.

Proof of Theorem 2: If the system of functions $\mathcal{F}$ consists of a single function $f : [a,b] \mapsto \mathbb{R}$ then the equi-integrability of $\mathcal{F}$ reduces to the integrability of f and the uniform Sklyarenko condition (b) from Theorem 3 is the same as the Sklyarenko condition (1) given in Theorem 1. Hence Theorem 1 is a corollary of Theorem 3. $\square$

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