On hereditary and product-stable quotient maps

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Abstract. It is shown that the quotient maps of a monotopological construct **A** which are preserved by pullbacks along embeddings, projections, or arbitrary morphisms, can be characterized by being quotient maps in appropriate extensions of **A**.

Keywords: hereditary quotient, product-stable quotient, pullback-stable quotient; extensional topological hull, CCT hull, topological universe hull; pretopological spaces, pseudotopological spaces

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A great part of research in topology has been devoted to questions of the type: Which properties of topological spaces are preserved by certain classes of continuous maps? Which classes of topological spaces occur as images under such maps? Under which conditions have the images certain specified properties? Among the classes of maps considered frequently in this context are the *pseudo-open* maps [Ar 63, Def. 2] and the *bi-quotient* maps [Ha 66], [Mi 68, 1.1, 2.2] (cf. Applications (1), (2)).

In a survey article on such questions [Ar 66, p. 127], Arhangel'skiĭ writes: "Many such irregularities can be explained by the fact that the property of being a quotient mapping is not hereditary." This is the importance of the pseudo-open maps: They are exactly those quotient maps in the category **Top** of topological spaces (and continuous maps) that are *hereditary* [Ar 63, Thm. 1]. (For a formal definition of hereditary quotient maps, see below.) The bi-quotient maps have a similar categorical property: Day and Kelly showed [DK 70, Thm. 1] that they are the *pullback-stable* quotients in **Top**.

A connection between this categorical characterization of pseudo-open, resp. bi-quotient maps and a description in terms of convergence was recently pointed out in [BHL 91, Props. 28, 35]—which, in fact, prompted our research for this paper: The pseudo-open maps of topology coincide with the quotient maps (between topological spaces) in the category **PrT** of pretopological spaces [Ke 69, Thm. 4], while the bi-quotient maps are just the quotient maps (between topological spaces) in the category **PsT** of pseudotopological spaces [Ke 69, Thm. 5] (which explains many of the properties of bi-quotient maps given in [Mi 68]). We will, in the following, establish that these facts follow a general categorical pattern. More precisely, we

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will show that for a well-fibred monotopological construct **A**, the quotient maps in **A** that are stable under certain pullbacks in **A** can be represented as the quotient maps between **A**-objects in certain extensions of **A** (subject to the existence of such extensions).

Convention. Throughout this paper, **A** will denote a *monotopological construct*, i.e., a concrete category over **Set** which has (unique) initial lifts w.r.t. point-separating structured sources. All constructs are assumed to be *well-fibred*, i.e., to be small-fibred and to have constants.

Recall that a surjective final **A**-morphism is called a *quotient map*. An injective initial **A**-morphism is called an *embedding*, and $B \in \mathbf{A}$ is called a *subspace* of $A \in \mathbf{A}$ if B, as a set, is contained in A and the inclusion map is an embedding.

The types of quotient maps in **A** that we will identify as the quotient maps between **A**-objects in some extension of **A** are listed in the following definition:

Definition. A quotient map $f : X \longrightarrow Y$ in **A** is called

(1) *hereditary* iff for each subspace B of Y, the domain-range restriction $f' : f^{-1}(B) \longrightarrow B$, where $f^{-1}(B)$ is considered as a subspace of X, is again a quotient;
(2) *product-stable* iff for each $A \in \mathbf{A}$, the product map $f \times 1 : X \times A \longrightarrow Y \times A$ is a quotient;
(3) *pullback-stable* iff every pullback of f along an arbitrary morphism h is a quotient.

Recall that the definitions of hereditary and product-stable quotients can be formulated in terms of pullbacks, by restricting the h in (3) to embeddings and projections, respectively.

The above-mentioned results that the hereditary, resp. pullback-stable, quotient maps in **Top** are exactly those quotients in **PrT**, resp. **PsT**, whose domain and range are topological spaces, are put into a general categorical perspective by the fact that **PrT** is the extensional topological hull of **Top** [He 88a, p. 259], and **PsT** is the topological universe hull of **Top** [Wy 76, 4.9]. Theorems 1 and 3 below will show that these connections remain true if **Top** is replaced by any monotopological construct (provided the respective hulls exist). Theorem 2 gives a similar result for product-stable quotients.

We will use the following terminology:

$Y^\sharp$	–	one-point extension of Y (in an extensional topological construct; underlying set $\lvert Y^\sharp\rvert$ is obtained from $\lvert Y\rvert$ by adjoining one new point ∞_Y);
$[X,Y]$	–	function space (in a cartesian closed topological construct; $\lvert[X,Y]\rvert$ is the set of all morphisms from X to Y);
ev	–	usual evaluation map $\mathrm{ev} : [X,Y] \times X \longrightarrow Y$, defined by $\mathrm{ev}(f,x) = f(x)$;
h^*	–	associated map $h^* : W \longrightarrow [X,Y]$ of $h : W \times X \longrightarrow Y$, defined by $h^*(w)(x) = h(w,x)$;
ETH**A**	–	extensional topological hull of **A** (least finally dense, extensional topological extension of **A**, characterized by the initial density of $\{ A^\sharp \mid A \in \mathbf{A} \}$);

CCTH**A** – cartesian closed topological hull of **A** (least finally dense, cartesian closed topological extension of **A**, characterized by the initial density of $\{ [A, B] \mid A, B \in \mathbf{A} \}$);

TUH**A** – topological universe hull of **A** (least finally dense topological universe extension of **A**, coincides with CCTH(ETH**A**)).

For more detailed information about these concepts, we refer, for example, to [Ad 86], [He 88a], [Sc 86] (extensional topological constructs and hulls, one-point extensions), [He 74], [HN 77] (cartesian closed topological constructs and hulls, function spaces), [Ad 86], [AH 86], [Sc 89], [ARS 91] (topological universes and topological universe hulls).

Theorem 1. *An* **A***-morphism* $f : X \longrightarrow Y$ *is a hereditary quotient in* **A** *iff it is a quotient in the extensional topological hull of* **A**.

Proof: The existence of ETH**A** is ensured by [He 88b, 4.2]. Since quotients in ETH**A** are hereditary, and **A** is closed under formation of subspaces in ETH**A**, it is clear that f is a hereditary quotient map in **A** whenever it is a quotient map in ETH**A**. For the converse implication, consider the object $Z \in$ ETH**A** which has the same underlying set as Y and makes $f : X \longrightarrow Z$ a quotient in ETH**A**. Then $1_{|Y|} : Z \longrightarrow Y$ is a morphism in ETH**A**; it remains to be shown that $1_{|Y|} : Y \longrightarrow Z$ is also an ETH**A**-morphism.

$$\begin{tikzcd}
f^{-1}(g^{-1}(A)) \arrow[r, hook] \arrow[d, "f'"] \arrow[ddd, bend right=60, "(g\circ f)'"'] & X \arrow[d, "f"] \arrow[dd, bend left=50, "f"] \arrow[ddd, bend left=80, "g\circ f"] \\
g^{-1}(A) \arrow[r, hook] \arrow[dd, dashed, "g'"'] & Y \arrow[d, dashed, "1_{|Y|}"] \arrow[dd, dashed, bend right=60, "g"'] \\
 & Z \arrow[d, "g"] \\
A \arrow[r, hook] & A^{\sharp}
\end{tikzcd}$$

Since the source $S = (g : Z \longrightarrow A^{\sharp} \mid A \in \mathbf{A},\ g \in \mathrm{Mor}(\mathrm{ETH}\mathbf{A}))$ is initial, it is sufficient to show that all maps $g : Y \longrightarrow A^{\sharp}$ with $g \in S$ are ETH**A**-morphisms. Consider the subspace $g^{-1}(A)$ of Y. By assumption, the restriction $f' : f^{-1}(g^{-1}(A)) \longrightarrow g^{-1}(A)$ of $f : X \longrightarrow Y$ is a quotient map in **A**. It follows that the restriction $g' : g^{-1}(A) \longrightarrow A$ of the map $g : Y \longrightarrow A^{\sharp}$ is a morphism, because $g \circ f : X \longrightarrow A^{\sharp}$ is an ETH**A**-morphism and consequently, $(g \circ f)' = g' \circ f'$ is an **A**-morphism. By the extensionality of ETH**A**, we obtain that $g : Y \longrightarrow A^{\sharp}$ is a morphism in ETH**A**, which completes the proof. $\square$

In giving similar characterizations of the product-stable and pullback-stable quotient maps of $\mathbf{A}$, one difference should be noted: While *any* well-fibred monotopological construct has an extensional topological hull [He 88b, 4.2], it is well-known that for CCT hulls and topological universe hulls, this is true only up to problems of "size"—though the constructions of these hulls can formally be carried out, they may lead to "huge" quasicategories, which, in particular, may fail to be small-fibred. We therefore include existence assumptions in the following two results.

Theorem 2. *An* $\mathbf{A}$*-morphism* $f : X \longrightarrow Y$ *is a product-stable quotient in* $\mathbf{A}$ *iff it is a quotient in the cartesian closed topological hull of* $\mathbf{A}$*, in case this hull exists.*

PROOF: If f is a quotient map in CCTH$\mathbf{A}$, then clearly f is a product-stable quotient in $\mathbf{A}$, because quotients in CCTH$\mathbf{A}$ are product-stable, and $\mathbf{A}$ is closed under formation of products in CCTH$\mathbf{A}$.

Conversely, assume that $f : X \longrightarrow Y$ is a product-stable quotient in $\mathbf{A}$. Let $Z \in$ CCTH$\mathbf{A}$ be the object with $|Z| = |Y|$ which makes $f : X \longrightarrow Z$ final in CCTH$\mathbf{A}$. We have to show that $Y = Z$, i.e., that $1_{|Y|} : Y \longrightarrow Z \in \mathrm{Mor}(\mathrm{CCTH}\mathbf{A})$. By the initial density of $\{ [A, B] \mid A, B \in \mathbf{A} \}$ in CCTH$\mathbf{A}$, it is sufficient to show that $g : Y \longrightarrow [A, B]$ is a CCTH$\mathbf{A}$-morphism whenever $g : Z \longrightarrow [A, B]$ is a CCTH$\mathbf{A}$-morphism with $A, B \in \mathbf{A}$.

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[rr, bend left, "f"] & Y \arrow[r, dashed, "1_{|Y|}"] \arrow[rr, dashed, bend right, "g"'] & Z \arrow[r, "g"] & {[A,B]} \end{tikzcd}$$

Since $g \circ f : X \longrightarrow [A, B] \in \mathrm{Mor}(\mathrm{CCTH}\mathbf{A})$, we obtain by the adjunction that $\mathrm{ev} \circ ((g \circ f) \times 1) : X \times A \longrightarrow B$ is an $\mathbf{A}$-morphism. By assumption, $f \times 1 : X \times A \longrightarrow Y \times A$ is a quotient map in $\mathbf{A}$. Consequently, the commutativity of the upper triangle in the diagram

$$\begin{tikzcd} X \times A \arrow[rr, "\mathrm{ev} \circ ((g \circ f) \times 1)"] \arrow[d, "f \times 1"] \arrow[dd, bend right, "(g \circ f) \times 1"'] & & B \\ Y \times A \arrow[urr, dashed, "\mathrm{ev} \circ (g \times 1)"] \arrow[d, dashed, "g \times 1"] & & \\ {[A,B] \times A} \arrow[uurr, "\mathrm{ev}"'] & & \end{tikzcd}$$

on the set level implies that $\mathrm{ev} \circ (g \times 1) : Y \times A \longrightarrow B$ is a morphism. Hence the associated map $g = (\mathrm{ev} \circ (g \times 1))^* : Y \longrightarrow [A, B]$ is a CCTH**A**-morphism, as desired. $\square$

Theorem 3. *An* **A***-morphism $f : X \longrightarrow Y$ is a pullback-stable quotient in* **A** *iff it is a quotient in the topological universe hull of* **A***, in case this hull exists.*

PROOF: Assume $f : X \longrightarrow Y$ to be a pullback-stable quotient in **A**. In order to show that f is a quotient in TUH**A**, observe first that $f \times 1 : X \times A \longrightarrow Y \times A$ is a hereditary quotient in **A** for each $A \in \mathbf{A}$, since $f \times 1 : X \times A \longrightarrow Y \times A$ is the pullback of $f : X \longrightarrow Y$ along the projection $p_Y : Y \times A \longrightarrow Y$, and pullbacks compose. By Theorem 1, it follows that $f \times 1 : X \times A \longrightarrow Y \times A$ is a quotient in ETH**A** for each $A \in \mathbf{A}$.

Now it is easy to see that $f : X \longrightarrow Y$ is a product-stable quotient in ETH**A**: Take any $Z \in$ ETH**A**. By the final density of **A** in ETH**A**, it follows that

$$\big(1 \times g : Y \times A \longrightarrow Y \times Z \;\big|\; A \in \mathbf{A},\, g : A \longrightarrow Z \in \mathrm{Mor}(\mathrm{ETH}\mathbf{A})\big)$$

is a final episink in ETH**A** [Ne 77, Lemma 3b]. Since $f \times 1 : X \times A \longrightarrow Y \times A$ is a quotient in ETH**A**, we obtain that the composition

$$\big((1 \times g) \circ (f \times 1) : X \times A \longrightarrow Y \times Z \;\big|\; A \in \mathbf{A},\, g : A \longrightarrow Z \in \mathrm{Mor}(\mathrm{ETH}\mathbf{A})\big)$$

is a final episink in ETH**A**. Because of $(1 \times g) \circ (f \times 1) = (f \times 1) \circ (1 \times g)$, this implies that $f \times 1 : X \times Z \longrightarrow Y \times Z$ is a quotient in ETH**A**.

It follows, by Theorem 2, that $f : X \longrightarrow Y$ is a quotient in CCTH(ETH**A**) = TUH**A**. $\square$

Other categorical characterizations of pullback-stable quotients can be found in [RT 91].

Remark.

(1) A well-fibred topological construct is extensional (cartesian closed, a topological universe) iff quotients *and coproducts* are preserved by pullbacks along embeddings (resp. projections, arbitrary morphisms). It is a special feature of **Top** that coproducts are pullback-stable. It is therefore worth pointing out that even in the general case, the characterizations in Theorems 1–3 do not in any way depend on coproducts.

(2) In fact, characterizations of hereditary, product-stable, and pullback-stable coproducts can be formulated (and proved) in analogy to Theorems 1–3. The same is true for the characterization of hereditary, product-stable, and pullback-stable final episinks.

(3) It is possible to generalize these results by dropping the assumption of small-fibredness.

Applications.

(1) For any epireflective subcategory **A** of **PrT** which does not consist of indiscrete spaces only, we know that TUH**A** = **PsT** by [Sc 90, Thm. 2].

Consequently, by Theorem 3, the pullback-stable quotient maps in all these categories **A** are characterized by the fact that they are quotients in **PsT**, i.e., surjections $f : X \longrightarrow Y$ satisfying the condition that an ultrafilter $\mathcal{U}$ converges to a point $y \in Y$ if and only if $\mathcal{U}$ is the image of an ultrafilter converging to some $x \in f^{-1}(y)$. (Equivalently, in terms of adherence points of filters, the latter condition reads: $y \in Y$ is an adherence point of a filter $\mathcal{F}$ iff some point in $f^{-1}(y)$ is an adherence point of $f^{-1}(\mathcal{F})$. Noting that in either condition, the "if" part is just the continuity of f, this is, for the case $\mathbf{A} = \mathbf{Top}$, exactly the description of a bi-quotient map as given in [Mi 68, 2.2].)

(2) Under the same assumptions on **A** as in (1), the extensional topological hull of **A** is given either by **PrT** or by R_0**PrT** (i.e. those pretopological spaces fulfilling the symmetry axiom

$$\dot{x} \longrightarrow y \iff \dot{y} \longrightarrow x\,)$$

depending on the Sierpinski space **2** being in **A** or not [Sc 90, Corollary]. Consequently, if $\mathbf{2} \in \mathbf{A}$, then the hereditary quotient maps of **A** are characterized by the fact that they are quotients in **PrT**, i.e. surjective maps $f : X \longrightarrow Y$ fulfilling

$$\mathcal{V}(y) = \bigcap_{x \in f^{-1}(y)} f(\mathcal{V}(x)) \quad \text{for all } y \in Y,$$

where $\mathcal{V}(z) = \bigcap\{\, \mathcal{F} \mid \mathcal{F} \longrightarrow z \,\}$. In case of $\mathbf{A} = \mathbf{Top}$, this coincides with the definition of a pseudo-open map as given in [Ar 63], since $\mathcal{V}(z)$ can be replaced by the neighborhood filter of z (and the inclusion $\subset$ is equivalent to the continuity of f).

If $\mathbf{2} \notin \mathbf{A}$—for example, if **A** is one of the categories of completely regular spaces, zerodimensional spaces, Hausdorff spaces or Tychonoff spaces—then the hereditary quotients in **A** are characterized by being quotients in R_0**PrT**. They can be described internally as those surjective maps $f : X \longrightarrow Y$ fulfilling

$$\mathcal{V}(y) = [B_y] \cap \bigcap_{x \in f^{-1}(y)} f(\mathcal{V}(x)) \quad \text{for all } y \in Y,$$

where

$$z \in B_y \iff \dot{y} \supset \bigcap_{a \in f^{-1}(z)} f(\mathcal{V}(a))$$

and $[B_y]$ is the filter on Y generated by B_y. A reformulation which bears closer resemblance to the definition of pseudo-open maps would read: f is a continuous surjection with the property that for each "neighborhood" $U \in \mathcal{V}(f^{-1}(y)) = \bigcap_{x \in f^{-1}(y)} \mathcal{V}(x)$ of $f^{-1}(y)$, there is some $V \in \mathcal{V}(y)$ such that for any point $z \in V \setminus f(U)$ and any neighborhood $W \in \mathcal{V}(f^{-1}(z))$, one has $y \in f(W)$.

Note that the map $f : [0,1] \longrightarrow \mathbf{I}_2$ from the unit interval (in its usual topology) to the indiscrete two-point space, defined by $f(x) = 0$ if $0 \leq x \leq 1/2$, $f(x) = 1$ if $1/2 < x \leq 1$, is a hereditary quotient in $\mathrm{R}_0\mathbf{Top}$ which is not pseudo-open.

(3) The product-stable quotient maps in **Top** and $\mathrm{T}_0\mathbf{Top}$ are characterized by being quotients in the category of Antoine spaces by Theorem 2. (For a discussion of Antoine spaces, see [Bo 75].) An internal characterization of these maps was given by Day and Kelly [DK 70, Thm. 2].

(4) Since the cartesian closed monotopological (CCMT) hull of a well-fibred construct **A** exists iff **A** has a CCT hull [Al 85, 4.3], Theorem 2 implies that the product-stable quotients in **A** can also be characterized by being quotients in the CCMT hull of **A**. For example, by [We 91, 5.5], the product-stable quotients in the category of Tychonoff spaces are characterized by being quotients in the category of c-embedded spaces (= ω-regular, *Hausdorff* pseudotopological spaces).

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