Four-dimensional curvature homogeneous spaces

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Abstract. We prove that a four-dimensional, connected, simply connected and complete Riemannian manifold which is curvature homogeneous up to order two is a homogeneous Riemannian space.

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1. Introduction and preliminaries.

Let (M, g) be an n-dimensional, connected Riemannian manifold with Levi Civita connection ∇ and Riemannian curvature tensor R defined by

$$R_{XY} = [\nabla_X, \nabla_Y] - \nabla_{[X,Y]}$$

for all smooth vector fields X, Y. Denote by $\nabla R, \dots, \nabla^k R, \dots$ its successive covariant derivatives and assume $\nabla^0 R = R$.

In his work on infinitesimally homogeneous spaces [11], I.M. Singer considered the following condition

$$\begin{aligned} P(\ell) \,:\, &\text{for every } x, y \in M \text{ there exists a linear isometry } \phi : T_xM \to T_yM \\ &\text{such that } \phi^*((\nabla^k R)_y) = (\nabla^k R)x \text{ for } k = 0, 1, \dots, \ell. \end{aligned}$$

A Riemannian manifold such that $P(0)$ holds is said to be *curvature homogeneous* and if $P(\ell)$ holds, we say that it is *curvature homogeneous up to order* ℓ. Now, for any point $x \in M$, let G^x_s be the Lie group

$$G^x_s = \{a \in O(T_xM) \mid (\nabla^i_x R)a = (\nabla^i R)_x, \quad i = 0, 1, \dots, s\}.$$

Its Lie algebra $\mathfrak{g}^x_s$ consists of all skew-symmetric endomorphisms A of T_xM such that $A \cdot (\nabla^i R)_x = 0$ for $i = 0, \dots, s$. Here A acts as a derivation of the tensor algebra. Clearly, there always exists a first integer k_x such that $\mathfrak{g}_{k_x} = \mathfrak{g}_{k_x+1}$.

Further, if $P(\ell)$ is satisfied, then $\mathfrak{g}^x_i$ and $\mathfrak{g}^y_i$ are conjugated for $0 \le i \le \ell$. Hence, if $P(k_x + 1)$ holds, k_x does not depend on x. In this case we put $\mathfrak{g}^x_i = \mathfrak{g}_i$, $k_M = k_x$ and a Riemannian manifold satisfying the condition $P(k_M + 1)$ is said to be *infinitesimally homogeneous* [11]. Singer's main result in [11] is then the following

Theorem. *A connected, simply connected, complete, infinitesimally homogeneous Riemannian manifold is a homogeneous Riemannian space.*

Note that $k_M \leq \frac{1}{2}n(n-1)-1$. M. Gromov gives in [3, p. 165] a better estimate, namely $k_M < \frac{3}{2}n - 1$.

From [11] we obtain also the following useful

Lemma. *If $P(r)$ is satisfied, then there exists a maximal principal subbundle F_r^b of the orthonormal frame bundle $O(M,g) \to M$ on which the components $R_{ijk\ell}$ and $R_{h_s \dots h_1, ijk\ell}$, $1 \leq h_1, \dots, h_s, i, j, k, \ell \leq n$, $1 \leq s \leq r$, are constants and which contains a given frame $b \in O(M,g)$. Moreover, the connected component of the identity of G_r^x, $x \in M$ being arbitrary, is the structure group of F_r^b.*

Here we use the notational convention

$$R_{ijk\ell} = g(R_{e_i e_j} e_k, e_\ell),$$
$$R_{h_s \dots h_1, ijk\ell} = g((\nabla^s_{h_s \dots h_1} R)_{e_i e_j} e_k, e_\ell),$$

where $\{e_i, i = 1, \dots, n\}$ is an orthonormal frame.

Many examples of non-homogeneous curvature homogeneous (i.e., satisfying $P(0)$) are known. We refer to [6], [7], [8], [9], [12], [13] for more details and further references. For three-dimensional manifolds Singer's estimate is $k_M + 1 \leq 3$, but in [10] the first author proved a better result:

Theorem A. *Let (M,g) be a three-dimensional, connected, simply connected, complete Riemannian manifold which is curvature homogeneous up to order 1. Then (M,g) is homogeneous and moreover, (M,g) is either symmetric or a group space with a left invariant metric.*

A short proof of the homogeneity has been given in [5].

For four-dimensional manifolds Singer's estimate gives $k_M + 1 \leq 6$ and Gromov's estimate $k_M + 1 < 6$. The main purpose of this note is to prove

Theorem B. *Let (M,g) be a four-dimensional, connected, simply connected, complete Riemannian manifold which is curvature homogeneous up to order two. Then (M,g) is homogeneous.*

Note that a result given in [1], [4] then yields that (M,g) is either symmetric or a group space with a left invariant metric.

2. Proof of Theorem B.

We will divide the proof into several lemmas.

First, let $u = (e_1, \dots, e_n)$ be a smooth local cross section of $O(M,g)$ and put

$$\nabla_{e_i} e_j = \sum_{k=1}^{n} \Gamma_{ijk} e_k\,, \quad i, j = 1, \dots, n.$$

Then the local functions Γ_{ijk} satisfy

$$\Gamma_{ijk} + \Gamma_{ikj} = 0, \quad i, j, k = 1, \dots, n.$$

Next, let (M,g) be a four-dimensional, connected, simply connected, complete Riemannian manifold. For $x \in (M,g)$, we may choose an orthonormal basis $\{e_i, i = 1,\dots,4\}$ such that

$$Qe_i = \lambda_i e_i, \quad 1 \le i \le 4,$$

where Q denotes the Ricci endomorphism. Then we have to consider the following five cases:

(I) four different Ricci eigenvalues,
(II) three different Ricci eigenvalues,
(III) two Ricci eigenvalues with multiplicity two,
(IV) three equal Ricci eigenvalues,
(V) four equal Ricci eigenvalues.

Note that, if (M,g) satisfies $P(0)$, all λ_i are constant functions on (M,g).

We start with

Lemma 2.1. *Let (M,g) be of type* (I). *If it satisfies $P(1)$, then it is homogeneous.*

Proof: For manifolds of this type, $\mathfrak{g}_0 = \{0\}$ or equivalently, $k_M = 0$. Then the result follows from Singer's theorem. □

Lemma 2.2. *A manifold (M,g) of type* (II) *satisfying $P(2)$ is homogeneous.*

Proof: The hypothesis implies that the following two cases are possible:

(i) $\mathfrak{g}_0 = \{0\}$,
(ii) $\mathfrak{g}_0 = \mathfrak{so}(2)$.

In the case (i), (M,g) is homogeneous by the same argument as in Lemma 2.1. For (ii) we have $\mathfrak{g}_1 = \mathfrak{so}(2)$ or $\mathfrak{g}_1 = \{0\}$. Then the result follows again by using Singer's result. □

Lemma 2.3. *A manifold (M,g) of type* (III) *satisfying $P(2)$ is homogeneous.*

Proof: Here, the following cases may occur:

(i) $\mathfrak{g}_0 = \{0\}$,
(ii) $\mathfrak{g}_0 = \mathfrak{so}(2)$,
(iii) $\mathfrak{g}_0 = \mathfrak{so}(2) \oplus \mathfrak{so}(2)$.

For the cases (i) and (ii) we may conclude as in Lemma 2.1 and Lemma 2.2. So, it suffices to consider the case (iii).

Then, the largest possible decreasing series of Lie algebras starting from $\mathfrak{g}_0$ which we have to consider, is

$$\mathfrak{g}_0 \supsetneqq \mathfrak{g}_1 = \mathfrak{so}(2) \supsetneqq \mathfrak{g}_2 = \{0\}.$$

This is the case of the condition $P(3)$. So, we cannot use Singer's theorem. To prove the result we shall use the Lemma given in Section 1 and put $\lambda_1 = \lambda_2 \neq \lambda_3 = \lambda_4$.

Since $\mathfrak{g}_0 = \mathfrak{so}(2) \oplus \mathfrak{so}(2)$, there exists a subbundle F_0 of $O(M,g)$ with structure group $SO(2) \times SO(2)$ such that all the functions $R_{abcd}(u)$ are constant on F_0. Now, let u be any smooth local cross section of F_0 on an open set U of M. Then, for any smooth functions θ, η on U, the section

$$\bar{u} = u \begin{pmatrix} \cos\theta & -\sin\theta & 0 & 0 \\ \sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & \cos\eta & -\sin\eta \\ 0 & 0 & \sin\eta & \cos\eta \end{pmatrix}$$

is also a local section of F_0 on U. From $R_{abcd}(u) = R_{abcd}(\bar{u})$ we get easily that, up to sign, the non-zero components of R are

$$\tag{2.1} \begin{cases} R_{1212}, R_{3434}, R_{1313} = R_{1414} = R_{2323} = R_{2424} (= k), \\ R_{1234}, R_{1324}, R_{1423} \quad \text{with} \quad R_{1324} + R_{1423} = 0. \end{cases}$$

Next, since (M, g) satisfies $P(1)$, all the functions $R_{h,ijkl}(u)$ are constant on the subbundle F_1 of F_0 with structure group $SO(2)$. (We may assume that $SO(2) \cong SO(2) \times 1$ in $SO(2) \times SO(2)$ without loss of generality.) Hence all the functions $\varrho_{h,jk}$, where ϱ denotes the Ricci tensor of type $(0,2)$, are constant on F_1. Since

$$\begin{aligned} \varrho_{i,13} &= \Gamma_{i13}(\lambda_1 - \lambda_3), & \varrho_{i,14} &= \Gamma_{i14}(\lambda_1 - \lambda_4), \\ \varrho_{i,23} &= \Gamma_{i23}(\lambda_2 - \lambda_3), & \varrho_{i,24} &= \Gamma_{i24}(\lambda_2 - \lambda_4), \end{aligned}$$

we see that Γ_{i13}, Γ_{i14}, Γ_{i23}, Γ_{i24}, $1 \leq i \leq 4$ are locally constant.

Next, using (2.1), a direct computation yields

$$\tag{2.2} \begin{aligned} R_{i,1213} &= \Gamma_{i23}(R_{1212} - k) + 3\Gamma_{i14} R_{1324}, \\ R_{i,1214} &= \Gamma_{i24}(R_{1212} - k) + 3\Gamma_{i13} R_{1324}, \\ R_{i,1223} &= \Gamma_{i13}(k - R_{1212}) + 3\Gamma_{i24} R_{1324}, \\ R_{i,1224} &= \Gamma_{i14}(k - R_{1212}) - 3\Gamma_{i23} R_{1324}, \\ R_{i,1334} &= \Gamma_{i14}(R_{3434} - k) + 3\Gamma_{i23} R_{1324}, \\ R_{i,1434} &= \Gamma_{i13}(k - R_{3434}) + 3\Gamma_{i24} R_{1324}, \\ R_{i,2334} &= \Gamma_{i24}(R_{3434} - k) - 3\Gamma_{i13} R_{1324}, \\ R_{i,2434} &= \Gamma_{i23}(k - R_{3434}) - 3\Gamma_{i14} R_{1324}, \end{aligned}$$

for $1 \leq i \leq 4$, the other components (up to sign) being zero.

Now, we proceed as above and take an arbitrary local cross section u of F_1 on an open set U of M. Let θ be a smooth function on U. Then

$$\bar{u} = u \begin{pmatrix} \cos\theta & -\sin\theta & 0 & 0 \\ \sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

is also a local section of F_1. Then, since $R_{i,abcd}(\bar{u})$, $1 \le i,a,b,c,d \le 4$, we get by a direct computation,

$$(2.3) \qquad \begin{aligned} R_{1,1213} &= R_{2,1223}, & R_{1,1214} &= R_{2,1224},\\ R_{2,1213} &= -R_{1,1223}, & R_{2,1214} &= -R_{1,1224},\\ R_{1,1334} &= R_{2,2334}, & R_{1,1434} &= R_{2,2434},\\ R_{2,1334} &= -R_{1,2334}, & R_{2,1434} &= -R_{1,2434}, \end{aligned}$$

and

$$(2.4) \qquad R_{3,abcd} = 0, \qquad R_{4,abcd} = 0, \qquad 1 \le a,b,c,d \le 4.$$

Using the second Bianchi identity, (2.4) yields

$$(2.5) \qquad R_{1,1334} = 0, \qquad R_{2,1334} = 0, \qquad R_{1,1434} = 0, \qquad R_{2,1434} = 0.$$

Further, the same identity and (2.3) yield

$$(2.6) \qquad R_{2,1213} = R_{1,1223} = 0, \qquad R_{2,1214} = R_{1,1224} = 0.$$

Hence, from (2.3), (2.5) and (2.6), we get

$$(2.7) \qquad \begin{cases} R_{1,1213} = R_{2,1223},\\ R_{1,1214} = R_{2,1224} \end{cases}$$

and, up to sign, all the other $\nabla_i R_{abcd}$ vanish.

Now, we first assume $R_{1212} = R_{3434}$. Then, from (2.2) we get

$$\nabla_i \varrho_{jk} = 0, \qquad 1 \le i,j,k \le 4.$$

So, using this and (2.7), we obtain that (M,g) is locally symmetric (and hence, symmetric).

Next, we assume $R_{1212} \ne R_{3434}$. In this case, (2.2) yields

$$(2.8) \qquad \Gamma_{i13} = \Gamma_{i24} = \Gamma_{i14} = \Gamma_{i23} = 0 \quad \text{for} \quad i = 3,4.$$

Further, taking account of (2.2) and

$$R_{2,1213} = R_{1,1223} = R_{2,1214} = R_{1,1224} = 0,$$

we obtain

$$(2.9) \qquad \begin{cases} \Gamma_{131} = \Gamma_{132} = \Gamma_{141} = \Gamma_{142} = 0,\\ \Gamma_{231} = \Gamma_{232} = \Gamma_{241} = \Gamma_{242} = 0. \end{cases}$$

Finally, (2.8) and (2.9) yield that the distribution corresponding to the eigenvalue $\lambda_1 = \lambda_2$ is parallel on M. So, also the distribution corresponding to $\lambda_3 = \lambda_4$ is parallel. In this case (M,g) is a direct product of two two-dimensional spaces of constant curvature and hence, symmetric. $\square$

Lemma 2.4. *A manifold* (M,g) *of type* (IV) *satisfying* $P(2)$ *is homogeneous.*

Proof: First, the hypothesis implies that we may have the following cases:

(i) $\mathfrak{g}_0 = \{0\}$,
(ii) $\mathfrak{g}_0 = \mathfrak{so}(2)$,
(iii) $\mathfrak{g}_0 = \mathfrak{so}(3)$.

For the cases (i) and (ii) the required result follows again from Singer's theorem. So, we are left with the case (iii) where we put $\lambda_1 = \lambda_2 = \lambda_3$.

Then there exists a subbundle F_0 of $O(M,g)$ with the structure group $SO(3)$ such that all the functions $R_{abcd}(u)$ are constant on F_0. Again, let u be an arbitrary fixed local cross section of F_0 on an open set U of M and let θ, η, φ be smooth functions on U. Then

$$u' = u \begin{pmatrix} \cos\theta & -\sin\theta & 0 & 0 \\ \sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad u'' = u \begin{pmatrix} \cos\eta & 0 & -\sin\eta & 0 \\ 0 & 1 & 0 & 0 \\ \sin\eta & 0 & \cos\eta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$
$$u''' = u \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\varphi & -\sin\varphi & 0 \\ 0 & \sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

are also local sections of F_0 on U. Using

$$R_{abcd}(u) = R_{abcd}(u') = R_{abcd}(u'') = R_{abcd}(u'''),$$

a direct computation yields

$$\text{(2.10)} \qquad \begin{cases} R_{1212} = R_{1313} = R_{2323} = k, \\ R_{1414} = R_{2424} = R_{3434} = k', \end{cases}$$

the other components (up to sign) being zero.

From (2.10) we obtain

$$\text{(2.11)} \qquad \begin{cases} R_{i,1214} = \Gamma_{i24}(k-k'), & R_{i,2324} = \Gamma_{i34}(k-k'), \\ R_{i,1314} = \Gamma_{i34}(k-k'), & R_{i,2334} = -\Gamma_{i24}(k-k'), \\ R_{i,1334} = -\Gamma_{i14}(k-k'), & R_{i,1224} = -\Gamma_{i14}(k-k'), \end{cases}$$

the other components (up to sign) being zero.

Next, using the second Bianchi identity, (2.11) yields

$$\text{(2.12)} \qquad \begin{cases} R_{1,2423} = R_{1,3423} = 0, \\ R_{2,1413} = R_{2,3413} = 0, \\ R_{3,2412} = R_{3,1214} = 0, \\ R_{4,1214} = R_{4,1224} = R_{4,1314} = 0. \end{cases}$$

Hence, since $k \neq k'$, we get from (2.11) and (2.12):

$$\Gamma_{234} = \Gamma_{324} = \Gamma_{424} = \Gamma_{134} = \Gamma_{341} = \Gamma_{441} = \Gamma_{241} = \Gamma_{434} = \Gamma_{142} = 0. \tag{2.13}$$

Using this and the second Bianchi identity, we then get

$$\begin{cases} R_{2,1412} + R_{1,4212} = -R_{4,2112} = 0, \\ R_{1,3413} + R_{3,4113} = -R_{4,1313} = 0, \\ R_{2,3423} + R_{3,4223} = -R_{4,2323} = 0 \end{cases} \tag{2.14}$$

and hence, with (2.11),

$$\Gamma_{224} = \Gamma_{141}, \quad \Gamma_{141} = \Gamma_{334}, \quad \Gamma_{242} = \Gamma_{334}, \tag{2.15}$$

which yields

$$\Gamma_{114} = \Gamma_{224} = \Gamma_{334} = 0. \tag{2.16}$$

Therefore, from (2.11), (2.13) and (2.16) we get that (M, g) is symmetric. So, M is a product of a three-dimensional space form and a real line. □

Lemma 2.5. *A manifold (M, g) of type (V) satisfying $P(0)$ is homogeneous.*

Proof: In this case (M, g) is a curvature homogeneous Einstein space, and hence symmetric, as follows from a still unpublished result of A. Derdziński [2]. □

The result in Theorem B follows now from Lemmas 1–5.

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