Comment.Math.Univ.Carolin. 33,2 (1992)269–276

On invariant operations on pseudo-Riemannian manifolds

Jan Slovák

Abstract. Invariant polynomial operators on Riemannian manifolds are well understood and the knowledge of full lists of them becomes an effective tool in Riemannian geometry, [Atiyah, Bott, Patodi, 73] is a very good example. The present short paper is in fact a continuation of [Slovák, 92] where the classification problem is reconsidered under very mild assumptions and still complete classification results are derived even in some non-linear situations. Therefore, we neither repeat the detailed exposition of the whole setting and the technical tools, nor we include all details of the proofs, the interested reader can find them in the above paper (or in the monograph [Kolář, Michor, Slovák]).

After a short introduction, we study operators homogeneous in weight on oriented pseudo-Riemannian manifolds. In particular, we are interested in those of weight zero. The results involve generalizations of some well known theorems by [Gilkey, 75] and [Stredder, 75].

Keywords: invariant operators, natural operators, bundle functors, Chern forms, Pontrjagin forms

Classification: 53A55, 58A20, 53C05, 53B20

1. Natural bundles and operators on Riemannian manifolds. The suitable explicit formulation of the classification problem for invariant operators on manifolds with some geometric structures is provided by the general theory of bundle functors and natural operators originated by Nijenhuis, see [Kolář, Michor, Slovák] for a detailed exposition. Roughly speaking, the bundle functors are direct generalizations of the usual tensor bundles (possibly non-linear and of higher orders) while the natural operators are some natural transformations between the infinite dimensional spaces of sections of the bundles. For our purposes, it suffices to deal only with tensor bundles of some type (p, q), i.e. p-times contravariant and q-times covariant, over pseudo-Riemannian manifolds with some fixed dimension m and fixed signature of the metric. Our operators should intertwine the natural action of local isometries on these tensor bundles. This setting covers all operators between natural vector bundles on Riemannian manifolds, which means the associated vector bundles to the pseudo-Riemannian linear frame bundles corresponding to some finite dimensional representation of $O(m', n, \mathbb{R})$ or $SO(m', n, \mathbb{R})$, $m' + n = m$, since each such representation is completely reducible and the irreducible ones live all in some tensor spaces. Moreover, we can really restrict ourselves to operators on the whole tensor bundles, since the invariant subbundles are subjects of both invariant projection onto them and invariant injection into the whole tensor bundles.

The author's research was supported by the University of Vienna during his visit in fall term 1991/1992

But dealing with the whole tensor bundles, we can add the pseudo-metric itself to the arguments of the operation in question and to solve the classification problem concerning the invariance with respect to all local diffeomorphisms (the latter are locally invertible but globally defined mappings).

We shall write $T^{(p,q)}f\colon T^{(p,q)}M \to T^{(p,q)}N$ for the action of a local diffeomorphism $f\colon M \to N$ of two m-dimensional manifolds on the p-times contravariant and q-times covariant tensor bundles, T and T^* are the tangent and cotangent functors, $S^2_{\text{reg}}T^*M$ is the bundle of non-degenerate metrics over M (or one of its connected components corresponding to the individual signatures). The natural operators which we shall discuss are systems of local smooth operators $D_M\colon C^\infty(T^{(p,q)}M) \to C^\infty(T^{(r,s)}M)$, i.e. their values depend only on the germs of the sections in the underlying points and smooth families of sections are transformed into smooth families, which commute with the natural actions of the local diffeomorphisms and which can depend on the metric on M. The whole system of operators D_M, i.e. the natural operator D in question, will be denoted by $D\colon S^2_{\text{reg}}T^* \times T^{(p,q)} \to T^{(r,s)}$ or briefly $D\colon T^{(p,q)} \to T^{(r,s)}$. In general, if E and F are two bundle functors, then $E \times F$ means the bundle functor with the values $(E \times F)M = EM \times_M FM$ and similarly on morphisms. For the sections of tensor bundles we shall use the usual notation with subscripts and superscripts indicating the type of the tensor and we also adopt the usual conventions for expressing the algebraic tensor operations. Since a distinguished non-degenerate (pseudo)-metric is always available, we can raise and lower the indices and so we consider the contraction over each couple of repeated indices.

2. In [Slovák, 92], the classification problems are solved with the help of some general tools which are proved in [Kolář, Michor, Slovák]. The most important ones are:

(1) the non-linear Peetre theorem stating that each local operator is "locally" of finite order
(2) the smooth version of the Schouten's reduction theorem which asserts that each operation depending on a connection factors through the curvature and its covariant derivatives
(3) the application of the latter reduction yields that operations depending on a pseudo-Riemannian metric factor through the values of the metric (i.e. no derivatives are explicitly involved), and through the curvature and its covariant derivatives of the Levi-Cività connection
(4) during the above reduction procedure, the polynomiality of the operations is preserved
(5) smooth functions with certain homogeneity properties are polynomials.

These tools apply to pseudo-Riemannian metrics with an arbitrary signature and the only difference which might effect the proofs from [Slovák, 92, Section 3] is that the Lie algebra valued curvature forms take their values in different Lie algebras. However, after lowering the subscript in the curvature and its covariant derivatives, i.e. considering only the quantities $R_{ijklm_1\dots m_s}$, we can repeat, step by step, all the above mentioned proofs. In particular we obtain the next three theorems.

Let us recall the definition of the operators homogeneous in the weight.

Definition. Let E and F be natural bundles over m-dimensional manifolds. We say that a natural operator $D\colon S^2_{\mathrm{reg}}T^*\times E\to F$ is *conformal*, if $D(c^2g,s)=D(g,s)$ for all metrics g, sections s, and all positive $c\in\mathbb{R}$. If F is a natural vector bundle and D satisfies $D(c^2g,s)=c^\lambda D(g,s)$, then D is said to be *homogeneous with weight* λ.

The weight of the metric g_{ij} is 2 (we consider the inclusion $g\colon S^2_{\mathrm{reg}}T^*\to S^2T^*$), that of its inverse g^{ij} is -2, while the curvature and all its covariant derivatives are conformal.

Let us point out that our "conformal operators" are independent of the deformation of the metric by a (constant) scalar multiplication. The conformally invariant operators from the conformal geometry are operators depending on the choice of metric up to the deformation by a scalar function.

3. Theorem. *All natural operators $D\colon T^{(s,r)}\to T^{(q,p)}$, $s<r$, on pseudo-Riemannian manifolds which are homogeneous in weight result from a finite number of the following steps:*

(a) *take tensor product of arbitrary covariant derivatives of the curvature tensor or the covariant derivatives of the tensor fields from the domain of D*
(b) *tensorize by the metric or by its inverse*
(c) *apply arbitrary $GL(m)$-equivariant operation*
(d) *take linear combinations.*

Let us notice that as a consequence of this theorem, the natural operators in question are polynomial operators involving the metric, the square root of its inverse, its derivatives and the derivatives of the tensor fields from the domain. The operators with the latter polynomiality properties are called the regular operators in [Atiyah, Bott, Patodi, 73].

The proof is analogous to that one in the positive definite case, see [Slovák, 92, subsection 3.3].

4. Theorem. *There are no non-zero homogeneous natural operators $D\colon S^2_{\mathrm{reg}}T^*\times T^{(0,r)}\to\Lambda T^*$ with a positive weight. The algebra of all conformal natural operators $D\colon S^2_{\mathrm{reg}}T^*\times T^{(0,r)}\to\Lambda T^*$ is generated by the Pontrjagin forms p_q, the alternation and the exterior differential. The operators which do not depend on the second argument are generated by the Pontrjagin forms.*

The proof is based on Theorem 3 and a discussion of the symmetries of the curvatures. It goes along lines of the proof of [Slovák, 92, Theorem 3.2] where the result appears in the positive definite case.

Theorem 4 generalizes the famous Gilkey theorem on the uniqueness of the Pontrjagin forms, see [Gilkey, 73], [Atiyah, Bott, Patodi, 73]. The Gilkey theorem describes the regular conformal natural forms in the Riemannian case, while we use no assumptions on the order or polynomiality or regularity, only the smoothness. In [Gilkey, 75], the uniqueness of the Pontrjagin forms is proved on pseudo-Riemannian

case as well. Let us remark, Gilkey proves his theorems directly discussing the derivatives of the metric.

5. Theorem. *There are no non-zero homogeneous natural operators* $D\colon S^2_{\mathrm{reg}}T^* \times T^{(0,0)} \to \Lambda T^*$ *with a positive weight. The algebra of all conformal natural operators* $D\colon S^2_{\mathrm{reg}}T^* \times T^{(0,0)} \to \Lambda T^*$ *is generated by the Pontrjagin forms* p_q*, the compositions with arbitrary smooth functions of one real variable and the exterior differential.*

The proof of this theorem is also analogous to that in the positive definite case, see [Slovák, 92, Theorem 3.6].

6. Linear operations on forms. The discussion leading to the above theorems can be continued for any fixed negative weight, but the number of the natural operators (and complexity of discussion) increases rapidly. But the simplest situation, i.e. the weight $\lambda = -2$, is extremely interesting.

Proposition. *All linear operators* $\Lambda^p T^* \to \Lambda^p T^*$ *on pseudo-Riemannian manifolds which are homogeneous with weight* -2 *are linearly generated by the following generators: the multiplication by scalar curvature, the contraction with the Ricci curvature, the contraction with the full pseudo-Riemannian curvature , the compositions* $\delta \circ d$ *and* $d \circ \delta$*.*

Up to scalar multiples, the formulas for these generators are:

$$v_{i_1\dots i_p} \mapsto \begin{cases} R_{abab}v_{i_1\dots i_p} \\ R_{aba[i_1}v_{i_2\dots i_p]b} \\ R_{ab[i_1i_2}v_{i_3\dots i_p]ab} \\ v_{i_1\dots i_paa} \\ v_{[i_1\dots i_{p-1}aai_p]}. \end{cases}$$

The codifferential δ is a homogeneous operator with weight -2, $\delta\colon \Lambda^p \to \Lambda^{p-1}$, $\delta = (-1)^{(p+1)(m+1)}(m-p+1)v_{i_1\dots i_{p-1}aa}$. The well known Laplace operator is involved as the linear combination $\delta \circ d + d \circ \delta$. On scalar functions, only $\delta \circ d$ is non-zero and we get the usual formula $f \mapsto f_{aa} = g^{ab}f_{ab}$, the usual Laplace operator for the Euclidean metric $g_{ab} = \delta_{ab}$, and the wave (Klein-Gordon) operator for the pseudo-Euclidean metric.

This result was originally proved by [Stredder, 75] under the stronger assumptions of regularity and only for the Riemannian manifolds. The proof of our proposition is rather easy, once we become familiar with the discussion on monomials in the curvature and its covariant derivatives used in the proofs of the above theorems, see also Section 8 below. We leave the details to the reader.

7. Operations on oriented pseudo-Riemannian manifolds. In the description of natural operators $S^2_{\mathrm{reg}}T^* \times E \to F$ we have to use the $O(m', n)$-invariance only at the very end of the proof of Theorem 3 and then only the symmetries of the ingredients from the list in 3 are to be exploited. Therefore, we can prove easily:

Theorem. *All natural operators $D\colon T^{(s,r)} \to T^{(q,p)}$, $s < r$, on oriented pseudo-Riemannian manifolds which are homogeneous in weight result from a finite number of the following steps:*

(a) *take tensor product of arbitrary covariant derivatives of the curvature tensor or the covariant derivatives of the tensor fields from the domain*
(b) *tensorize by the metric or by its inverse*
(c) *tensorize by the (pseudo)-Riemannian volume form ν*
(d) *apply arbitrary $GL(m)$-equivariant operation*
(e) *take linear combinations.*

PROOF: Since the $SO(m', n, \mathbb{R})$ invariant tensors are generated by the $O(m', n, \mathbb{R})$ invariant ones and the (pseudo)-Riemannian volume form ν (this is the classical Weyl theory, see [Weyl, 39] or [Stredder, 75] in the positive definite case), we have only to prove that the covariant derivatives of the volume form ν cannot be involved. But the latter are obviously zero. □

Let us remark that the latter theorem, as well as Theorem 3 are valid also without the requirement $s < r$ if we add the polynomiality assumption.

8. Conformal operators on forms. The volume form ν is defined by the expression $\nu_{i_1 \dots i_m} = ((-1)^n \det(g_{ij}))^{1/2} \varepsilon_{i_1 \dots i_m}$ (the signature of the pseudo-metric is (m', n)) and so it is evidently homogeneous with weight m. Thus, the homogeneous weight of $*\colon \Omega^p \to \Omega^{m-p}$ is $m - 2p$. In general, there exist more conformal natural operators in the oriented case. First of all, if the dimension $m = 2p$ is even, then $**\colon \Omega^p \to \Omega^p$ is identity up to sign and we can split the space of p-forms, $\Omega^p = \Omega^p_+ \oplus \Omega^p_-$, where $\Omega_\pm$ are the eigen spaces for $*$ (the eigen values are ± 1 or $\pm\sqrt{-1}$). If we compose the exterior differential d with the projections, we get the operators $d = d_+ + d_-$ and the compositions $d \circ d_\pm$ are no more zero. Further, it might happen that composing enough d's and $*$'s together, we get a conformal operator. Let us write $\delta_q = *d* \dots d*\colon \Omega^{q+1} \to \Omega^{m-q-1}$, $q < p$, with $m - 2q - 1$ stars involved, and $D_q = d \circ \delta_q \circ d\colon \Omega^q \to \Omega^{m-q}$.

Proposition. *If the dimension $m = 2p$ is even, then each operator D defined by $D = D_q = d \circ \delta_q \circ d$ or $D = \delta_q \circ d$ or $D = \delta_q$ is a conformal natural operator on oriented pseudo-Riemannian manifolds. In particular, $D_{p-1}\colon \Omega^{p-1} \to \Omega^{p+1}$ equals $d * d = d \circ d_+ - d \circ d_-$.*

Up to the constant multiples, these operators D are the only non-zero conformal linear natural operators on exterior forms on flat pseudo-Riemannian manifolds beside the identities and the exterior differentials d, $d_\pm$.

If the dimension m is odd, then the only non-zero conformal linear natural operators on forms on flat pseudo-Riemannian manifolds are the exterior differentials and the identities and their constant multiples.

PROOF: Clearly each operator D is natural. If we start in Ω^{q+1} and apply $*d*$, then the mappings go: $\Omega^{q+1} \mapsto \Omega^{m-q-1} \mapsto \Omega^{m-q} \mapsto \Omega^q$ while the weights which are added are: $0 \mapsto m - 2q - 2 \mapsto m - 2q - 2 \mapsto -2$ (the total is obvious — the weight of δ). Hence if $m = 2p$, $q < p$ and if we start at Ω^{q+1} we reach weight zero

exactly after composing $(m-2q-2)$-times $d*$ and applying $*$ at the very end. In all other cases we never get weight zero, for each turn around decreases the weight by 2 and once we get back to the initial position with a negative weight in all three last positions the hope is lost.

Let us now perform the general discussion in our special situation and let us restrict ourselves to the natural operators on the whole category of (not oriented) pseudo-Riemannian manifolds. If we want to get a linear operator $D\colon \Omega^q \to \Omega^{q'}$ which is non-zero on flat manifolds, then the only monomials which make sense are of the form $v_{i_1\dots i_q l_1\dots l_s}$. Since we are in the flat case, the covariant derivatives l_k are symmetric. Thus at most one index among the l's may remain uncontracted and at most one can be contracted with some of the i's, for the alternation of the remaining uncontracted indices would kill the expression otherwise. Hence what we only can do is to involve $2s$ or $2s+1$ or $2s+2$ derivatives, to choose s pairs, to contract them and to contract one of the remaining indices (if any) with some of the i's. Hence, up to constant multiples and linear combinations, $D = d\circ\delta\cdots\circ d$ or $D = \delta\circ d\cdots\circ d$ or $D = d\circ\delta\cdots\circ\delta$ or $D = \delta\circ d\cdots\circ\delta$, and we get $q'-q = 1$ or 0 or 0 or -1, respectively.

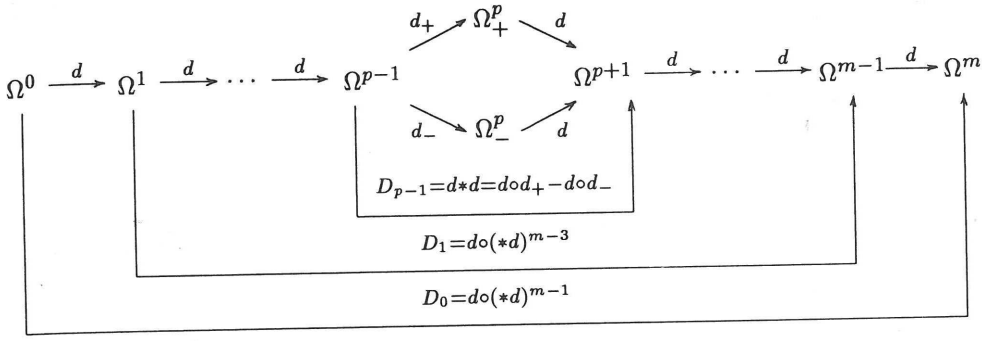
On the space of all natural operators $D\colon \Omega^q \to \Omega^{q'}$, there is the canonical action of $O(m',n)/SO(m',n) = \mathbb{Z}_2$ and so each such operator is a sum $D = D_+ + D_-$ where D_+ is invariant with respect to the change of orientation while D_- changes the sign. If D is natural and conformal, then also both D_+ and D_- are natural and possibly conformal. Now, notice that $*D_-$ is invariant with respect to the change of orientation and $D_- = \pm **D_-$. Thus, $*D_-\colon \Omega^q \to \Omega^{m-q'}$ and, up to constant multiples and linear combinations, either $m-q'-q = 1$ and $**D_- = *d\circ\delta\cdots\circ d$, or $m-q'-q = 0$ and $**D_- = *\delta\circ d\cdots\circ d$ or $**D_- = *d\circ\delta\cdots\circ\delta$, or $m-q'-q = -1$ and $**D_- = *\delta\circ d\cdots\circ\delta$, respectively. The last Hodge star in these operators acts on $\Omega^{m-q'}$ and so its weight is $2q'-m$. If m is odd then this can never kill the even negative weight appearing through δ's. Thus, there is no codifferential involved in the expression, $D_- = 0$ and D is either exterior differential or identity (up to constant multiples). This proves the last statement of the proposition.

If $m = 2p$ is even and $2q'-m < 0$, then the weight of $*$ is negative and we get the same result as in the odd-dimensional case. If $2q'-m \geq 0$, then a simple discussion shows that the only possible operators are those listed in the proposition. □

While the operators in the odd dimensional case form the well known de Rham resolvent

$$0 \longrightarrow \Omega^0 \xrightarrow{d} \Omega_1 \xrightarrow{d} \cdots \xrightarrow{d} \Omega^{m-1} \xrightarrow{d} \Omega_m \longrightarrow 0,$$

the most interesting operators in the even dimension m are described in the diagram



The diagram is not commutative! The horizontal line is exact, but not the arrows in the central diamond. The diagram does not exhaust all operators from the proposition, but notice that the operators indicated on the arrows are unique, up to multiples.

It is interesting that the latter operators are exactly the conformally invariant operators on forms on conformally flat manifolds, but they do not commute with conformal local isomorphisms of general conformal manifolds. On the other hand, they all admit extensions to all conformal manifolds beside D_0, cf. [Baston, Eastwood, 90].

9. Remark. Analogous results to Theorems 4 and 5 can be also formulated for oriented pseudo-Riemannian manifolds. The only new ingredient is the distinguished volume form, but since this has a positive weight, we get much more natural operators with positive weights.

References

Atiyah M., Bott R., Patodi V.K., *On the heat equation and the index theorem*, Inventiones Math. **19** (1973), 279–330.

Baston R.J., Eastwood M.G., *Invariant operators*, Twistors in mathematics and physics, Lecture Notes in Mathematics 156, Cambridge University Press, 1990.

Gilkey P.B., *Curvature and the eigenvalues of the Laplacian for elliptic complexes*, Advances in Math. **10** (1973), 344–382.

Gilkey P.B., *Local invariants of a pseudo-Riemannian manifold*, Math. Scand. **36** (1975), 109–130.

Kolář I., Michor P.W., Slovák J., *Natural operations in differential geometry*, to appear in Springer-Verlag, 1992.

Nijenhuis A., *Natural bundles and their general properties*, in Differential Geometry in Honor of K. Yano, Kinokuniya, Tokyo, 1972, 317–334.

Slovák J., *On invariant operations on a manifold with connection or metric*, (to appear), J. Diff. Geometry (1992).

Stredder P., *Natural differential operators on Riemannian manifolds and representations of the orthogonal and special orthogonal group*, J. Diff. Geom. **10** (1975), 647–660.

Terng C.L., *Natural vector bundles and natural differential operators*, American J. of Math. **100** (1978), 775–828.
Weyl H., *The classical groups*, Princeton University Press, Princeton, 1939.

Mathematical Institute of the ČSAV, branch Brno, Mendlovo n. 1, 662 82 Brno, Czechoslovakia

(*Received* February 10, 1992)