Strong Fubini axioms from measure extension axioms

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Abstract. It is shown that measure extension axioms imply various forms of the Fubini theorem for nonmeasurable sets and functions in Radon measure spaces.

Keywords: Fubini theorem, Product Measure Extension Axiom, Radon measure

Classification: Primary 03E35, 28C15, 28A35; Secondary 03E05, 03E65

1. Introduction.

Fubini's theorem asserts that if $(X, \mathcal{A}, \mu)$ and $(Y, \mathcal{B}, \nu)$ are σ-finite measure spaces and $f : X \times Y \to \mathbb{R}$ is a measurable function, then the iterated integrals $\int [\int f(x,y)\, d\mu(x)]\, d\nu(y)$ and $\int [\int f(x,y)\, d\nu(y)]\, d\mu(x)$ exist and are equal.

Strong Fubini axioms are statements about the existence and equality of iterated integrals of functions which are not necessarily measurable. The simplest one asserts that the iterated integrals of a nonnegative function (the restriction made to avoid trivial counterexamples — see [9]) are equal, if only they can be defined.

It is easy to give in ZFC an example showing that the above quoted strong Fubini axiom (SFA) is in general false. On the other hand, in the case when $X = Y = \mathbb{R}$ and $\mu = \nu$ is Lebesgue measure, SFA is false under CH (Sierpiński), but it is consistent with ZFC (Friedman [7]). Laczkovich proved that it follows from the assumption that if κ is the least cardinality of a nonmeasurable subset of $\mathbb{R}$, then the union of κ-many null sets does not cover $\mathbb{R}$ ($\operatorname{Non} \mathbb{L} < \operatorname{Cov} \mathbb{L}$).

The connections between various strengthenings of SFA for $\mathbb{R} \times \mathbb{R}$ and other cardinal conditions were investigated by Shipman [9].

In this note we consider the more general case in which $(X, \mathcal{A}, \mu)$ and $(Y, \mathcal{B}, \nu)$ are Radon measure spaces.

Our aim is to present an alternative approach to strong Fubini axioms via measure extension axioms. It is shown that various instances of the Product Measure Extension Axiom imply strengthenings of SFA for $X \times Y$.

The main result of this paper is Theorem 3.1. Its proof is a modification of arguments due to Kunen and Kamburelis concerning related but different problems (see [8, Corollary 6.(2)]. The main contribution of the author is noting that the same ideas may be applied to settle strong Fubini axioms under measure extension axioms.

After the results of Section 3 were presented at the 15th Summer Symposium in Real Analysis (see [10]), I was given a copy of an early version of D.H. Fremlin's

*This paper was written while the author was visiting the Technische Universität Berlin as a research fellow of the Alexander von Humboldt Foundation.

article [6] in which, among others, strong Fubini axioms are discussed in connection with the existence and properties of real-valued-measurable cardinals. Section 4 consists mostly of strengthenings of relevant results from that paper, which can be obtained by modifications of proofs given there.

I am grateful to David Fremlin for giving me the above mentioned copy of his article and for fruitful discussions and correspondence that followed.

2. Definitions and preliminaries.

Our set theoretic terminology and notations are standard.

Ordinals are identified with sets of their predecessors and cardinals with initial ordinals. In particular, $2 = \{0, 1\}$ and $\omega = \{0, 1, \ldots\}$. If I is any set, then 2^I denotes the collection of functions from I to 2 and $[I]^\omega$ the family of countable infinite subsets of I. The letters κ, λ and ϱ are reserved for infinite cardinals.

Our measure theoretic terminology agrees essentially with Fremlin [4].

A σ-finite Radon measure space is a triple $(X, \mathcal{A}, \mu)$, where

(1) X is a topological space with a Hausdorff topology $\mathfrak{T}$;
(2) $(X, \mathcal{A}, \mu)$ is a complete, σ-finite measure space;
(3) $\mathfrak{T} \subseteq \mathcal{A}$;
(4) $\mu(A) = \sup\{\mu(D) : D \subseteq A, D \text{ is compact}\}$ for every $A \in \mathcal{A}$;
(5) every $x \in X$ has an open neighborhood U such that $\mu(U) < \infty$.

One particular Radon measure space is of a special importance, the product space $(2^I, \mathscr{L}_I, m_I)$ for an infinite set I.

Let $\mathrm{Seq}(I)$ be the set of functions from a finite subset of I to $\{0, 1\}$ and for each $s \in \mathrm{Seq}(I)$ define $[s]_I = \{f \in 2^I : s \subseteq f\}$. Let $\mathcal{B}_I$ be the σ-algebra generated by $\{[s]_I : s \in \mathrm{Seq}(I)\}$ and let $\mathcal{N}_I = \{A \subseteq 2^I : m_I(A) = 0\}$.

The measure m_I, defined on the σ-algebra $\mathscr{L}_I$ generated by $\mathcal{B}_I \cup \mathcal{N}_I$, is the completion of the usual product measure on 2^I. By well-known regularity properties, $(2^I, \mathscr{L}_I, m_I)$ is a Radon probability space (see [4, 1.5]).

For each $S \subseteq I$ let $\pi_S : 2^I \to 2^S$ be the canonical projection. If $B \in \mathcal{B}_I$, then there exists $S \in [I]^\omega$ such that $B = \pi_S^{-1}[A]$ for a certain $A \in \mathcal{B}_S$. We call any such S a support of B.

Let $(X, \mathcal{A}, \mu)$ be a σ-finite measure space.

Its Maharam type is the least cardinality of a collection $\mathcal{D} \subseteq \mathcal{A}$ such that for every $\varepsilon > 0$ and $A \in \mathcal{A}$ there exists $D \in \mathcal{D}$ with $\mu((A \setminus D) \cup (D \setminus A)) < \varepsilon$.

If $(Y, \mathcal{B}, \nu)$ is another measure space, then a function $\phi : X \to Y$ is inverse-measure-preserving if $\phi^{-1}[B] \in \mathcal{A}$ and $\mu(\phi^{-1}[B]) = \nu(B)$ for every $B \in \mathcal{B}$. In particular, the projection $\pi_S : 2^I \to 2^S$ is an inverse-measure-preserving function between the spaces $(2^I, \mathscr{L}_I, m_I)$ and $(2^S, \mathscr{L}_S, m_S)$.

The following folklore-like result is one of the reasons why the space $(2^I, \mathscr{L}_I, m_I)$ plays a special role in our considerations (for its proof see e.g. [4, Corollary 3.12 and Theorem 4.12]).

Proposition 2.1. *Let $(X, \mathcal{A}, \mu)$ be a Radon probability space of Maharam type $\leq \kappa$. Then there exists an inverse-measure-preserving function $\phi : 2^\kappa \to X$.* □

Let $(X, \mathcal{A}, \mu)$ and $(Y, \mathcal{B}, \nu)$ be σ-finite Radon measure spaces.

For every $f : X \times Y \to \mathbb{R}$, $D \subseteq X \times Y$, $x \in X$ and $y \in Y$, let $f_x : Y \to \mathbb{R}$, $f^y : X \to \mathbb{R}$, $D_x \subseteq Y$, and $D^y \subseteq X$ be defined as follows: $f_x(y) = f^y(x) = f(x,y)$, $D_x = \{y \in Y : (x,y) \in D\}$ and $D^y = \{x \in X : (x,y) \in D\}$.

The following strong Fubini axioms are considered in this paper.

The Strong Fubini Axiom (SFA) for $X \times Y$ asserts that if $f : X \times Y \to [0,\infty)$ is such that:

(1) for μ-a.a. $x \in X$ and for ν-a.a. $y \in Y$ the functions f_x and f^y are measurable,
(2) the functions $x \to \int f_x \, d\nu$ and $y \to \int f^y \, d\mu$ are measurable,

then the iterated integrals $\int (\int f_x \, d\nu) \, d\mu$ and $\int (\int f^y \, d\mu) \, d\nu$ are equal.

The Super Strong Fubini Axiom (SSFA) for sets in $X \times Y$ asserts that for every $D \subseteq X \times Y$ such that:

for μ-a.a. $x \in X$ and for ν-a.a. $y \in Y$ the sets D_x and D^y are measurable,

if $\mu(\{x \in X : \nu(D_x) > 0\}) = 0$, then $\nu(\{y \in Y : \mu(D^y) > 0\}) = 0$.

The next auxiliary result establishes a connection between the above two assertions.

Proposition 2.2. *Let $(X, \mathcal{A}, \mu)$ and $(Y, \mathcal{B}, \nu)$ be σ-finite Radon measure spaces of Maharam types $\leq \lambda$ and $\leq \kappa$, respectively. Then* SSFA *for sets in $2^\lambda \times 2^\kappa$ implies* SFA *for $X \times Y$.*

Proof: Suppose that SFA for $X \times Y$ is false. Without loss of generality assume that $\mu(X) = \nu(Y) = 1$.

By a direct generalization of an argument due to Freiling [3] (see also [9]), there exists a set $B \subseteq X^\omega \times Y^\omega$ such that $\forall \bar{x} \in X^\omega \ \nu^\omega(B_{\bar{x}}) = 0$ and $\forall \bar{y} \in Y \ \mu^\omega(B^{\bar{y}}) = 1$, where $(X^\omega, \mathcal{A}^\omega, \mu^\omega)$ and $(Y^\omega, \mathcal{B}^\omega, \nu^\omega)$ are Radon probability spaces, the products of ω-many copies of spaces $(X, \mathcal{A}, \mu)$ and $(Y, \mathcal{B}, \nu)$, respectively (see [5, A7E]).

Since the Maharam types of the spaces $(X^\omega, \mathcal{A}^\omega, \mu^\omega)$ and $(Y^\omega, \mathcal{B}^\omega, \nu^\omega)$ are $\leq \lambda$ and $\leq \kappa$, respectively, by Proposition 2.1 there are inverse-measure-preserving functions $\phi_X : 2^\lambda \to X^\omega$ and $\phi_Y : 2^\kappa \to Y^\omega$.

Let $D = (\phi_X \times \phi_Y)^{-1}[B] \subseteq 2^\lambda \times 2^\kappa$. Then D contradicts SSFA for sets in $2^\lambda \times 2^\kappa$. □

If $(X, \mathcal{A}, \mu)$ is a σ-finite measure space, then μ is ϱ-additive if the union of less then ϱ-many μ-null sets is μ-null.

We consider the following measure extension axioms.

The Product Measure Extension Axiom (PMEA) asserts that for every set I, the measure m_I can be extended to a 2^ω-additive measure defined on the power set of 2^I.

PMEA$(\lambda, \kappa, \varrho)$, where $\varrho \leq 2^\omega$, is the weaker version of PMEA asserting that for any family $\mathscr{S}$ of κ-many subsets of 2^λ, the measure m_λ can be extended to a ϱ-additive measure $\bar{m}_\lambda : \overline{\mathscr{L}}_\lambda \to [0,1]$ with $\mathscr{S} \subseteq \overline{\mathscr{L}}_\lambda$.

Kunen proved Con (ZFC + PMEA) starting from Con (ZFC +$\exists$ strongly compact cardinal) (for a proof see [2, Theorem 3.4]).

The consistency results concerning $\mathrm{PMEA}(\lambda, \kappa, \varrho)$ are due to Carlson [1]. In particular he proved Con ($\mathrm{ZFC} + \forall\, \kappa < 2^\omega\, \mathrm{PMEA}(2^\omega, \kappa, \kappa^+)$) without large cardinal assumptions.

3. Results.

The main result of this note is the following

Theorem 3.1. $\mathrm{PMEA}(\lambda, \kappa, \kappa^+)$ *implies* SSFA *for sets in* $2^\lambda \times 2^\kappa$.

PROOF: Take an arbitrary set $D \subseteq 2^\lambda \times 2^\kappa$ such that $D_x \in \mathcal{N}_\kappa$ and $D^y \in \mathscr{L}_\lambda$ for every $x \in 2^\lambda$ and $y \in 2^\kappa$.

It follows that there exists a set $B \subseteq 2^\lambda \times 2^\kappa$ such that $D \subseteq B$ and $B_x \in \mathcal{N}_\kappa \cap \mathcal{B}_\kappa$ for every $x \in 2^\lambda$; let S_x be a support of B_x and $B_x = \pi_{S_x}^{-1}[E_x]$ for a certain set $E_x \in \mathcal{N}_{S_x}$. For each x and $n > 0$, find an open subset $(H_n)_x$ of 2^{S_x} with $E_x \subseteq (H_n)_x$ and $m_{S_x}((H_n)_x) < \frac{1}{n}$.

Set $(G_n)_x = \pi_{S_x^{-1}}[(H_n)_x]$. Then $D_x \subseteq (G_n)_x$ and $m_\kappa((G_n)_x) < \frac{1}{n}$ for each x.

For $s \in \mathrm{Seq}(\kappa)$ and $n > 0$, let $K_{n,s} = \{x \in 2^\lambda : [s]_\kappa \subseteq (G_n)_x\}$.

Claim. There exists a κ^+-additive extension $\bar{m}_\lambda : \overline{\mathscr{L}}_\lambda \to [0, 1]$ of m_λ satisfying the following conditions:

(1) $\{K_{n,s} : n > 0,\ s \in \mathrm{Seq}(\kappa)\} \subseteq \overline{\mathscr{L}}_\lambda$,

(2) there exist sets $W \in \overline{\mathscr{L}}_\lambda$ and $S \in [\kappa]^\omega$ such that $\bar{m}_\lambda(W) = 1$ and $S_x \subseteq S$ for every $x \in W$.

To prove the claim, for each $x \in 2^\lambda$, enumerate S_x as $\{y_{x,k} : k < \omega\}$, and then for each $k < \omega$, consider the function $F_k : 2^\lambda \to \kappa$ defined by $F_k(x) = y_{x,k}$. For every $k < \omega$ and $\alpha < \kappa$ let $X_{k,\alpha} = F_k^{-1}[\{\alpha\}]$.

Using $\mathrm{PMEA}(\lambda, \kappa, \kappa^+)$ extend m_λ to a κ^+-additive measure $\bar{m}_\lambda$ which measures all sets $K_{n,s}$ and $X_{k,\alpha}$, $n > 0$, $s \in \mathrm{Seq}(\kappa)$, $k < \omega$ and $\alpha < \kappa$.

Then (1) is clear and the checking of (2) is routine (see [8, Lemma 4]; this point requires the κ^+-additivity of $\bar{m}_\lambda$).

Without loss of generality assume that $S_x = S$ for every $x \in W$.

Since $\{[s]_S : s \in \mathrm{Seq}(S)\}$ is a base for the topology of 2^S and $K_{n,s} \cap W = \{x \in W : [s]_S \subseteq (H_n)_x\}$, it is easy to compute that $H_n \cap (W \times 2^S) = \bigcup_{s \in \mathrm{Seq}(S)}((K_{n,s} \cap W) \times [s]_S)$.

It follows that $G_n \cap (W \times 2^\kappa) = \bigcup_{s \in \mathrm{Seq}(S)}((K_{n,s} \cap W) \times [s]_\kappa)$ is the union of countably many elements of the product σ-algebra $\overline{\mathscr{L}}_\lambda \otimes \mathscr{L}_\kappa$.

Let $G = \bigcap_{n<\omega} G_n$.

Then by the above, $G \cap (W \times 2^\kappa) \in \overline{\mathscr{L}}_\lambda \otimes \mathscr{L}_\kappa$.

Recall that, moreover, $\bar{m}_\lambda(W) = 1$, $D \subseteq G$ and $m_\kappa((G_n)_x) < \frac{1}{n}$ for each x.

Hence, by the Fubini theorem, $m_\kappa(\{y \in 2^\kappa : \bar{m}_\lambda(D^y) > 0\}) = 0$. But $D^y \in \mathscr{L}_\lambda$ for each y, so $\bar{m}_\lambda(D^y) > 0$ iff $m_\lambda(D^y) > 0$. Hence $\{y \in 2^\kappa : D^y \notin \mathcal{N}_\lambda\} \in \mathcal{N}_\kappa$ as required. $\square$

Corollary 3.2. PMEA *implies* SSFA *for sets in* $2^\lambda \times 2^\kappa$, *whenever* λ *is arbitrary and* $\kappa < 2^\omega$. □

The above result is, in a sense, optimal, as the following example shows. Its idea is taken from [8] (see [8, the remark after Corollary 6]).

Proposition 3.3. SSFA *for sets in* $2^\omega \times 2^{2^\omega}$ *is false.*

PROOF: Let $\{P_x : x < 2^\omega\}$ be a partition of 2^ω into countable infinite sets.

For each x, let $B_x = \{y \in 2^{2^\omega} : y(\beta) = 0 \text{ for every } \beta \in P_x\}$.

Note that $B_x \in \mathcal{N}_{2^\omega}$ since $B_x = \pi_{P_x}^{-1}[\langle 0, \dots 0, \dots\rangle)$.

Let $H = \{y \in 2^{2^\omega} : \exists\, P \in [2^\omega]^\omega \forall\, \beta \notin P\ y(\beta) = 0\}$.

Finally define $D = B \cap (2^\omega \times H)$.

Clearly, $D_x \in \mathcal{N}_{2^\omega}$ for every $x \in 2^\omega$.

Also, $D^y \in \mathscr{L}_\omega$ for every $y \in 2^{2^\omega}$.

Indeed, if $y \in H$, then there exists a set $P \in [2^\omega]^\omega$ with $\{x \in 2^\omega : P_x \cap P = \emptyset\} \subseteq D^y$, so D^y is co-countable. If $y \notin H$, then $D^y = \emptyset$.

On the other hand, $H = \{y \in 2^{2^\omega} : D^y \notin \mathcal{N}_\omega\} \notin \mathcal{N}_{2^\omega}$.

Indeed, if $H \subseteq H' \in \mathcal{B}_{2^\omega}$ and S is a support of H', then $2^S = \pi_S[\{y \in 2^{2^\omega} : \forall\, \beta \notin S\ y(\beta) = 0\}] \subseteq \pi_S[H] \subseteq \pi_S[H']$. Hence $H' = 2^{2^\omega}$, which shows that H is not contained in any set from $\mathcal{B}_{2^\omega} \cap \mathcal{N}_{2^\omega}$. □

Combining Theorem 3.1 with Proposition 2.2 gives the final result of this section.

Theorem 3.4. $\mathrm{PMEA}(\lambda, \kappa, \kappa^+)$ *implies* SFA *for* $X \times Y$, *whenever* $(X, \mathcal{A}, \mu)$ *and* $(Y, \mathcal{B}, \nu)$ *are* σ*-finite Radon measure spaces of Maharam types* $\leq \lambda$ *and* $\leq \kappa$*, respectively.*

□

Corollary 3.5. PMEA *implies* SFA *for* $X \times Y$, *whenever* $(X, \mathcal{A}, \mu)$ *and* $(Y, \mathcal{B}, \nu)$ *are* σ*-finite Radon measure spaces such that the Maharam type of at least one of them is less than* 2^ω. □

4. Generalizations.

One way to strengthen the results concerning SSFA for sets in $X \times Y$, where $(X, \mathcal{A}, \mu)$ and $(Y, \mathcal{B}, \nu)$ are σ-finite Radon measure spaces, is to consider the following Super Strong Fubini Axiom (SSFA) for *functions* on $X \times Y$:

If $f : X \times Y \to [0, \infty)$ is such that for μ-a.e. $x \in X$ and for ν-a.e. $y \in Y$ the functions f_x and f^y are measurable, then the functions $\to \int f_x \, d\nu$ and $y \to \int f^y \, d\mu$ are also measurable and the iterated integrals are equal.

Woodin proved that SSFA for functions on $\mathbb{R} \times \mathbb{R}$ is consistent with ZFC. Shipman claims (see [9, p. 480]) that it follows already from the inequality $\mathrm{Non}\,\mathbb{L} < \mathrm{Cov}\,\mathbb{L}$.

The next result strengthens Theorem 3.1 (compare [6, Proposition 6K]).

Theorem 4.1. *Assume* $\mathrm{PMEA}(\lambda, \kappa, \kappa^+)$ *and let* $(X, \mathcal{A}, \mu)$ *and* $(Y, \mathcal{B}, \nu)$ *be* σ*-finite Radon measure spaces.*

(i) *Suppose that the Maharam type of* X *is* $\leq \lambda$ *and the topological weight of* Y *is* $\leq \kappa$.

If $f : X \times Y \to [0,\infty)$ is such that for μ-a.a. $x \in X$ and for ν-a.a. $y \in Y$ the functions f_x and f^y are measurable, then the function $y \to \int f^y \, d\mu$ is measurable and

$$\int_{-} \Big(\int f_x \, d\nu \Big) \, d\mu \le \int \Big(\int f^y \, d\mu \Big) \, d\nu \le \int^{-} \Big(\int f_x \, d\nu \Big) \, d\mu.$$

(ii) *If both X and Y have Maharam types $\le \lambda$ and topological weights $\le \kappa$, then* SSFA *for functions on $X \times Y$ is true.*

Moreover, for every function $f : X \times Y \to [0,\infty)$ such that for μ-a.a. $x \in X$ and for ν-a.a. $y \in Y$ the functions f_x and f^y are measurable, there exists a function $h : X \times Y \to [0,\infty)$ measurable with respect to the product of measures μ and ν, such that $f_x = h_x$ ν-a.e. for μ-a.a. $x \in X$ and $f^y = h^y$ μ-a.e. for ν-a.a. $y \in Y$.

Proof: Modify the proofs of Theorem 3A, Proposition 6J and 6K of [6]. Instead of measures defined on $P(X)$ and $P(Y)$, use suitable extensions of μ and ν measuring enough sets to carry out the argument at hand. □

Note that, because of the result by Carlson [1] mentioned in Section 2, as a corollary we obtain the consistency of "ZFC + SSFA for functions on $X \times Y$, whenever $(X, \mathcal{A}, \mu)$ and $(Y, \mathcal{B}, \nu)$ are σ-finite Radon measure spaces with topological weights less than 2^{ω}" from the consistency of ZFC alone.

Note also that, because of Proposition 3.3, SSFA for functions on $2^{\omega} \times 2^{2^{\omega}}$ is false in ZFC.

An alternative approach to establish SFA is suggested by the result of Laczkovich stated in the introduction.

Let $\operatorname{Non} \mathcal{N}_{\lambda}$ denote the least cardinality of a subset of 2^{λ} not in $\mathcal{N}_{\lambda}$ and $\operatorname{Cov} \mathcal{N}_{\kappa}$ be the least cardinality of a family of m_{κ}-null sets whose union covers 2^{κ}.

The following generalizes Laczkovich's result (compare [9, Theorem 1] and [6, Proposition 6I]).

Proposition 4.2. $\operatorname{Non} \mathcal{N}_{\lambda} < \operatorname{Cov} \mathcal{N}_{\kappa}$ *implies* SSFA *for $X \times Y$, whenever $(X, \mathcal{A}, \mu)$ and $(Y, \mathcal{B}, \nu)$ are σ-finite Radon measure spaces of Maharam types $\le \lambda$ and $\le \kappa$, respectively.* □

The above implies that the conclusion of Corollary 3.5 follows from the weaker assumption that 2^{ω} is real-valued-measurable. This is due to the fact that under this assumption, if $\lambda < 2^{\omega}$ and $\kappa \ge \omega$, then $\operatorname{Non} \mathcal{N}_{\lambda} = \omega_1$ and $\operatorname{Cov} \mathcal{N}_{\kappa} = 2^{\omega}$ (see [6, Corollary 6G]).

We conclude with the formulation of an open problem which naturally arises from the results of this paper.

Problem. Is it possible to find in ZFC an example of σ-finite Radon measure spaces $(X, \mathcal{A}, \mu)$ and $(Y, \mathcal{B}, \nu)$ such that SFA for $X \times Y$ is false?

Note that, by Corollary 3.5, the first case to check is $2^{2^{\omega}} \times 2^{2^{\omega}}$.

Also note that if $\operatorname{Non}(\mathcal{N}_{\kappa}) = 2^{\kappa}$, then SFA for $2^{\kappa} \times 2^{\kappa}$ is false, the characteristic function of a well-ordering of 2^{κ} in type 2^{κ} providing a counterexample. This holds, in particular, if κ is a strong limit cardinal of cofinality ω such that $2^{\kappa} = \kappa^{+}$ (see [4, Theorem 6.17 (v)]). The consistency of the non-existence of a κ with the latter property requires very strong set-theoretic assumptions of large-cardinal type. So, the complementary question is:

Is it possible to find in ZFC a cardinal κ with $\mathrm{Non}(\mathcal{N}_\kappa) = 2^\kappa$?

References

[1] Carlson T., *Extending Lebesgue measure by infinitely many sets*, Pac. J. Math. **115** (1984), 33–45.

[2] Fleissner W.G., *The normal Moore space conjecture*, in: Handbook of set-theoretic topology, ed. by K. Kunen and J.E. Vaughan, North-Holland, 1984.

[3] Freiling C., *Axioms of symmetry: throwing darts at the real number line*, J. Symbolic Logic **51** (1986), 190–200.

[4] Fremlin D.H., *Measure algebras*, in: Handbook of Boolean algebras, ed. by J.D. Monk, Elsevier Science Publishers B.V., 1989.

[5] ———, *Consequences of Martin's Axiom*, Cambridge University Press, 1984.

[6] ———, *Real-valued-measurable cardinals*, to appear.

[7] Friedman H., *A consistent Fubini-Tonelli theorem for nonmeasurable functions*, Illinois J. Math. **24** (1980), 390–395.

[8] Kamburelis A., *A new proof of the Gitik-Shelah theorem*, Israel J. Math. **72** (1990), 373–380.

[9] Shipman J., *Cardinal conditions for strong Fubini theorems*, Trans. Amer. Math. Soc. **321** (1990), 465–481.

[10] Zakrzewski P., *Strong Fubini theorems from measure extension axioms*, an abstract of the talk given at the 15th Summer Symposium in Real Analysis, Real Analysis Exchange **17** (1991–92), 65–66.

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(*Received* December 13, 1991)