Multipliers of Hankel transformable generalized functions

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Abstract. Let $\mathcal{H}_\mu$ be the Zemanian space of Hankel transformable functions, and let $\mathcal{H}'_\mu$ be its dual space. In this paper $\mathcal{H}_\mu$ is shown to be nuclear, hence Schwartz, Montel and reflexive. The space $\mathcal{O}$, also introduced by Zemanian, is completely characterized as the set of multipliers of $\mathcal{H}_\mu$ and of $\mathcal{H}'_\mu$. Certain topologies are considered on $\mathcal{O}$, and continuity properties of the multiplication operation with respect to those topologies are discussed.

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1. Introduction.

Let $\mu \in \mathbb{R}$. The space $\mathcal{H}_\mu$, introduced by A.H. Zemanian [7], consists of all those infinitely differentiable functions $\phi = \phi(x)$ defined on $I =]0,\infty[$ such that the quantities

$$\lambda^\mu_{m,k}(\phi) = \sup_{x\in I} |x^m(x^{-1}D)^k x^{-\mu-1/2}\phi(x)| \quad (m,k \in \mathbb{N})$$

are finite. Endowed with the topology generated by the family of seminorms $\{\lambda^\mu_{m,k}\}_{(m,k)\in\mathbb{N}\times\mathbb{N}}$, $\mathcal{H}_\mu$ is a Fréchet space.

We note that this topology of $\mathcal{H}_\mu$ can be also defined by means of the seminorms

$$\tau^\mu_{m,k}(\phi) = \sup_{x\in I} |(1+x^2)^m(x^{-1}D)^k x^{-\mu-1/2}\phi(x)| \quad (m,k \in \mathbb{N},\ \phi \in \mathcal{H}_\mu).$$

The vector space $\mathcal{O}$ of all those $\theta \in C^\infty(I)$ such that for every $k \in \mathbb{N}$ there exist $n_k \in \mathbb{N}$, $A_k > 0$ satisfying

$$|(x^{-1}D)^k\theta(x)| \le A_k(1+x^2)^{n_k} \quad (x \in I)$$

was shown in [7] to be a space of multipliers for $\mathcal{H}_\mu$. Here we prove that $\mathcal{O}$ is precisely the space of multipliers of $\mathcal{H}_\mu$ (Section 2) and of $\mathcal{H}'_\mu$ (Section 4). In characterizing $\mathcal{O}$ as the space of multipliers for $\mathcal{H}'_\mu$ we use the reflexivity of $\mathcal{H}_\mu$, which derives from the fact, previously established in that section, that $\mathcal{H}_\mu$ is nuclear.

Sections 3 and 5 mainly deal with the problem of topologizing $\mathcal{O}$. We show that this can be done in such a way that the bilinear maps $(\theta,\vartheta) \mapsto \theta\vartheta$ from $\mathcal{O}\times\mathcal{O}$ into $\mathcal{O}$, $(\theta,\phi) \mapsto \theta\phi$ from $\mathcal{O}\times\mathcal{H}_\mu$ into $\mathcal{H}_\mu$, and $(\theta,T) \mapsto \theta T$ from $\mathcal{O}\times\mathcal{H}'_\mu$ into $\mathcal{H}'_\mu$, are separately continuous (Section 3) or even hypocontinuous with respect to bounded subsets (Section 5).

We note that most of the properties established here for $\mathcal{H}_\mu$, $\mathcal{H}'_\mu$, and $\mathcal{O}$ are similar to the corresponding ones for the Schwartz space $\mathcal{S}$, its dual $\mathcal{S}'$ (the space of tempered distributions), and their space of multipliers $\mathcal{O}_M$. A difference between $\mathcal{O}$ and $\mathcal{O}_M$ should be pointed out, however: $\mathcal{O}$ is not a normal space of distributions (see the remark following Proposition 3.5).

2. Multipliers of $\mathcal{H}_\mu$.

A function $\theta = \theta(x)$ defined on I is said to be a *multiplier* for $\mathcal{H}_\mu$ if the map $\phi \mapsto \theta\phi$ is continuous from $\mathcal{H}_\mu$ into $\mathcal{H}_\mu$. Our purpose in this section is to characterize the space of multipliers of $\mathcal{H}_\mu$. This will be done in Theorem 2.3; some preliminary results are needed.

Lemma 2.2 below provides certain useful examples of functions in $\mathcal{H}_\mu$. The following particular case of Peetre's Inequality (see, e.g., [1, Lemma 5.2]) is helpful in constructing such functions.

Lemma 2.1. *For every $\xi, \eta \in \mathbb{R}$, there holds:*

$$\frac{1+\xi^2}{1+\eta^2} \leq 2\big(1+|\xi-\eta|^2\big).$$

Lemma 2.2. *Let $\alpha \in \mathcal{D}(I)$ be such that $0 \leq \alpha \leq 1$, $\operatorname{supp}\alpha = [1/2, 3/2]$ and $\alpha(1) = 1$. Also, let $\{x_j\}_{j\in\mathbb{N}}$ be a sequence of real numbers satisfying $x_0 > 1$ and $x_{j+1} > x_j + 1$. Define*

$$\phi(x) = x^{\mu+1/2} \sum_{j=0}^{\infty} \frac{\alpha(x-x_j+1)}{(1+x_j^2)^j} \quad (x \in I). \tag{2.1}$$

Then $\phi \in \mathcal{H}_\mu$.

Proof: It should be noted that the sum on the right-hand side of (2.1) is finite, because the functions $\alpha(x - x_j + 1)$ have pairwise disjoint supports. In fact, if $m, k \in \mathbb{N}$ and $x_j - 1/2 \leq x \leq x_j + 1/2$, we may write:

$$(1+x^2)^m (x^{-1}D)^k x^{-\mu-1/2}\phi(x) = \Big(\frac{1+x^2}{1+x_j^2}\Big)^m \frac{(x^{-1}D)^k \alpha(x-x_j+1)}{(1+x_j^2)^{j-m}}.$$

Lemma 2.1 guarantees that $\tau^\mu_{m,k}(\phi) < +\infty$, thus showing that $\phi \in \mathcal{H}_\mu$, as asserted. □

We are now in a position to characterize the multipliers of $\mathcal{H}_\mu$.

Theorem 2.3. *Any one of the following statements is equivalent to the other two:*

(i) *The function $\theta = \theta(x)$ belongs to $C^\infty(I)$, and for every $k \in \mathbb{N}$ there exists $n_k \in \mathbb{N}$ such that*

$$(1+x^2)^{-n_k}(x^{-1}D)^k\theta(x)$$

is bounded on I.

(ii) *The product $\theta\phi$ lies in $\mathcal{H}_\mu$ whenever $\phi \in \mathcal{H}_\mu$, and the map $\phi \mapsto \theta\phi$ is a continuous endomorphism of $\mathcal{H}_\mu$.*

(iii) *The function θ is infinitely differentiable on I, for every $k \in \mathbb{N}$ and every $\phi \in \mathcal{H}_\mu$ the function $\phi(x)(x^{-1}D)^k\theta(x)$ belongs to $\mathcal{H}_\mu$, and the map $\phi(x) \mapsto \phi(x)(x^{-1}D)^k\theta(x)$ is a continuous endomorphism of $\mathcal{H}_\mu$.*

PROOF: That (i) implies (ii) has already been proved by Zemanian ([7, p. 134]).

To show that (ii) implies (iii), let us consider the function $\phi \in \mathcal{H}_\mu$ defined by

$$\phi(x) = x^{\mu+1/2}e^{-x^2}\,. \tag{2.2}$$

According to (ii),

$$\psi(x) = x^{\mu+1/2}\theta(x)e^{-x^2} \tag{2.3}$$

lies in $\mathcal{H}_\mu$, so that

$$\theta(x) = x^{\mu+1/2}\psi(x)e^{-x^2} \tag{2.4}$$

is infinitely differentiable on I. At this point, it suffices to show that $(x^{-1}D)^k\theta(x)$ is a multiplier of $\mathcal{H}_\mu$ whenever θ is. But this can be easily established by induction on k, taking into account the formula

$$\begin{aligned}&\phi(x)(x^{-1}D)\theta(x) =\\ &\qquad = x^{\mu+1/2}(x^{-1}D)x^{-\mu-1/2}\theta(x)\phi(x) - \theta(x)x^{\mu+1/2}(x^{-1}D)x^{-\mu-1/2}\phi(x)\end{aligned}$$

along with the fact that if ϕ is in $\mathcal{H}_\mu$ then so is

$$x^{\mu+1/2}(x^{-1}D)^k x^{-\mu-1/2}\phi(x).$$

Finally, let $\theta(x)$ satisfy (iii). Since (2.2) belongs to $\mathcal{H}_\mu$, so does (2.3). Then $\theta(x)$ can be represented by (2.4), and, in particular, the limit $\lim_{x\to 0+}\theta(x)$ exists. According to (iii), each $(x^{-1}D)^k\theta(x)$ is a multiplier of $\mathcal{H}_\mu$, and we conclude that $\lim_{x\to 0+}(x^{-1}D)^k\theta(x)$ exists for all $k \in \mathbb{N}$.

Arguing by contradiction, let us assume that (i) is false. Then there exist $k \in \mathbb{N}$ and a sequence $\{x_j\}_{j\in\mathbb{N}}$ of real numbers, which, by what has been just proved, may be chosen so that $x_0 > 1$ and $x_{j+1} > x_j + 1$, such that:

$$\left|(x^{-1}D)^k\theta(x)_{|x=x_j}\right| > \left(1 + x_j^2\right)^j\,.$$

The function $\phi \in \mathcal{H}_\mu$ constructed by means of $\{x_j\}_{j\in\mathbb{N}}$ as in Lemma 2.2 plainly satisfies

$$\left|x_j^{-\mu-1/2}\phi(x_j)(x^{-1}D)^k\theta(x)_{|x=x_j}\right| > \alpha(1) = 1 \quad (j \in \mathbb{N}),$$

contradicting (iii). □

3. Topology and properties of the space of multipliers.

Following [7], we denote by $\mathcal{O}$ the linear space of all those $\theta \in C^\infty(I)$ such that for every $k \in \mathbb{N}$ there exist $n_k \in \mathbb{N}$, $A_k > 0$ satisfying

$$\left|(x^{-1}D)^k\theta(x)\right| \leq A_k\left(1+x^2\right)^{n_k} \quad (x \in I).$$

The equivalence between the conditions (i) and (ii) in Theorem 2.3 above characterizes $\mathcal{O}$ as the space of multipliers of $\mathcal{H}_\mu$, with independence of the value of the real parameter μ. However, once μ has been fixed, the condition (iii) suggests to introduce on $\mathcal{O}$ the (separating) family of seminorms

$$\Gamma_\mu = \{\gamma^\mu_{\phi,k} : \phi \in \mathcal{H}_\mu,\ k \in \mathbb{N}\},$$

where

$$\gamma^\mu_{\phi,k}(\theta) = \sup_{x\in I}\left|x^{-\mu-1/2}\phi(x)(x^{-1}D)^k\theta(x)\right|.$$

Since the map $\phi(x) \mapsto x^{\nu-\mu}\phi(x) = \varphi(x)$ establishes an isomorphism between $\mathcal{H}_\mu$ and $\mathcal{H}_\nu$ for any $\mu, \nu \in \mathbb{R}$, the equality $\gamma^\mu_{\phi,k}(\theta) = \gamma^\nu_{\varphi,k}(\theta)$ holds whenever $k \in \mathbb{N}$ and $\theta \in \mathcal{O}$. Therefore, all families Γ_μ ($\mu \in \mathbb{R}$) define one and the same topology on $\mathcal{O}$. In the sequel, unless otherwise stated, it will always be assumed that $\mathcal{O}$ is endowed with this topology, and μ will be any real number.

Remarks. (i) If $\theta \in C^\infty(I)$ is such that $\gamma^\mu_{\phi,k}(\theta) < +\infty$ for every $\phi \in \mathcal{H}_\mu$ and $k \in \mathbb{N}$, then $\theta \in \mathcal{O}$. In fact, fix $\phi \in \mathcal{H}_\mu$, $m, k \in \mathbb{N}$ and for $0 \leq p \leq k$ define $\phi_p \in \mathcal{H}_\mu$ by

$$\phi_p(x) = (1+x^2)^m x^{\mu+1/2}(x^{-1}D)^{k-p}x^{-\mu-1/2}\phi(x) \quad (x \in I).$$

Since

$$(1+x^2)^m(x^{-1}D)^k x^{-\mu-1/2}(\theta\phi)(x) = \sum_{p=0}^{k}\binom{k}{p}x^{-\mu-1/2}\phi_p(x)(x^{-1}D)^p\theta(x) \quad (x \in I),$$

necessarily

$$\tau^\mu_{m,k}(\theta\phi) \leq \sum_{p=0}^{k}\binom{k}{p}\gamma^\mu_{\phi_p,p}(\theta). \tag{3.1}$$

In general

$$\tau^\mu_{m,k}(\phi(x)(\frac{1}{x}D)^k\theta(x)) \leq \sum_{p=0}^{k}\binom{k}{p}\gamma^\mu_{\phi_p,p+n}(\theta), \quad (n \in \mathbb{N}).$$

Our assertion now follows as in the proof that (iii) implies (i) in Theorem 2.3.

(ii) The topology of $\mathcal{O}$ may be also generated by means of the family of seminorms $\{\gamma^{\mu}_{m,k;\phi} : (m,k) \in \mathbb{N} \times \mathbb{N},\ \phi \in \mathcal{H}_\mu\}$, where

$$\gamma^{\mu}_{m,k;\phi}(\theta) = \tau^{\mu}_{m,k}(\theta\phi) \quad (m,k \in \mathbb{N},\ \phi \in \mathcal{H}_\mu).$$

Certainly, let $k \in \mathbb{N}$ and, for every $\phi \in \mathcal{H}_\mu$ and every $p \in \mathbb{N}$ with $0 \leq p \leq k$, define $\phi_p \in \mathcal{H}_\mu$ by

$$\phi_p(x) = x^{\mu+1/2}(x^{-1}D)^p x^{-\mu-1/2}\phi(x) \quad (x \in I).$$

If $\phi \in \mathcal{H}_\mu$ and $\theta \in \mathcal{O}$, the equality

$$x^{-\mu-1/2}\phi(x)(x^{-1}D)^k\theta(x) = \sum_{p=0}^{k}(-1)^p\binom{k}{p}(x^{-1}D)^{k-p}x^{-\mu-1/2}(\theta\phi_p)(x) \quad (x \in I)$$

then shows that

$$\gamma^{\mu}_{\phi,k}(\theta) \leq \sum_{p=0}^{k}\binom{k}{p}\gamma^{\mu}_{0,k-p;\phi_p}(\theta).$$

Along with (3.1), this estimate proves our assertion.

Proposition 3.1. *The identity map $\mathcal{O} \hookrightarrow \mathcal{E}(I)$ is continuous.*

Proof: It is enough to observe that

$$D^k\theta(x) = \frac{1}{x^{-\mu-1/2}\phi(x)}\sum_{p=0}^{k}C_p\,x^{\alpha(p)}x^{-\mu-1/2}\phi(x)(x^{-1}D)^{\beta(p)}\theta(x) \quad (x \in I)$$

for every $k \in \mathbb{N}$ and every $\theta \in \mathcal{O}$, where $\phi(x) = x^{\mu+1/2}e^{-x^2}$ $(x \in I)$ belongs to $\mathcal{H}_\mu$, $C_p > 0$ $(0 \leq p \leq k)$ are suitable constants, and $\alpha(p) \leq k$, $\beta(p) \leq k$ $(0 \leq p \leq k)$ denote nonnegative integers, with $C_k = 1$ and $\alpha(k) = \beta(k) = k$. $\square$

Proposition 3.2. *The linear topological space $\mathcal{O}$ is locally convex, Hausdorff, nonmetrizable, and complete.*

Proof: The only property that needs to be checked out is completeness.

Let $\{\theta_\iota\}_{\iota\in J}$ be a Cauchy net in $\mathcal{O}$. Since $\mathcal{O}$ injects continuously into $\mathcal{E}(I)$ (Proposition 3.1), $\{\theta_\iota\}_{\iota\in J}$ is also a Cauchy net in $\mathcal{E}(I)$. $\mathcal{E}(I)$ being complete, $\{\theta_\iota\}_{\iota\in J}$ converges to some $\theta \in \mathcal{E}(I)$ in $\mathcal{E}(I)$. We must show that $\theta \in \mathcal{O}$ and that $\{\theta_\iota\}_{\iota\in J}$ converges to θ in the topology of $\mathcal{O}$.

Fix $\phi \in \mathcal{H}_\mu$, $k \in \mathbb{N}$, $\varepsilon > 0$. By hypothesis, there exists $\iota_0 = \iota_0(\phi,k,\varepsilon) \in J$ such that

$$\gamma^{\mu}_{\phi,k}(\theta_\iota - \theta_{\iota'}) < \varepsilon \quad (\iota, \iota' \geq \iota_0). \tag{3.2}$$

Let us consider $x \in I$, $\eta > 0$. Since $\{\theta_\iota\}_{\iota \in J}$ converges to θ in $\mathcal{E}(I)$, there holds

$$\left|x^{-\mu-1/2}\phi(x)(x^{-1}D)^k(\theta - \theta_{\iota'})(x)\right| < \eta \tag{3.3}$$

for some $\iota' = \iota'(\phi, x, \eta) \geq \iota_0$. The combination of (3.2) and (3.3) yields

$$\left|x^{-\mu-1/2}\phi(x)(x^{-1}D)^k(\theta - \theta_\iota)(x)\right| < \varepsilon + \eta \quad (\iota \geq \iota_0),$$

and from the arbitrariness of x and η, we infer that

$$\gamma^\mu_{\phi,k}(\theta - \theta_\iota) \leq \varepsilon \quad (\iota \geq \iota_0).$$

With the inequality

$$\gamma^\mu_{\phi,k}(\theta) \leq \gamma^\mu_{\phi,k}(\theta - \theta_\iota) + \gamma^\mu_{\phi,k}(\theta_\iota) \quad (\iota \geq \iota_0)$$

we finally prove that $\theta \in \mathcal{O}$ and $\{\theta_\iota\}_{\iota \in J}$ converges to θ in $\mathcal{O}$. ☐

The next Proposition 3.3 collects several continuity properties of certain operators on $\mathcal{O}$.

Proposition 3.3. *The following holds:*

(i) *The bilinear map*

$$\begin{aligned} \mathcal{O} \times \mathcal{O} &\to \mathcal{O} \\ (\theta, \vartheta) &\mapsto \theta\vartheta \end{aligned}$$

is separately continuous.

(ii) *If $R(x) = P(x)/Q(x)$, where $P(x)$ and $Q(x)$ are polynomials and Q does not vanish in $[0, \infty[$, then the map $\theta(x) \mapsto R(x^2)\theta(x)$ is continuous from $\mathcal{O}$ to $\mathcal{O}$.*

(iii) *For every $k \in \mathbb{N}$, the map $\theta(x) \mapsto (x^{-1}D)^k\theta(x)$ is continuous from $\mathcal{O}$ to $\mathcal{O}$.*

Proof: Let $\theta \in \mathcal{O}$, $k \in \mathbb{N}$, and for $0 \leq p \leq k$ let $n_p \in \mathbb{N}$, $A_p > 0$ be such that

$$\left|(x^{-1}D)^p\theta(x)\right| \leq A_p(1 + x^2)^{n_p} \quad (x \in I).$$

If $\phi \in \mathcal{H}_\mu$, set

$$\phi_p(x) = (1 + x^2)^{n_p}\phi(x) \quad (x \in I).$$

Note that $\phi_p \in \mathcal{H}_\mu$. The formula

$$\begin{aligned} &x^{-\mu-1/2}\phi(x)(x^{-1}D)^k(\theta\vartheta)(x) = \\ &\qquad = \sum_{p=0}^{k} \binom{k}{p} x^{-\mu-1/2}\phi_p(x)\frac{(x^{-1}D)^p\theta(x)}{(1 + x^2)^{n_p}}(x^{-1}D)^{k-p}\vartheta(x), \end{aligned}$$

valid for all $x \in I$, leads to the inequality

$$\gamma^\mu_{\phi,k}(\theta\vartheta) \leq \sum_{p=0}^{k} \binom{k}{p} A_p \gamma^\mu_{\phi_p,k-p}(\vartheta),$$

which proves (i).

Assertion (ii) may be immediately deduced from (i) and from Lemma 5.3.1 in [7], whereas (iii) derives from the relationship

$$\gamma^\mu_{\phi,p}\big((x^{-1}D)^k\theta(x)\big) = \gamma^\mu_{\phi,k+p}(\theta).$$

☐

Proposition 3.4. *The bilinear map*

$$\begin{gathered}\mathcal{O} \times \mathcal{H}_\mu \to \mathcal{H}_\mu\\(\theta, \phi) \mapsto \theta\phi\end{gathered}$$

is separately continuous.

Proof: See Theorem 2.3 and part (i) of the remark preceding Proposition 3.1. □

Proposition 3.5. *The map $\varphi(x) \mapsto x^{-\mu-1/2}\varphi(x)$ is continuous from $\mathcal{H}_\mu$ into $\mathcal{O}$.*

Proof: There holds:

$$\gamma^\mu_{\phi,k}(x^{-\mu-1/2}\varphi(x)) \leq \sup_{x\in I}|x^{-\mu-1/2}\phi(x)|\lambda^\mu_{0,k}(\varphi) \quad (\varphi,\phi \in \mathcal{H}_\mu,\ k\in\mathbb{N}).$$

□

Remark. We claim that the test space $\mathcal{D}(I)$ is not dense in $x^{-\mu-1/2}\mathcal{H}_\mu$ with respect to the topology of $\mathcal{O}$. Admitting for the moment the veracity of this assertion, it follows from Proposition 3.5 that $\mathcal{D}(I)$ is not dense in $\mathcal{O}$, which prevents $\mathcal{O}$ from being a normal space of distributions. This differs from the case of Schwartz multipliers (cf. [1, Theorem 4.7]).

To prove the claim, take $\varphi \in \mathcal{H}_\mu$ and assume (to reach a contradiction) that $\{x^{-\mu-1/2}\alpha_\iota(x)\}_{\iota\in J}$ is a net in $\mathcal{D}(I)$, converging to $x^{-\mu-1/2}\varphi(x)$ in the topology of $\mathcal{O}$. Given $k \in \mathbb{N}$, $\varepsilon > 0$, there exists $\iota_0 = \iota_0(k,\varepsilon) \in J$, with

$$|e^{-x^2}(x^{-1}D)^k x^{-\mu-1/2}(\alpha_{\iota_0} - \varphi)(x)| < \varepsilon/e \quad (x \in I).$$

For $x \in]0,1[$, we may write:

$$|(x^{-1}D)^k x^{-\mu-1/2}(\alpha_{\iota_0} - \varphi)(x)| \leq e|e^{-x^2}(x^{-1}D)^k x^{-\mu-1/2}(\alpha_{\iota_0} - \varphi)(x)| < \varepsilon.$$

Therefore, to every $k \in \mathbb{N}$ and every $n = 1,2,3,\dots$ there corresponds $\iota_n \in J$, $x_n \in]0,1/n[$, such that

$$\begin{aligned}|(x^{-1}D)^k x^{-\mu-1/2}\varphi(x)_{|x=x_n}| &\leq |(x^{-1}D)^k x^{-\mu-1/2}(\alpha_{\iota_n} - \varphi)(x)_{|x=x_n}|\\&\quad + |(x^{-1}D)^k x^{-\mu-1/2}\alpha_{\iota_n}(x)_{|x=x_n}| < 1/n,\end{aligned}$$

whence

$$\lim_{n\to\infty}(x^{-1}D)^k x^{-\mu-1/2}\varphi(x)_{|x=x_n} = 0.$$

However, the particularizations $\varphi(x) = x^{\mu+1/2}e^{-x^2}$ and $k = 0$ lead to

$$\lim_{x\to 0^+}(x^{-1}D)^k x^{-\mu-1/2}\varphi(x) = 1,$$

thus yielding a contradiction, as expected.

Proposition 3.6. *Set $\mu \geq -1/2$. Given $\theta \in \mathcal{O}$, the function $x^{\mu+1/2}\theta(x)$ defines an element in $\mathcal{H}'_\mu$ by the formula*

$$\langle x^{\mu+1/2}\theta(x), \phi(x)\rangle = \int_0^\infty x^{\mu+1/2}\theta(x)\phi(x)\,dx \quad (\phi \in \mathcal{H}_\mu), \tag{3.4}$$

and the map $\theta(x) \mapsto x^{\mu+1/2}\theta(x)$ is continuous from $\mathcal{O}$ into $\mathcal{H}'_\mu$.

PROOF: Take $\theta \in \mathcal{O}$, $\phi \in \mathcal{H}_\mu$, and choose $r \in \mathbb{N}$, $A_r > 0$ satisfying

$$|\theta(x)| \leq A_r(1+x^2)^r \quad (x \in I).$$

Also, let $s \in \mathbb{N}$, $s > \mu + 1$, be such that

$$C_s^\mu = \int_0^\infty \frac{x^{2\mu+1}}{(1+x^2)^s}\,dx < +\infty.$$

Upon multiplying and dividing the integrand in (3.4) by $x^{-\mu-1/2}(1+x^2)^s$ we find that:

$$|\langle x^{\mu+1/2}\theta(x), \phi(x)\rangle| \leq A_r C_s^\mu \tau^\mu_{r+s,0}(\phi),$$

and that:

$$|\langle x^{\mu+1/2}\theta(x), \phi(x)\rangle| \leq C_s^\mu \gamma^\mu_{\psi,0}(\theta),$$

where $\psi(x) = (1+x^2)^s\phi(x) \in \mathcal{H}_\mu$. □

4. Multipliers of $\mathcal{H}'_\mu$.

Next we aim to characterize $\mathcal{O}$ as the space of multipliers of $\mathcal{H}'_\mu$ ($\mu \in \mathbb{R}$). The reflexivity of $\mathcal{H}_\mu$ will be needed for that purpose. In Proposition 4.2 we prove that $\mathcal{H}_\mu$ is nuclear ([4, Definition III.50.1]), a property stronger than reflexivity; to this end, the following is useful.

Lemma 4.1. *Let $m, k \in \mathbb{N}$, and let $\phi \in \mathcal{H}_\mu$. There holds:*

$$\sum_{k=0}^m \sup_{x\in I} |(1+x^2)^m(x^{-1}D)^k x^{-\mu-1/2}\phi(x)| \leq$$
$$\leq (m+1)\sum_{k=0}^{m+1}\int_0^\infty |(1+t^2)^{m+1}(t^{-1}D)^k t^{-\mu-1/2}\phi(t)|\,dt.$$

PROOF: In fact, we have:

$$\begin{aligned}(1+x^2)^m(x^{-1}D)^k x^{-\mu-1/2}\phi(x) &= -\int_x^\infty D((1+t^2)^m(t^{-1}D)^k t^{-\mu-1/2}\phi(t))\,dt \\ &= -\int_x^\infty 2mt(1+t^2)^{m-1}(t^{-1}D)^k t^{-\mu-1/2}\phi(t)\,dt \\ &\quad -\int_x^\infty t(1+t^2)^m(t^{-1}D)^{k+1} t^{-\mu-1/2}\phi(t)\,dt \quad (x \in I).\end{aligned}$$

Since $2t \leq 1 + t^2$ $(t \in I)$, it follows that

$$|(1+x^2)^m (x^{-1}D)^k x^{-\mu-1/2}\phi(x)| \leq m \int_0^\infty |(1+t^2)^m (t^{-1}D)^k t^{-\mu-1/2}\phi(t)|\,dt$$
$$+ \int_0^\infty |(1+t^2)^{m+1} (t^{-1}D)^{k+1} t^{-\mu-1/2}\phi(t)|\,dt \quad (x \in I),$$

whence the lemma. □

Proposition 4.2. *The space $\mathcal{H}_\mu$ is nuclear.*

PROOF: Let $m, k \in \mathbb{N}$, and let $\phi \in \mathcal{H}_\mu$. For $t \in I$ and $0 \leq k \leq m+2$, define $u_{t,k} \in \mathcal{H}'_\mu$ by the formula:

$$\langle u_{t,k}, \phi\rangle = (1+t^2)^{m+2} (t^{-1}D)^k t^{-\mu-1/2}\phi(t) \quad (\phi \in \mathcal{H}_\mu),$$

and consider

$$V = \{\phi \in \mathcal{H}_\mu : \sum_{k=0}^{m+2} \sup_{t\in I} |(1+t^2)^{m+2} (t^{-1}D)^k t^{-\mu-1/2}\phi(t)| < 1\}.$$

Note that V is a neighborhood of the origin in $\mathcal{H}_\mu$, and that each $u_{t,k}$ $(t \in I,\ 0 \leq k \leq m+2)$ belongs to V°, the polar set of V. Thus, a positive Radon measure μ may be defined on V° by the equation:

$$\langle \mu, \varphi\rangle = \int_{V^\circ} \varphi\, d\mu = (m+1) \sum_{k=0}^{m+2} \int_0^\infty \varphi(u_{t,k})(1+t^2)^{-1}\,dt \quad (\varphi \in C(V^\circ)).$$

Lemma 4.1 now implies:

$$\sum_{k=0}^{m} \sup_{x\in I} |(1+x^2)^m (x^{-1}D)^k x^{-\mu-1/2}\phi(x)| \leq$$
$$\leq (m+1) \sum_{k=0}^{m+2} \int_0^\infty |(1+t^2)^{m+1} (t^{-1}D)^k t^{-\mu-1/2}\phi(t)|\,dt$$
$$= (m+1) \sum_{k=0}^{m+2} |\langle u_{t,k}, \phi\rangle| (1+t^2)^{-1}\,dt$$
$$= \int_{V^\circ} |\langle u, \phi\rangle|\, d\mu(u) \quad (\phi \in \mathcal{H}_\mu).$$

Since the sets

$$V(m,\varepsilon) =$$
$$= \{\phi \in \mathcal{H}_\mu : \sum_{k=0}^{m} \sup_{x\in I} |(1+x^2)^m (x^{-1}D)^k x^{-\mu-1/2}\phi(x)| < \varepsilon\} \quad (m \in \mathbb{N},\ \varepsilon > 0)$$

form a basis of neighborhoods of the origin in $\mathcal{H}'_\mu$, the nuclearity of this space follows from [3, Proposition 4.1.5]. □

Once that Proposition 4.2 has been established, a number of consequences may be deduced by applying general properties of nuclear spaces.

Corollary 4.3. *The space $\mathcal{H}'_\mu$ is nuclear with respect to its strong topology.*

PROOF: See [4, Proposition III.50.6]. □

Corollary 4.4. *$\mathcal{H}_\mu$ (with its usual topology) and $\mathcal{H}'_\mu$ (with the strong topology) are Schwartz spaces.*

PROOF: This is derived from [5, Proposition 3.2.5]. □

Corollary 4.5. *The space $\mathcal{H}_\mu$ is Montel, hence reflexive.*

PROOF: Fréchet-Schwartz spaces are Montel ([2, Corollary to Proposition 3.15.4]), and Montel spaces are reflexive ([2, Corollary to Proposition 3.9.1]). □

We turn to the study of the multipliers of $\mathcal{H}'_\mu$.

Definition 4.6. For $\theta \in \mathcal{O}$ and $T \in \mathcal{H}'_\mu$, θT is defined by transposition:

$$\langle \theta T, \phi \rangle = \langle T, \theta\phi \rangle \quad (\phi \in \mathcal{H}_\mu).$$

Proposition 3.4 implies that $\theta T \in \mathcal{H}'_\mu$ and that each map $T \mapsto \theta T$ is continuous from $\mathcal{H}'_\mu$ to $\mathcal{H}'_\mu$. By applying the universal property of initial topologies, we also find that the map $\theta \mapsto \theta T$ is continuous from $\mathcal{O}$ into $\mathcal{H}'_\mu$ if the latter is equipped with its weak* topology. We are thus led to the following.

Proposition 4.7. *The bilinear map*

$$\begin{gathered} \mathcal{O} \times \mathcal{H}'_\mu \to \mathcal{H}'_\mu \\ (\theta, T) \mapsto \theta T \end{gathered}$$

is separately continuous when $\mathcal{H}'_\mu$ is endowed with its weak topology.*

Given $a > 0$ and $\mu \in \mathbb{R}$, $\mathcal{B}_{\mu,a}$ (see [6]) is the subspace of $\mathcal{H}_\mu$ formed by all those functions $\psi = \psi(x)$ infinitely differentiable on I such that $\psi(x) = 0$ $(x \geq a)$, for which the quantities

$$\lambda_k^\mu(\psi) = \sup_{x \in I} |(x^{-1}D)^k x^{-\mu-1/2}\psi(x)| \quad (k \in \mathbb{N})$$

are finite. When equipped with the topology generated by the family of seminorms $\{\lambda_k^\mu\}_{k\in\mathbb{N}}$, $\mathcal{B}_{\mu,a}$ becomes a Fréchet space. It is easy to see that $\mathcal{B}_{\mu,a} \subset \mathcal{B}_{\mu,b}$ if $0 < a < b$, and that $\mathcal{B}_{\mu,a}$ inherits from $\mathcal{B}_{\mu,b}$ its own topology. These facts allow us to define $\mathcal{B}_\mu = \bigcup_{a>0} \mathcal{B}_{\mu,a}$ as the inductive limit of the family $\{\mathcal{B}_{\mu,a}\}_{a>0}$. The space $\mathcal{B}_\mu$ turns out to be dense in $\mathcal{H}_\mu$.

Definition 4.8. Let $\theta \in C^\infty(I)$ be such that $(x^{-1}D)^k\theta(x)$ is bounded in a neighborhood of zero for every $k \in \mathbb{N}$. If $T \in \mathcal{H}'_\mu$ then T lies in $\mathcal{B}'_\mu$, the dual space of $\mathcal{B}_\mu$, and $\theta T \in \mathcal{B}'_\mu$ may be consistently defined by the formula

$$\langle \theta T, \psi \rangle = \langle T, \theta\psi \rangle \quad (\psi \in \mathcal{B}_\mu).$$

We are now ready to prove that the space of multipliers of $\mathcal{H}'_\mu$ is precisely $\mathcal{O}$:

Theorem 4.9. *Assume that $\theta \in C^\infty(I)$ is such that each $(x^{-1}D)^k\theta(x)$ $(k \in \mathbb{N})$ is bounded in a neighborhood of zero. If, for every $T \in \mathcal{H}'_\mu$, the functional $\theta T \in \mathcal{B}'_\mu$ (given by Definition 4.8) can be (a fortiori uniquely) extended up to $\mathcal{H}_\mu$ as a member of $\mathcal{H}'_\mu$ in such a way that the map $\theta \mapsto \theta T$ is continuous from $\mathcal{H}'_\mu$ into itself, then $\theta \in \mathcal{O}$.*

PROOF: Let $\phi \in \mathcal{H}_\mu$. Our hypotheses imply that the linear functional $T \mapsto \langle \theta T, \phi\rangle$ is continuous on $\mathcal{H}'_\mu$. By the reflexivity of $\mathcal{H}_\mu$ (Corollary 4.5), there exists $\varphi \in \mathcal{H}_\mu$ satisfying

$$\langle \theta T, \phi\rangle = \langle T, \varphi\rangle \quad (T \in \mathcal{H}'_\mu).$$

In particular:

$$\langle \theta\phi, \psi\rangle = \langle \theta\psi, \phi\rangle = \langle \psi, \varphi\rangle = \langle \varphi, \psi\rangle \quad (\psi \in \mathcal{B}_\mu).$$

Thus, $\theta\phi = \varphi \in \mathcal{H}_\mu$. Since the space of multipliers of $\mathcal{H}_\mu$ is $\mathcal{O}$ (Theorem 2.3), we conclude that $\theta \in \mathcal{O}$. □

5. Another topology on $\mathcal{O}$.

Let μ be any real number, and let $\mathfrak{B}_\mu$ denote the family of all bounded subsets of $\mathcal{H}_\mu$. Throughout this section we shall assume that $\mathcal{O}$ is endowed with the topology generated by the family of seminorms

$$\gamma^\mu_{B,k} = \sup\{\gamma^\mu_{\phi,k} : \phi \in B\} \quad (B \in \mathfrak{B}_\mu,\ k \in \mathbb{N}). \tag{5.1}$$

Remark. Clearly, the topology just defined on $\mathcal{O}$ is finer than that introduced in Section 3. As before, any two spaces $\mathcal{H}_\mu$ and $\mathcal{H}_\nu$ being isomorphic, this topology does not depend on the parameter μ.

Proposition 5.1. *The topological vector space $\mathcal{O}$ is locally convex, Hausdorff, nonmetrizable, and complete.*

PROOF: Again, the only property to be checked out is completeness.

Let $\{\theta_\iota\}_{\iota\in J}$ be a Cauchy net in $\mathcal{O}$. Since $\{\theta_\iota\}_{\iota\in J}$ is also Cauchy with respect to the topology considered on $\mathcal{O}$ in Section 3 above (see the preceding remark), there exists $\theta \in \mathcal{O}$ such that $\{\theta_\iota\}_{\iota\in J}$ converges to θ in that topology.

Take $B \in \mathfrak{B}_\mu$, $k \in \mathbb{N}$, $\varepsilon > 0$. By hypothesis, there exists $\iota_0 = \iota_0(B, k, \varepsilon) \in J$ such that

$$\gamma^\mu_{B,k}(\theta_\iota - \theta_{\iota'}) < \varepsilon/2 \quad (\iota, \iota' \geq \iota_0).$$

Moreover, as just observed, to every $\phi \in B$ there corresponds $\iota' = \iota'(\phi, k, \varepsilon) \geq \iota_0$ satisfying

$$\gamma^\mu_{\phi,k}(\theta_{\iota'} - \theta) < \varepsilon/2.$$

A combination of the last two inequalities shows that

$$\gamma^\mu_{B,k}(\theta_\iota - \theta) < \varepsilon \quad (\iota \geq \iota_0).$$

Therefore, $\{\theta_\iota\}_{\iota\in J}$ converges to θ in $\mathcal{O}$. □

Proposition 5.2. *The bilinear map*

$$\begin{aligned} \mathcal{O} \times \mathcal{H}_\mu &\to \mathcal{H}_\mu \\ (\theta, \phi) &\mapsto \theta\phi \end{aligned} \tag{5.2}$$

is hypocontinuous.

Proof: That (5.2) is separately continuous follows from Proposition 3.4 and from the remark preceding Proposition 5.1 above.

Since $\mathcal{H}_\mu$ is a Fréchet space, the uniform boundedness principle guarantees the hypocontinuity with respect to the bounded subsets of $\mathcal{O}$. On the other hand, take $m, k \in \mathbb{N}$, and for every $\phi \in \mathcal{H}_\mu$ and every $p \in \mathbb{N}$, $0 \le p \le k$, define $\phi_p \in \mathcal{H}_\mu$ by

$$\phi_p(x) = (1+x^2)^m x^{\mu+1/2}(x^{-1}D)^{k-p} x^{-\mu-1/2}\phi(x) \quad (x \in I).$$

Leibniz's rule shows that the map $\phi \mapsto \phi_p$ is continuous from $\mathcal{H}_\mu$ into $\mathcal{H}_\mu$. Denoting by $B_p \in \mathfrak{B}_\mu$ the image of $B \in \mathfrak{B}_\mu$ through this map, it can be proved, as in the part (i) of the remark preceding Proposition 3.1 that

$$\tau^\mu_{m,k}(\theta\phi) \le \sum_{p=0}^{k} \binom{k}{p} \gamma^\mu_{B_p,p}(\theta) \quad (\theta \in \mathcal{O},\ \phi \in B). \tag{5.3}$$

Thus, (5.2) is $\mathfrak{B}_\mu$-hypocontinuous. □

It should be observed that the topology generated on $\mathcal{O}$ by the seminorms (5.1) is compatible with the family

$$\gamma^\mu_{m,k;B}(\theta) = \sup\{\tau^\mu_{m,k}(\theta\phi) : \phi \in B\} \quad (m, k \in \mathbb{N},\ B \in \mathfrak{B}_\mu).$$

In fact, let $k \in \mathbb{N}$. For every $p \in \mathbb{N}$ with $0 \le p \le k$, the map $\phi \mapsto \phi_p$, defined from $\mathcal{H}_\mu$ into $\mathcal{H}_\mu$ by the formula

$$\phi_p(x) = x^{\mu+1/2}(x^{-1}D)^p x^{-\mu-1/2}\phi(x) \quad (x \in I),$$

is continuous; as before, we denote by $B_p \in \mathfrak{B}_\mu$ the image of $B \in \mathfrak{B}_\mu$ through this map. Now, the argument in the part (ii) of the remark preceding Proposition 3.1 shows that

$$\gamma^\mu_{B,k}(\theta) \le \sum_{p=0}^{k} \binom{k}{p} \gamma^\mu_{0,k-p;B_p}(\theta) \quad (B \in \mathfrak{B}_\mu,\ k \in \mathbb{N},\ \theta \in \mathcal{O}).$$

Along with (5.3), this estimate proves our assertion.

Proposition 5.3. *The bilinear map*

$$\begin{gathered}\mathcal{O} \times \mathcal{H}'_\mu \to \mathcal{H}'_\mu \\ (\theta, T) \mapsto \theta T\end{gathered}$$

is separately continuous when $\mathcal{H}'_\mu$ is endowed either with its weak or with its strong topology.*

Proof: The continuity in the second variable follows from [4, Propositions II.19.5 and II.35.8]. On the other hand, let $T \in \mathcal{H}'_\mu$, $\theta \in \mathcal{O}$, $B \in \mathfrak{B}_\mu$. There exist $r \in \mathbb{N}$ and a constant $C > 0$ such that

$$|\langle T, \varphi\rangle| \leq C \max_{0 \leq m,\ k \leq r} \tau^\mu_{m,k}(\varphi) \quad (\varphi \in \mathcal{H}_\mu),$$

Hence

$$|\langle \theta T, \phi\rangle| = |\langle T, \theta\phi\rangle| \leq C \max_{0 \leq m,\ k \leq r} \tau^\mu_{m,k}(\theta\phi) \quad (\phi \in B),$$

which leads to the inequality

$$\sup\{|\langle \theta T, \phi\rangle| : \phi \in B\} \leq C \max_{0 \leq m,\ k \leq r} \gamma^\mu_{m,k;B}(\theta).$$

□

Proposition 5.4. *The bilinear map*

$$\begin{gathered}\mathcal{O} \times \mathcal{O} \to \mathcal{O} \\ (\theta, \vartheta) \mapsto \theta\vartheta\end{gathered}$$

is hypocontinuous.

Proof: Let $\mathfrak{B}$ denote the family of all bounded subsets of $\mathcal{O}$. If $A \in \mathcal{B}$ and $B \in \mathcal{B}_\mu$, a fortiori $AB \in \mathfrak{B}_\mu$ (Proposition 5.2 and [2, Proposition 4.7.2]). Fix $m, k \in \mathbb{N}$, $\theta \in A$, $\vartheta \in \mathcal{O}$, $\phi \in B$; then

$$\gamma^\mu_{m,k;B}(\theta\vartheta) \leq \gamma^\mu_{m,k;AB}(\vartheta).$$

□

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