On the paracompactness of frames

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Abstract. Through the study of frame congruences, new characterizations of the paracompactness of frames are obtained.

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There are some frame properties whose corresponding topological properties have important characterizations in terms of subsets that are not necessarily open. For instance, local connectedness and paracompactness are such properties. To get frame counterparts of these valuable characterizations, we turn to frame congruences, which provide sufficient tools for us to translate directly some topological properties pertaining to general subsets, especially, closed subsets of spaces. Advantages of such considerations have been shown through the study of the local connectedness of frames in [2].

It is well known that the paracompactness of spaces can be characterized by employing one of the following refinements: (1) locally finite; (2) cushioned; (3) closure-preserving; (4) σ-locally finite open; (5) σ-closure preserving open; and (6) σ-cushioned open. Through the study of congruences, and applying the results of [3], we obtain the frame versions of the above classical characterizations, and thus extend the related results of [3], [6] and [7]. The topological intuition behind our arguments concerning congruences can be easily traced by knowing the correspondence between congruences and subspaces.

Throughout this paper, L always represents a frame.

1. Basic facts.

For general background of frames, we refer to [1] and [4].

1.1. For a frame L, its *top* (*bottom*) element will be denoted by e (0). For $a \in L$, its **pseudocomplement** is denoted by a^* and is given by $a^* = \bigvee\{x \in L \mid x \wedge a = 0\}$. For any $X \subseteq L$, we use X^* to denote the set $\{x^* \mid x \in X\}$.

The relation $\prec$ on L is defined such that $b \prec a$ if and only if $b^* \vee a = e$. A frame L is called **regular** if $a = \bigvee\{x \in L \mid x \prec a\}$ for each $a \in L$.

A frame L is called **normal** if for $a_1, a_2 \in L$ such that $a_1 \vee a_2 = e$ there are b_1, b_2 such that $a_1 \vee b_1 = a_2 \vee b_2 = e$ and $b_1 \wedge b_2 = 0$.

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1.2. The lattice of frame congruences on L under set inclusion is a frame, denoted by $\mathfrak{C}L$. The top and bottom of $\mathfrak{C}L$ are denoted by ∇ and $\triangle$. We reserve small Greek letters to denote congruences.

The following properties, extracted from [1], are very useful:

(1.2.1) For any $a \in L$, $\nabla_a = \{(x,y) \mid x \vee a = y \vee a\}$, called **closed**, is the least congruence containing $(0,a)$; $\triangle_a = \{(x,y) \mid x \wedge a = y \wedge a\}$, called **open**, is the least congruence containing (e,a).

(1.2.2) Each ∇_a is complemented in $\mathfrak{C}L$ with complement $\triangle_a$.

(1.2.3) For any $a \in L$, the map $a \rightsquigarrow \nabla_a$ is a frame embedding $L \longrightarrow \mathfrak{C}L$, whereas, the map $a \rightsquigarrow \triangle_a$ is a dual poset embedding $L \longrightarrow \mathfrak{C}L$ taking finitary $\wedge$ to finitary $\vee$ and arbitrary $\bigvee$ to arbitrary $\bigwedge$.

(1.2.4) For any $\theta \in \mathfrak{C}L$ and any $a \in L$,

$$\nabla_a \vee \theta = \{(x,y) \mid (x \vee a, y \vee a) \in \theta\} \text{ and } \triangle_a \vee \theta = \{(x,y) \mid (x \wedge a, y \wedge a) \in \theta\}.$$

1.3. For any $\theta \in \mathfrak{C}L$, the **closure** of θ, denoted by $\overline{\theta}$, is defined to be the ∇_a where a is the largest element such that $(0,a) \in \theta$. It is obvious that, for any $a \in L$, $\overline{\triangle_a} = \nabla_{a^*}$.

2. Covers and envelopes.

2.1. A subset A of L is said to be a **cover** of L if $\bigvee A = e$. For any two covers A and B, A is called a **refinement** of B if for each $a \in A$ there exist $b \in B$ such that $a \leq b$. Notation $A \leq B$.

A subset T of L is called **locally finite** if there exists a cover A such that, for each $a \in A$, $\{t \in T \mid t \wedge a \neq 0\}$ is finite. Such a cover A is said to **finitize** T.

A subset U of L is said to be an **envelope** if $x = \bigwedge\{x \vee u \mid u \in U\}$ for each $x \in L$. (Such a U is also called a closed covering of L by [3]). An envelope U is called a **corefinement** of an envelope V if for each $u \in U$ there exist $v \in V$ such that $v \leq u$.

A subset T of L is called **conservative** if for each $S \subseteq T$ and each $x \in L$, $x \vee \bigwedge S = \bigwedge\{x \vee t \mid t \in S\}$.

An envelope U is called a **dual-refinement** of a cover A if for each $u \in U$ there exist $a \in A$ such that $u \vee a = e$.

2.2. Lemma. *For any $T \subseteq L$ and any $\theta \in \mathfrak{C}L$,*

$$\theta \vee \bigwedge\{\triangle_t \mid t \in T\} = \bigwedge\{\theta \wedge \triangle_t \mid t \in T\}.$$

Therefore, $O(T) = \{\triangle_t \mid t \in T\}$ is conservative in $\mathfrak{C}L$.

PROOF: By applying (1.2.4). $\square$

2.3. Lemma. *For any $T \subseteq L$, $\bigwedge\{\nabla_t \mid t \in T\} = \nabla_{\bigwedge T}$ if and only if $\bigwedge T \vee x = \bigwedge\{t \vee x \mid t \in T\}$ for all $x \in L$.*

PROOF: ($\Longleftarrow$) Take any $(x,y) \in \bigwedge\{\nabla_t \mid t \in T\}$, $x \vee t = y \vee t$ for all $t \in T$, then $x \vee \bigwedge T = y \vee \bigwedge T$, which means $(x,y) \in \nabla_{\bigwedge T}$.

($\Longrightarrow$) Consider $(\bigwedge T \vee x, \bigwedge\{t \vee x \mid t \in T\})$. For each $t \in T$, $x \vee t = \bigwedge T \vee x \vee t = \bigwedge\{t \vee x \mid t \in T\} \vee t$, hence $(x, \bigwedge\{t \vee x \mid t \in T\}) \in \bigwedge\{\nabla_t \mid t \in T\} = \nabla_{\bigwedge T}$, which means $\bigwedge T \vee x = \bigwedge\{t \vee x \mid t \in T\}$ for all $x \in L$. $\square$

2.4. A set Θ of congruences of L is called **locally cofinite** if there is a cover A of L such that, for each $a \in A$, $\nabla_a \leq \theta$ for all but finitely many $\theta \in \Theta$. Such a cover A is said to **co-finitize** Θ.

The following are trivial observations:

(2.4.1) $\Theta \subseteq \mathfrak{C}L$ *is locally cofinite if and only if* $\bar{\Theta} = \{\bar{\theta} \mid \theta \in \Theta\}$ *is locally cofinite.*

(2.4.2) *A subset T of L is locally finite if and only if* $O(T) = \{\triangle_t \mid t \in T\}$ *is locally cofinite.*

2.5. Lemma. *For any $T \subseteq L$, if $C(T) = \{\nabla_t \mid t \in T\}$ is locally cofinite, then*

(1) *$C(T)$ is conservative in $\mathfrak{C}L$.*
(2) $\bigwedge\{\nabla_t \mid t \in T\} = \nabla_{\bigwedge T}$.
(3) $\bigvee\{\triangle_t \mid t \in T\} = \triangle_{\bigwedge T}$.

Proof: Let A be a cover of L co-finitizing $C(T)$. Take an arbitrary element $a \in A$. Let $T_0 = \{t \in T \mid a \not\leq t\}$, T_0 is finite and $a \leq \bigwedge(T - T_0)$.

For any $(x, y) \in \bigwedge\{\theta \vee \nabla_t \mid t \in T\}$, $(x \vee t, y \vee t) \in \theta$ for all $t \in T$. Then

$$\begin{aligned}\bigwedge T \vee (x \wedge a) &= \bigwedge(T - T_0) \wedge \bigwedge\{t \vee (x \wedge a) \mid t \in T_0\} \\ &\equiv_\theta \bigwedge(T - T_0) \wedge \bigwedge\{t \vee (y \wedge a) \mid t \in T_0\} = \bigwedge T \vee (y \wedge a).\end{aligned}$$

Therefore, $\bigwedge T \vee x \equiv_\theta \bigwedge T \vee y$, that is, $(x, y) \in \theta \vee \nabla_{\bigwedge T} \leq \theta \vee \bigwedge\{\nabla_t \mid t \in T\}$. This proves (1).

(2) and (3) are easy now. $\square$

2.6. Lemma. *If $\Theta \subseteq \mathfrak{C}L$ is locally cofinite, then*

$$\overline{\bigwedge\{\theta \mid \theta \in \Theta\}} = \bigwedge\{\bar{\theta} \mid \theta \in\mid \Theta\}.$$

Proof: By (2.4.1) and (2.5), $\bigwedge\{\bar{\theta} \mid \theta \in\mid \Theta\}$ is closed. $\square$

2.7. Proposition. *For any locally finite $T \subseteq L$, $T^* = \{t^* \mid t \in T\}$ is conservative.*

Proof: That $O(T) = \{\triangle_t \mid t \in T\}$ is locally cofinite implies that $\overline{O(T)} = \{\nabla_{t^*} \mid t \in T\}$ is locally cofinite. By (2.5) and (2.3),

$$x \vee \bigwedge\{t^* \mid t \in T\} = \bigwedge\{x \vee t^* \mid t \in T\} \text{ for all } x \in L.$$

$\square$

2.8. For two subsets T, S of L, we use $T \prec S$ to mean that there is a function $f : T \longrightarrow S$ such that, for each subset $X \subseteq T$, $\bigvee X \prec \bigvee f[X]$. Recall that T is said to be **cushioned** in S (see [3]) if there is a function $f : T \longrightarrow S$ such that, for each subset $X \subseteq T$, $\bigwedge X \vee \bigvee f[X] = e$. Therefore, $T \prec S$ means that T^* is cushioned in S.

2.9. Proposition. *For $T, S \subseteq L$, if for each $t \in T$ there exist $s \in S$ such that $t \prec s$, and T^* is conservative, then $T \prec S$.*

Proof: From the given condition, we obtain a function $f : T \longrightarrow S$ such that $t \prec f(t)$ for all $t \in T$. Then for any subset X of T,

$$\bigvee f[X] \vee (\bigvee X)^* = \bigvee f[X] \vee \bigwedge\{t^* \mid t \in X\} = \bigwedge\{\bigvee f[X] \vee t^* \mid t \in X\} = e,$$

that is, $\bigvee X \prec \bigvee f[X]$. Hence $T \prec S$. □

2.10. Lemma. *For every countable cover $\{a_n \mid n = 1, 2, \cdots\}$ of L, there is a locally cofinite $\{\theta_n \mid n = 1, 2, \cdots\}$ of $\mathfrak{C}L$ such that*

$$\bigwedge_{n=1}^{\infty} \theta_n = \triangle \quad \textit{and} \quad \triangle_{a_n} \leq \theta_n, \quad n = 1, 2, \cdots .$$

Proof: Take

$$\theta_1 = \triangle_{a_1}, \quad \theta_n = \triangle_{a_n} \vee \bigvee\{\nabla_{a_i} \mid i = 1, \ldots, n-1\},$$

then $\{\theta_n \mid n = 1, 2, \cdots\}$ is locally cofinite.

Suppose $x \neq y$ and $(x, y) \in \bigwedge\{\theta_n \mid n = 1, 2, \cdots\}$. Suppose k is the smallest number such that $(x, y) \notin \triangle_{a_k}$. Then $(x, y) \in \bigwedge\{\triangle_{a_i} \mid i = 1, \cdots k-1\}$. Then

$$(x, y) \in (\triangle_{a_k} \vee \bigvee\{\nabla_{a_i} \mid i = 1, \ldots, k-1\}) \wedge \bigwedge\{\triangle_{a_i} \mid i = 1, \cdots k-1\} \leq \triangle_{a_k},$$

a contradiction. Hence $\bigwedge\{\theta_n \mid n = 1, 2, \cdots\} = \triangle$. □

3. Paracompactness.

3.1. A frame is called **paracompact** if each cover has a locally finite refinement.

3.2. For a cover $A = \{a_i \mid i \in I\}$ of L, we call

(1) A **shrinkable** if there is a cover $B = \{b_i \mid i \in I\}$ such that $b_i \prec a_i$ for all $i \in I$ (see [3]).

(2) A **strongly shrinkable** if there is a cover B such that $B \prec A$.

(3) A **σ-strongly shrinkable** if A has a refinement $B = \bigcup_{n=1}^{\infty} B_n$ such that $B_n \prec A$.

Lemma. *Every strongly shrinkable cover is shrinkable.*

Proof: Let $A = \{a_i \mid i \in I\}$ be a strongly shrinkable cover, then there is a cover B satisfying $B \prec A$ with a corresponding function $f : B \longrightarrow A$. For each $i \in I$, put $b_i = \bigvee\{b \in B \mid f(b) = a_i\}$, which satisfies $b_i \prec a_i$. □

3.3. Theorem. *The following properties are equivalent:*

(i) *L is paracompact and normal.*

(ii) *Each cover of L has a conservative envelope as a dual-refinement.*

(iii) *For each cover A of L, there is an envelope U being cushioned in A.*

(iv) *For each cover A of L, there exists a $\Theta \subseteq \mathfrak{C}L$ satisfying $\bigwedge \Theta = \triangle$ and a function $f : \Theta \longrightarrow A$ such that for every subset Θ' of Θ,*

$$\bigwedge\{\triangle_{f(\theta)} \mid \theta \in \Theta'\} \leq \overline{\bigwedge\{\theta \mid \theta \in \Theta'\}}.$$

(v) *Every cover of L is σ-strongly shrinkable.*

PROOF: The equivalences for **(i)**, **(ii)** and **(iii)** have been proved by Dowker and Papert Strauss [3].

(i) $\Longrightarrow$ (v). For each cover A of L, there is a locally finite refinement B of A. By [3, Proposition 1], B is shrinkable, so there is a cover C such that for each $c \in C$ there exist $b \in B$ such that $c \prec b$. Again C has a locally finite refinement D, then $D \prec B$ by (2.7) and (2.9), implies $D \prec A$. Thus A is strongly shrinkable.

(v) $\Longrightarrow$ (iv). Consider a cover A of L and a refinement $B = \bigcup_{n=1}^{\infty} B_n$ of A such that $B_n \prec A$ with corresponding functions $f_n : B_n \longrightarrow A$ for $n = 1, 2, \cdots$.

Put

$$d_n = \bigvee B_n, \quad n = 1, 2, \cdots .$$

By (2.10), there exists a locally cofinite $\{\theta_n \mid n = 1, 2, \cdots\}$ of $\mathfrak{C}L$ such that $\bigwedge_{n=1}^{\infty} \theta_n = \triangle$ and $\triangle_{d_n} \leq \theta_n$ for all n.

Put

$$\Phi = \{\theta_n \vee \triangle_b \mid b \in B_n, n = 1, 2, \cdots\}.$$

(1) $\bigwedge \Phi = \bigwedge_{n=1}^{\infty} \bigwedge\{\theta_n \vee \triangle_b \mid b \in B_n\} = \bigwedge_{n=1}^{\infty}(\theta_n \vee \triangle_{d_n}) = \bigwedge_{n=1}^{\infty} \theta_n = \triangle$, where the second equality holds since $\theta \wedge \triangle_{\vee X} = \bigwedge\{\theta \vee \triangle_x \mid x \in X\}$ for any $X \subseteq L$ and $\theta \in \mathfrak{C}L$.

(2) Define $f : \Phi \longrightarrow A$ by $f(\theta_n \vee \triangle_b) = f_n(b)$. For any subset $\Phi' \subseteq \Phi$ with $\Phi' = \{\theta_n \vee \triangle_b \mid b \in B'_n,\ n = 1, 2, \cdots\}$ where $B'_n \subset B_n$,

$$\begin{aligned}
\overline{\bigwedge \Phi'} &= \overline{\bigwedge\{\theta_n \vee \triangle_{\bigvee B'_n} \mid n = 1, 2, \cdots\}} \\
&= \bigwedge\{\overline{\theta_n \vee \triangle_{\bigvee B'_n}} \mid n = 1, 2, \cdots\} \quad \text{by (2.6)} \\
&\geq \bigwedge\{\nabla_{(\bigvee B'_n)^*} \mid n = 1, 2, \cdots\} \\
&\geq \bigwedge\{\triangle_{\bigvee f_n[B'_n]} \mid n = 1, 2, \cdots\} \\
&= \bigwedge\{\triangle_{f_n(b)} \mid b \in B'_n, n = 1, 2, \cdots\}.
\end{aligned}$$

Thus Φ has the property required for Θ in (iv).

(iv) $\Longrightarrow$ (iii). Since for every $\Theta' \subseteq \Theta$,

$$\overline{\bigwedge\{\theta \mid \theta \in \Theta'\}} = \overline{\bigwedge\{\bar{\theta} \mid \theta \in \Theta'\}},$$

we may choose $\Theta = \{\nabla_b \mid b \in B\}$ consisting of closed congruences of L. Then B is an envelope of L by (2.3). Define $g : B \longrightarrow A$ by $g(b) = f(\nabla_b)$. For every $B' \subseteq B$,

$$\bigwedge\{\triangle_{g(b)} \mid b \in B'\} \leq \overline{\bigwedge\{\nabla_b \mid b \in B'\}}$$

implies $\bigvee g[B'] \vee \bigwedge B' = e$. Hence B is cushioned in A. □

3.4. Remark. Recall a classical result (see [5]) that *for a T_1-space X, the following properties are equivalent:*

- **(i)** *X is paracompact.*
- **(ii)** *Every open covering X has a cushioned (not necessarily open) refinement.*
- **(iii)** *Every covering of X has a σ-cushioned open refinement.*

The equivalences of (i), (iv) and (v) of (3.3) can be regarded as the frame counterpart of this classical results.

3.5. Theorem. *For a regular frame L, the following are equivalent:*

- **(i)** *L is paracompact.*
- **(ii)** *Each cover of L has a σ-locally finite refinement.*
- **(iii)** *Each cover A of L has a refinement $B = \bigcup_{n=1}^{\infty} B_n$ such that B_n^* is conservative for every n.*
- **(iv)** *For each A of L, $O(A) = \{\triangle_a \mid a \in A\}$ has a locally cofinite corefinement Θ.*

Proof: (i) $\Longrightarrow$ (ii) $\Longrightarrow$ (iii) and (i) $\Longrightarrow$ (iv) are obvious.

(iii) $\Longrightarrow$ (i). Let A be a cover of L. Put $C = \{c \mid c \prec a$ for some $a \in A\}$, which is also a cover of L. Then C has a refinement $B = \bigcup_{n=1}^{\infty} B_n$ such that B_n^* is conservative. By (3.5), $B_n \prec A$ for all n. Therefore L is paracompact by (v) of (3.3).

(iv) $\Longrightarrow$ (i). Let A be a cover of L. Put $C = \{c \mid c \prec a$ for some $a \in A\}$. By assumption, $O(C)$ has a locally cofinite corefinement Θ. Then for each $\theta \in \Theta$ there exist $c \in C$ and $a \in A$ such that $\triangle_a \leq \nabla_{c^*} \leq \triangle_c \leq \theta$, implying $\triangle_a \leq \bar{\theta}$. This correspondence gives a function $f : \Theta \longrightarrow A$ such that for every $\Theta' \subseteq \Theta$,

$$\bigwedge\{\triangle_{f(\theta)} \mid \theta \in \Theta'\} \leq \bigwedge\{\bar{\theta} \mid \theta \in \Theta'\} = \overline{\bigwedge\{\theta \mid \theta \in \Theta'\}}.$$

Therefore L is paracompact by (iv) of (3.3). □

3.6. Proposition. *Every complete Boolean algebra is paracompact.*

A proof of this has been given by Sun [7]. Here we can give a simple proof based on (3.5).

Proof: For every cover A of a complete Boolean algebra, A^* is conservative. Thus the claim holds by (iii) of (3.5). □

3.7. Proposition. *If L is paracompact and regular, then L/θ is paracompact whenever θ has a form of $\bigwedge\{\nabla_{t_n} \mid n = 1, 2, \cdots\}$.*

Proof: Let $q : L \longrightarrow L/\theta$ be the quotient map. For each cover A of L/θ, we can find a subset S of L such that $q[S] = A$, which implies

$$(\bigvee S, e) \in \theta = \bigwedge\{\nabla_{t_n} \mid n = 1, 2, \cdots\},$$

that is, $\bigvee S \vee t_n = e$ for $n = 1, 2, \cdots$. Then for each n there is a locally finite refinement C_n of $S \cup \{t_n\}$. Put

$$B_n = \{x \in C_n \mid x \nleq t_n\},$$

then $\bigvee B_n \vee t_n = e$. Thus $(\bigvee(\bigcup_{n=1}^{\infty} B_n), e) \in \theta$, which means $\bigcup_{n=1}^{\infty} q[B_n]$ is a cover of L/θ. Moreover each $q[B_n]$ is locally finite and $\bigcup_{n=1}^{\infty} q[B_n] \leq q[S] = A$. Hence A has a σ-locally finite refinement. By (3.5), we conclude that L/θ is paracompact. □

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