Non-compact perturbations of m-accretive operators in general Banach spaces

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Abstract. In this paper we deal with the Cauchy problem for differential inclusions governed by m-accretive operators in general Banach spaces. We are interested in finding the sufficient conditions for the existence of integral solutions of the problem $x'(t) \in -Ax(t) + f(t, x(t))$, $x(0) = x_0$, where A is an m-accretive operator, and f is a continuous, but non-compact perturbation, satisfying some additional conditions.

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1. Introduction.

The main goal of the present paper is to prove a local existence result for a class of nonlinear evolution equations of the form

$$(1) \qquad \begin{cases} x'(t) \in -Ax(t) + f(t, x(t)) \\ x(0) = x_0 \end{cases} , \; t \in [0, T],$$

where A is an m-accretive operator acting on a real Banach space E and f is a continuous function satisfying some additional conditions.

This problem has been intensively studied over the past several years mainly because of a great practical interest, for example in the synthesis of the optimal control, differential games and population dynamic (cf. [11], [13] and the references therein). The case, when $-A$ generates a compact semigroup is well known (see [4], [9], [11], [13]), for example, if E is finite dimensional, then each m-accretive operator is such that $-A$ generates a compact semigroup (hence equicontinuous as well, cf. [3], [5]). However, there exists a lot of m-accretive operators for which $-A$ generates equicontinuous, but not compact semigroups ([13]). In this case, the authors of many papers ([8], [9], [12], [13]) considered compact perturbations of m-accretive operators.

Our purpose is to generalize the last concept. The perturbations are not compact, but so-called k-set contractions. It is well known that this is a very large class of mappings (see [1], [10] for instance). Moreover, for a recent account of this theory we refer the reader to [9] and [13].

2. Main result.

Throughout this paper we will denote by E a real Banach space with the norm $\|\cdot\|$. Let $I := [0,T] \subset \mathbb{R}_+$, and let $L^1(I,E)$ denote the space of all integrable functions from I to E with the standard norm $\|\cdot\|_1$. Moreover by $(C(I,E), \|\cdot\|_c)$ we will denote a space of all continuous functions from I to E.

We begin with a definition that we need in the statement of the main result.

Definition 1. *An operator $A : D(A) \subset E \to 2^E$ is called accretive if $[x-\tilde{x}, y-\tilde{y}]_+ \geq 0$ for each $x, \tilde{x} \in D(A)$, $y \in Ax$ and $\tilde{y} \in A\tilde{x}$. If, in addition, the range of $Id + tA$ is the whole E (for each $t > 0$), then A is called m-accretive.*

Here $[u,v]_+$ denotes the normalized upper semi-inner product on E, i.e. $[u,v]_+ := \lim_{h \searrow 0} \frac{1}{h}(\|u+hv\| - \|u\|)$ (see [2], [10], [13]). Let $\{S(t) : S(t) : \overline{D(A)} \to \overline{D(A)}, t > 0\}$ be the semigroup of nonexpansive mappings generated by $-A$ on $\overline{D(A)}$ via the formula of Crandall and Liggett ([2, Theorem 1.3], [13, Theorem 1.8.8]). This semigroup is called compact if $S(t)$ is a compact operator for each $t > 0$, and it is called equicontinuous if for each bounded subset M of $\overline{D(A)}$, the family of functions $\{S(\cdot)x : x \in M\}$ is equicontinuous at each $t > 0$ (see [9], [13]). It is well known that if a semigroup of nonexpansive mappings on $\overline{D(A)}$ is compact, then it is equicontinuous (see [13, Theorem 2.2.1]). For the examples, we refer the reader to [13].

We omit the definition of an integral solution of our problem, because it is well known (see [2], [11], [13] for instance).

The next result due to Bénilan is one of the main ingredients in the proof of our main theorem.

Proposition 1 ([2, Theorem 2.1], [13, Corollary 1.7.1]). *If $A : D(A) \to 2^E$ is m-accretive operator, then for each $(x_0, f) \in \overline{D(A)} \times L^1(I,E)$ the following problem*

$$(1') \qquad \begin{cases} x'(t) \in -Ax(t) + f(t) \\ x(0) = x_0 \end{cases}, \quad t \in I,$$

has a unique integral solution $H(f,x_0) : \overline{D(A)} \to \overline{D(A)}$, such that if $H(g,y_0)$ is an integral solution to $(1')$ corresponding to (y_0, g), then

$$\|H(f,x_0)(t) - H(g,y_0)(t)\| \leq \\ \leq \|H(f,x_0)(s) - H(g,y_0)(s)\| + \int_s^t \|f(u) - g(u)\|\, du$$

for each $0 \leq s \leq t \leq T$.

This theorem exhibits the Lipschitz-continuous dependence of integral solutions of $(1')$ on the data. For abbreviation, we will write $H(f)$ instead of $H(f,x_0)$.

And now, we can recall the next important theorem.

Proposition 2 ([8], [13, Theorem 2.5.1]). *Let $A : D(A) \to 2^E$ be an m-accretive operator so that $-A$ generates an equicontinuous semigroup, and let $x_0 \in \overline{D(A)}$. Then for each uniformly integrable subset K in $L^1(I, E)$ the set $H(K) := \{H(k) : k \in K\}$ is bounded and equicontinuous on I.*

For completeness, we must recall the definition of Kuratowski measure of noncompactness α [Hausdorff mnc β].

Definition 2. *Let B be a bounded subset of E. Then*:

$$\alpha(B) = \inf\{\varepsilon > 0 : B \subset \bigcup_{i=1}^{n(\varepsilon)} M_i^\varepsilon \ \ \textit{for some} \ \ M_i^\varepsilon \subset E,\ i = 1, \dots, n(\varepsilon),\ \textit{with} \ \ \operatorname{diam}(M_i^\varepsilon) \le \varepsilon\}$$

and

$$\beta(B) = \inf\{\varepsilon > 0 : B \subset \{x_1^\varepsilon, \dots, x_{n(\varepsilon)}^\varepsilon\} + \varepsilon \cdot B^0 \ \ \textit{for some} \ \ x_i^\varepsilon \in E,\ i = 1, \dots, n(\varepsilon)\}.$$

For the properties of these measures we refer the reader to [1] and [10]. For example, if F is a subspace of E and W is a bounded subset of F, then

$$\beta(W) \le \beta^F(W) \le \alpha(W) \le 2\beta(W),$$

where β^F denotes the Hausdorff mnc in F. Furthermore, we have the following proposition.

Proposition 3 (Ambrosetti's lemma, [1, Theorem 11.3]). *If M is bounded and equicontinuous subset of $C(I, E)$, then*

$$\alpha_c(M) = \sup\{\alpha(M(t)) : t \in I\},$$

where α_c is the Kuratowski measure of noncompactness in $C(I, E)$.

Another main ingredient in the proof of our existence result is the following fixed point theorem due to Sadovskii.

Proposition 4 ([1, Theorem 5.1], [6], [10, Theorem 3.2]). *Let C denote a nonempty, convex, closed and bounded subset of a Banach space X. Let $F : C \to C$, and assume that there exists $k < 1$, that $\mu(F(W)) \le k \cdot \mu(W)$, for each bounded subset W of X, where μ denotes an arbitrary measure of noncompactness in X. Then the set of all fixed points of F is nonempty and compact.*

We will use the following lemma.

Lemma 1. *Let $A : D(A) \to 2^E$ be an m-accretive operator and let $H(g)$ denote a (unique) integral solution of*

$$(2) \qquad \begin{cases} x'(t) \in -Ax(t) + g(t) \\ x(0) = x_0 \end{cases}, \; t \in I, \; g \in L^1(I,E).$$

Then for each bounded subset W of $L^1(I,E)$ we have

$$\beta_c(H(w)) \leq \beta_1(W),$$

where β_c, β_1 denote Hausdorff measure of noncompactness in $C(I,E)$, and $L^1(I,E)$, respectively.

Proof: Let $g \in H(W)$, so there exists $v \in W$ that $g = H(v)$. Fix arbitrary $\varepsilon > 0$. Let $\{x_1, \dots, x_n\}$ be a finite $(\beta_1(W)+\varepsilon)$-net in W. Then there exists a number k, $1 \leq k \leq n$, such that $\|v - x_k\|_1 \leq \beta_1(W) + \varepsilon$. Let $t \in I$. We have

$$\begin{aligned} \|g(t) - H(x_k)(t)\| &\leq \int_0^t \|v(s) - x_k(s)\|\, ds \\ &\leq \int_0^T \|v(s) - x_k(s)\|\, ds \\ &= \|v - x_k\|_1 \leq \beta_1(W) + \varepsilon. \end{aligned}$$

Therefore

$$\begin{aligned} \|g(t) - H(x_k)(t)\| &\leq \beta_1(W) + \varepsilon, \\ \sup\{\|g(t) - H(x_k)(t) : t \in I\|\} &\leq \beta_1(W) + \varepsilon, \\ \|g - H(x_k)\|_c &\leq \beta_1(W) + \varepsilon, \end{aligned}$$

and we see that $\{H(x_1), \dots, H(x_k)\}$ is a $(\beta_1(W)+\varepsilon)$-net in $H(W)$, so $\beta_c(H(W)) \leq \beta_1(W) + \varepsilon$. But $\varepsilon > 0$ is arbitrary, and finally

$$\beta_c(H(W)) \leq \beta_1(W).$$

□

Now, we are able to state the main result in this paper.

Theorem 1. *Assume that:*

(A1) *$A : D(A) \to 2^E$ is an m-accretive operator which generates an equicontinuous semigroup,*

(A2) *$f : I \times \overline{D(A)} \to E$ is a locally uniformly continuous function, such that*

  (i) *for each bounded subset W of E, there exists $M > 0$, that $\sup\{\|f(t,x)\| : x \in W\} \leq M$ for each $t \in I$,*

  (ii) *$\alpha(f(t,W)) \leq k \cdot \alpha(W)$, $k \in [0, 1/(2 \cdot T))$, where α denotes the Kuratowski measure of noncompactness in E, and W is an arbitrary bounded subset of E.*

Under the above assumptions for each $x_0 \in \overline{D(A)}$ *there exists* $T_0 = T(x_0) \in (0, T]$ *such that the problem* (1) *has at least one integral solution on* $[0, T_0]$.

PROOF: Let $x_0 \in \overline{D(A)}$. Fix $r > 0$, choose $M > 0$ and $T_0 \in (0, T]$ such that

$$\sup\{\|f(t,x)\| : x \in B(x_0, r)\} \leq M \quad \text{on} \;\; J := [0, T_0], \tag{3}$$

and

$$\|H(0)(t) - x_0\| + T_0 M \leq r \quad \text{for each} \;\; t \in J. \tag{4}$$

We see that it is possible because, in view of (A2) (i), there exists such a number M satisfying (3) on I as well. In addition $\|H(0)(t) - x_0\| \to 0$, when $t \to 0_+$, so we may choose T_0 satisfying (3) and (4).

Next, let us define $P := \{x \in L^1(J, E) : \|x(t)\| \leq M \text{ a.e. on } J\}$, and it is clear that this set is uniformly integrable in $L^1(J, E)$. Moreover, we denote by Q the following set $Q := H(P) = \{H(x) : x \in P\}$. By Proposition 2, this set is bounded and equicontinuous in $C(J, E)$. Consequently, for $t \in J$ and $x \in P$, we have

$$\begin{aligned}\|H(x)(t) - x_0\| &\leq \|H(x)(t) - H(0)(t)\| + \|H(0)(t)x_0\| \\ &\leq \|H(0)(t) - x_0\| + \int_0^t \|x(s)\|\, ds \\ &\leq \|H(0)(t) - x_0\| + \int_0^t \|h(s)\|\, ds,\end{aligned}$$

and by (4)

$$H(x)(t) \in B(x_0, r).$$

Hence, for every $w \in Q$

$$\|w(t) - x_0\| \leq r. \tag{5}$$

Set $K_0 := \{y \in C(J, E) : y(\cdot) = f(\cdot, u(\cdot)),\ u \in Q\}$. If $y \in K_0$ then by (3) and (5) $\|y(t)\| \leq M$ for each $t \in J$. From the uniform continuity of f, the set K_0 is equicontinuous in $C(J, E)$. However, the set $K := \overline{\operatorname{conv}} K_0$ is nonempty, closed, convex, bounded and equicontinuous in $C(J, E)$. Indeed, the set P is convex and closed, by (3) and (5) $K_0 \subset P$, and we see that $K \subset P$.

Thus, we can define an operator $F : K \to C(J, E)$ as follows

$$F(u)(t) = f(t, H(u)(t)),\ t \in J,\ u \in K.$$

In addition, if $v \in K$, then $F(v)(t) = f(t, H(v)(t))$, $t \in J$, and $K \subset P$, so $H(v) \in Q$, and consequently $F(v) \in K_0 \subset K$. In conclusion, $F(K) \subset K$.

Furthermore F is continuous as a superposition of two continuous functions $f(\cdot, \cdot)$ and $H(\cdot)$.

Let W be a bounded subset of K, and $t \in J$. Hence by (A2) (ii), $\alpha(F(w)(t)) = \alpha(f(t, H(W)(t))) \leq k \cdot \alpha(H(w)(t))$. But $H(W) \subset Q$, and by Proposition 3 and Lemma 1 we have that $\alpha(F(W)(t)) \leq k \cdot \alpha_c(H(w)) \leq 2k \cdot \beta_c(H(W)) \leq 2k \cdot \beta_1(W)$. The set W, as a subset of K, is equicontinuous, and so

$$\alpha_c(F(W)) \leq 2k \cdot \beta_1(W) \leq 2k \cdot \beta_1^{C(J,E)}(W).$$

Denote by B_c^0 and B_1^0 the unit balls with the norms $\|\cdot\|_c$ and $\|\cdot\|_1$, respectively. Since $\|\cdot\|_1 \leq T \cdot \|\cdot\|_c$, then we see that for each fixed $\varepsilon > 0$ there exists a finite set $\{u_1, \ldots, u_m\} \subset C(J,E)$, that for a bounded set W in E $W \subset \{u_1, \ldots, u_m\} + (\beta_c(W) + \varepsilon) \cdot B_c^0 \subset \{u_1, \ldots, u_m\} + (\beta_c(W) + \varepsilon) \cdot T \cdot B_c^0$ and $\beta_1^{C(J,E)}(W) \leq (\beta_c(W) + \varepsilon) \cdot T$. Finally, $\beta_1^{C(J,e)}(W) \leq T \cdot \beta_c(W) \leq T \cdot \alpha_c(W)$.

Now, we can write that

$$\alpha_c(F(W)) \leq 2k \cdot T \cdot \alpha_c(W),$$

and since $2k \cdot T \leq 1$, then f satisfies all the assumptions of Proposition 4. Finally, there exists a fixed point theorem of F, i.e. $w_0 \in K$, such that

$$F(w_0) = w_0.$$

Equivalently, w_0 is an integral solution of (1) on J. □

Theorem 2. *The set of all integral solutions of the problem* (1) *on J is nonempty and compact.*

This is an immediate consequence of our Theorem 1 and Theorem 5.1 of [1].

The class of all functions satisfying the condition (A2) (ii) is very large (see [10] for instance). However, it is well known that if E is a finite dimensional, then each m-accretive operator is such that $-A$ generates a compact semigroup, so we can use the previous results ([4], [9], [11]). But the case of infinite dimensional Banach space is more delicate (cf. [9]). For example, the operator $Ax \equiv 0$ generates a semigroup $S(t) \equiv Id$, $t \geq 0$, which is equicontinuous, but not compact. Thus, this is one of the special cases of our theorem (see [10]). The applications of the results of this type in PDE's are due to Vrabie [13] for instance.

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