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$\mathcal{T}_0$- and $\mathcal{T}_1$-reflections

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Abstract. In an abstract category with suitable notions of subobject, closure and point, we discuss the separation axioms T_0 and T_1. Each of the arising subcategories is reflective. We give an iterative construction of the reflectors and present characteristic examples.

Keywords: factorization system, closure operator, separation axioms, prereflection, reflection

Classification: 18A40, 18B30, 54A05, 54D10

Introduction.

Given a category $\mathcal{X}$ and a fixed 'well behaved' class of monomorphisms $\mathcal{M}$, we distinguish some $\mathcal{M}$-morphisms and we think of them as points. Moreover, if a closure operator C, with respect to $\mathcal{M}$, is defined on $\mathcal{X}$ (in the sense of [6]), there is a natural way of considering separation axioms T_0 and T_1.

The objects satisfying each of these separation axioms, as well as the (full) subcategories they define, have interesting properties. We are mainly interested in those concerned with reflections. Actually, under mild conditions on $\mathcal{X}$, these subcategories are extremal epireflective. When it is the case, their reflections can be obtained in a natural way by a transfinite process. The idea comes from an easy construction of the $\mathcal{T}op_0$-reflection and suggests a more detailed study of the $\mathcal{T}op_1$-reflection. Indeed, one can see that, with respect to this process, the reflection in $\mathcal{T}op_1$ is far from being as easily obtained as the $\mathcal{T}op_0$-reflection. Furthermore, in this general setting, the process of constructing the $\mathcal{T}_0$-reflection can be as complex as in the case of $\mathcal{T}op_1$.

Section 1 presents the basic concepts that are used throughout the paper.

In Section 2 we give the notions of T_0- and T_1-object and we analyse the immediate consequences for the subcategories $\mathcal{T}_0$ and $\mathcal{T}_1$ of T_0- and T_1-objects, respectively.

The properties of $\mathcal{T}_0$ and $\mathcal{T}_1$ described in 2.2 lead us to a first attempt of obtaining the $\mathcal{T}_0$- and $\mathcal{T}_1$-reflections. Although we only get, at a first stage, prereflections, which, in general, are not the $\mathcal{T}_0$- and $\mathcal{T}_1$-reflections, they direct us to a transfinite process of defining the requested reflections. This is studied in Section 3.

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In Section 4 we study examples that clarify some situations. Example 1 illustrates the behaviour of the $\mathcal{T}op_1$-reflection in the sense that it shows that the $\mathcal{T}op_1$-reflection is not obtained at any fixed step of the iteration of the corresponding prereflection. For an arbitrary $\mathcal{T}_0$-reflection, the same conclusion follows from the remaining examples.

In the last section we briefly discuss connections between the notions of T_0- and T_1-objects we introduce and known concepts of separation.

1. Preliminaries.

Throughout we consider a category $\mathcal{X}$ and a class $\mathcal{M}$ of morphisms of $\mathcal{X}$ containing all isomorphisms and closed under composition. Moreover, we assume that $\mathcal{X}$ is $\mathcal{M}$-complete, that is, pullbacks of $\mathcal{M}$-morphisms along arbitrary morphisms and multiple pullbacks of (possibly large) families of $\mathcal{M}$-morphisms exist and belong to $\mathcal{M}$.

Then we have that (cf. [18] in the dual situation):

- $\mathcal{M}$ is a class of monomorphisms of $\mathcal{X}$.
- There exists a class $\mathcal{E}$ of $\mathcal{X}$-morphisms such that $(\mathcal{E},\mathcal{M})$ is a factorization system in $\mathcal{X}$, that is, every $\mathcal{X}$-morphism has an $(\mathcal{E},\mathcal{M})$-factorization, and, for each commutative diagram

$$\begin{tikzcd} \cdot \arrow[r, "e"] \arrow[d, "u"'] & \cdot \arrow[d, "v"] \\ \cdot \arrow[r, "m"'] & \cdot \end{tikzcd}$$

  where $e \in \mathcal{E}$ and $m \in \mathcal{M}$, there exists a unique morphism d such that $m \cdot d = v$ and $d \cdot e = u$.
- Given an $\mathcal{X}$-object X and $\mathcal{M}$-morphisms m and n with codomain X, we shall say that $m \leq n$ if there exists a morphism m_n such that $n \cdot m_n = m$. The comma category $\mathcal{M}/_X$ of $\mathcal{M}$-morphisms with codomain X equipped with this preorder is a complete class.
- For each morphism $f : X \to Y$, there exist functors $f^{-1}(-) : \mathcal{M}/_Y \to \mathcal{M}/_X$ and $f(-) : \mathcal{M}/_X \to \mathcal{M}/_Y$ given by pullback and $(\mathcal{E},\mathcal{M})$-factorization, respectively, $f(-)$ being left adjoint to $f^{-1}(-)$.

We also consider a closure operator C on $\mathcal{X}$, with respect to $\mathcal{M}$, in the sense of [6]. We recall that, in order to define C, one only needs to give, for each $\mathcal{X}$-object X, a functor $c_X : \mathcal{M}/_X \to \mathcal{M}/_X$ such that, for each $m, n \in \mathcal{M}/_X$ and $f \in \mathcal{X}(X,Y)$, $m \leq c_X(m)$ and $f(c_X(m)) \leq c_Y(f(m))$.

The closure operator C is said to be *idempotent* if, for each $m \in \mathcal{M}/_X$, $c_X(m) \cong c_X(c_X(m))$ and *additive* if, for each $m, n \in \mathcal{M}/_X$, $c_X(m \vee n) \cong c_X(m) \vee c_X(n)$.

We shall denote $c_X(m)$ by $[m]_X$, or simply by $[m]$, when its meaning is clear from the context.

More on closure operators can be found on [6] and [7].

Throughout all subcategories are assumed to be full and replete (i.e. closed under isomorphisms).

2. $\mathcal{T}_0$- and $\mathcal{T}_1$-objects.

From now on we consider a fixed $\mathcal{X}$-object P such that any two parallel $\mathcal{E}$-morphisms with domain P are equal.

We shall denote by $\mathcal{P}$ the class of $\mathcal{X}$-objects which are codomains of $\mathcal{E}$-morphisms with domain P, that is,

$$\mathcal{P} := \{Q \in \operatorname{Ob}\mathcal{X} \mid \text{ there exists } e : P \to Q \text{ in } \mathcal{E}\}.$$

Moreover, for each $\mathcal{X}$-object X, the class of $\mathcal{M}/_X$-morphisms with domain in $\mathcal{P}$ will be denoted by $\mathcal{P}_X$. In our approach these morphisms are thought of as points of X. A detailed study of the behaviour of these 'points' is presented in [5].

Definition 2.1. (a) An $\mathcal{X}$-object X is said to be a $\mathcal{T}_0$-*object* if, for each pair x, y of $\mathcal{P}_X$-morphisms, x and y are isomorphic whenever $x \leq [y]$ and $y \leq [x]$.

(b) An $\mathcal{X}$-object X is said to be a $\mathcal{T}_1$-*object* if, for each pair x, y of $\mathcal{P}_X$-morphisms, x and y are isomorphic whenever $x \leq [y]$.

We shall denote by $\mathcal{T}_0$ (resp. $\mathcal{T}_1$) the subcategory of $\mathcal{X}$ whose objects are the $\mathcal{T}_0$-objects (resp. $\mathcal{T}_1$-objects).

Proposition 2.2. (a) *If $f : X \to Y$ is an $\mathcal{X}$-morphism and Y belongs to $\mathcal{T}_0$, then, for each pair x, y of $\mathcal{P}_X$-morphisms, $x \leq [y]$ and $y \leq [x]$ implies that $f(x) \cong f(y)$.*

(b) *If $f : X \to Y$ is an $\mathcal{X}$-morphism and Y belongs to $\mathcal{T}_1$, then, for each pair x, y of $\mathcal{P}_X$-morphisms, $x \leq [y]$ implies that $f(x) \cong f(y)$.*

PROOF: For $f \in \mathcal{X}(X, Y)$ and $m, n \in \mathcal{M}/_X$, we always have that, if $m \leq [n]$, then $f(m) \leq f([n]) \leq [f(n)]$, by definition of closure operator. The assertions (a) and (b) follow easily from this fact and the definitions of $\mathcal{T}_0$- and $\mathcal{T}_1$-object, respectively. □

These results lead us to the following

Theorem 2.3. *$\mathcal{T}_0$ and $\mathcal{T}_1$ are closed under monosources.*

PROOF: Let $(f_i : X \to X_i)_I$ be a monosource, where X_i is a $\mathcal{T}_0$-object, for each $i \in I$, and let $x : Q \to X$ and $y : R \to X$ be $\mathcal{P}_X$-morphisms such that $x \leq [y]$ and $y \leq [x]$. From 2.2 (a) it follows that $f_i(x) \cong f_i(y)$ (i.e. there exists an isomorphism h_i such that $f_i(y) \cdot h_i = f_i(x)$), for each $i \in I$. Let e and a be the $\mathcal{E}$-morphisms from P to Q and R, respectively. If $f_i(x) \cdot e_i$ and $f_i(y) \cdot a_i$ are the $(\mathcal{E}, \mathcal{M})$-factorizations of $f_i \cdot x$ and $f_i \cdot y$, respectively,

$$\begin{tikzcd}
& Q \arrow[rr, "e_i"] \arrow[dr, "x"] & & Q_i \arrow[dd, "h_i"] \arrow[dr, "f_i(x)"] & \\
P \arrow[ur, "e"] \arrow[dr, "a"'] & & X \arrow[rr, "f_i"] & & X_i \\
& R \arrow[ur, "y"'] \arrow[rr, "a_i"'] & & R_i \arrow[ur, "f_i(y)"'] &
\end{tikzcd}$$

then, since $h_i \cdot e_i \cdot e$ and $a_i \cdot a$ are $\mathcal{E}$-morphisms from P to R_i, $h_i \cdot e_i \cdot e = a_i \cdot a$, for each $i \in I$, by our assumption on P. Hence, $f_i \cdot x \cdot e = f_i \cdot y \cdot a$, for each $i \in I$, which implies that $x \cdot e = y \cdot a$. From this equality it follows that $x \cong y$, therefore, X belongs to $\mathcal{T}_0$.

Similarly for $\mathcal{T}_1$. □

3. Construction of the $\mathcal{T}_0$- and $\mathcal{T}_1$-reflection.

A *prereflection* on $\mathcal{A}$ consists of an endofunctor $T : \mathcal{A} \to \mathcal{A}$ and a natural transformation $\eta : \mathrm{Id}_{\mathcal{A}} \to T$ such that, for each $\mathcal{A}$-morphism $f : X \to Y$, $Tf : TX \to TY$, is the only morphism rendering the diagram

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "\eta_X"'] & Y \arrow[d, "\eta_Y"] \\
TX \arrow[r, "Tf"'] & TY
\end{tikzcd}$$

commutative (cf. [2], [18] and [20]).

A prereflection (T, η) is called a *reflection* whenever η_T is pointwise an isomorphism.

We shall denote by $\mathcal{F}ix(T, \eta)$ the subcategory of $\mathcal{A}$ whose objects are the $\mathcal{A}$-objects A such that η_A is an isomorphism.

Proposition 3.1. *For a prereflection (T, η), the following assertions are equivalent:*

(i) *(T, η) is a reflection.*

(ii) *$\mathcal{F}ix(T, \eta)$ is reflective, with reflections $(\eta_A : A \to TA)_{A \in \mathrm{Ob}\,\mathcal{A}}$.*

Proof: Cf. [18, Proposition 4.2]. □

From now on we assume that $\mathcal{X}$ has coequalizers and multiple pushouts of (possibly large) families of regular epimorphisms.

For each $\mathcal{X}$-object X, consider all pairs $x : Q \to X$, $y : R \to X$ of $\mathcal{P}_X$-morphisms such that $x \le [y]$ and $y \le [x]$, the $\mathcal{E}$-morphisms $e : P \to Q$ and $a : P \to R$, and the coequalizer $c_{(x,y)}$ of $x \cdot e$ and $y \cdot a$, and form the multiple pushout of $(c_{(x,y)})$, $\zeta_X : X \to SX$.

Proposition 3.2. $(\zeta_X : X \to SX)_{X \in \mathrm{Ob}\,\mathcal{X}}$ *defines a prereflection* (S, ζ) *such that* $\mathcal{F}ix(S,\zeta) = \mathcal{T}_0$.

Proof: First we shall define the endofunctor $S : \mathcal{X} \to \mathcal{X}$. Let $f : X \to Y$ be an $\mathcal{X}$-morphism and $x : Q \to X$ and $y : R \to X$ be $\mathcal{P}_X$-morphisms with $x \le [y]$ and $y \le [x]$. Consider the following diagram

$$\begin{tikzcd}
 & Q \arrow[dr, "x"] \arrow[rr, "e'"] & & Q' \arrow[dr, "f(x)"] & \\
P \arrow[ur, "e"] \arrow[dr, "a"'] & & X \arrow[rr, "f"] \arrow[dd, "\zeta_X"'] & & Y \arrow[dd, "\zeta_Y"] \\
 & R \arrow[ur, "y"'] \arrow[rr, "a'"] & & R' \arrow[ur, "f(y)"'] & \\
 & & SX & & SY
\end{tikzcd}$$

with e, a, e' and a' in $\mathcal{E}$. Since $f(x) \le [f(y)]$ and $f(y) \le [f(x)]$, by definition of ζ_Y we have that $\zeta_Y \cdot f(x) \cdot e' \cdot e = \zeta_Y \cdot f(y) \cdot a' \cdot a$, that is, $\zeta_Y \cdot f \cdot x \cdot e = \zeta_Y \cdot f \cdot y \cdot a$. By definition of ζ_X it follows that there exists a morphism $Sf : SX \to SY$ such that $Sf \cdot \zeta_X = \zeta_Y \cdot f$. This morphism is unique, since ζ_X is an epimorphism.

It is easily verified that we have defined an endofunctor $S : \mathcal{X} \to \mathcal{X}$. From the way we defined S it follows immediately that (S, ζ) is a prereflection.

So, it remains to prove that $\mathcal{F}ix(S,\zeta) = \mathcal{T}_0$. If ζ_X is an isomorphism, then, for each $x, y \in \mathcal{P}_X$ with $x \le [y]$ and $y \le [x]$, we have that $c_{(x,y)}$ is an isomorphism, hence, $x \cdot e = y \cdot a$, with e and a the $\mathcal{E}$-morphisms from P to the domains of x and y, respectively. Hence, $x \cong y$, and then it follows that X is a $\mathcal{T}_0$-object. Conversely, if X belongs to $\mathcal{T}_0$, then, for each pair x, y of $\mathcal{P}_X$-morphisms such that $x \le [y]$ and $y \le [x]$, we have that $x \cong y$, and this implies that $x \cdot e = y \cdot a$ (e and a as above). Therefore $c_{(x,y)}$ is an isomorphism, and then ζ_X is an isomorphism too, that is, X belongs to $\mathcal{F}ix(S,\zeta)$. $\square$

As we shall see in Section 4, (S, ζ) is not always a reflection. However, under mild conditions on $\mathcal{X}$, (S,ζ) enables us to construct the $\mathcal{T}_0$-reflection, using a natural transfinite construction given in [19], and which we describe below.

By a transfinite construction over the class Ord of ordinals, we can define prereflections $(S^\alpha, \zeta^\alpha)_{\alpha\in\mathrm{Ord}}$ as follows:

(i) $S^0 = \mathrm{Id}_{\mathcal{X}}$ and $\zeta^0 = \mathrm{Id}$;
(ii) if $\beta = \alpha + 1$, then $S^\beta = S \cdot S^\alpha$ and $\zeta^\beta = \zeta_{S^\alpha} \cdot \zeta^\alpha$;
(iii) if β is a limit ordinal, then S^β is the colimit of $(S^\alpha)_{\alpha<\beta}$ and ζ^β is the natural transformation induced by $(\zeta^\alpha)_{\alpha<\beta}$.

If $\mathcal{X}$ is weakly cowellpowered (i.e. cowellpowered with respect to strong epimorphisms), then, for each $\mathcal{X}$-object X, there exists a least ordinal α_X such that $\zeta_{S^{\alpha_X} X}$ is an isomorphism, that is, $S^{\alpha_X}X$ is a $\mathcal{T}_0$-object.

Theorem 3.3. *Let $\mathcal{X}$ be weakly cowellpowered. The reflection (S^∞, ζ^∞) defined by*

$$S^\infty X := S^{\alpha_X} X \quad \text{and} \quad \zeta_X^\infty := \zeta_X^{\alpha_X},$$

for each $\mathcal{X}$-object X, is the $\mathcal{T}_0$-reflection.

PROOF: $\mathcal{F}ix(S^\infty, \zeta^\infty) = \mathcal{T}_0$ follows from 3.2. Using 2.2 (a) and the definition of (S^∞, ζ^∞), it is easily seen that, for each $\mathcal{X}$-object X, $\zeta_X^\infty : X \to T^\infty X$ is its $\mathcal{T}_0$-reflection. □

In an analogous way we can also construct the $\mathcal{T}_1$-reflection. In order to do that, for each $\mathcal{X}$-object X, we consider all pairs $x : Q \to X$, $y : R \to X$ of $\mathcal{P}_X$-morphisms such that $x \le [y]$, the $\mathcal{E}$-morphisms $e : P \to Q$ and $a : P \to R$, the coequalizer of $x \cdot e$ and $y \cdot a$, $c_{(x,y)}$, and the multiple pushout of $(c_{(x,y)})$, $\eta_X : X \to TX$.

Proposition 3.4. $(\eta_X : X \to TX)_{X\in\mathrm{Ob}\,\mathcal{X}}$ *defines a prereflection (T, η) such that $\mathcal{F}ix(T, \eta) = \mathcal{T}_1$.*

PROOF: It is analogous to the proof of 3.2. □

By transfinite construction over the class of ordinals, we get prereflections $(T^\alpha, \eta^\alpha)_{\alpha\in\mathrm{Ord}}$.

Furthermore, if $\mathcal{X}$ is weakly cowellpowered, then, for each $\mathcal{X}$-object X, there exists a least ordinal λ_X such that $\eta_{T^{\lambda_X} X}$ is an isomorphism.

Theorem 3.5. *Let $\mathcal{X}$ be weakly cowellpowered. The reflection (T^∞, η^∞) defined by*

$$T^\infty X := T^{\lambda_X} X \quad \text{and} \quad \eta_X^\infty := \eta_X^{\lambda_X},$$

for each $\mathcal{X}$-object X, is the $\mathcal{T}_1$-reflection.

PROOF: It is proved analogously to 3.3. □

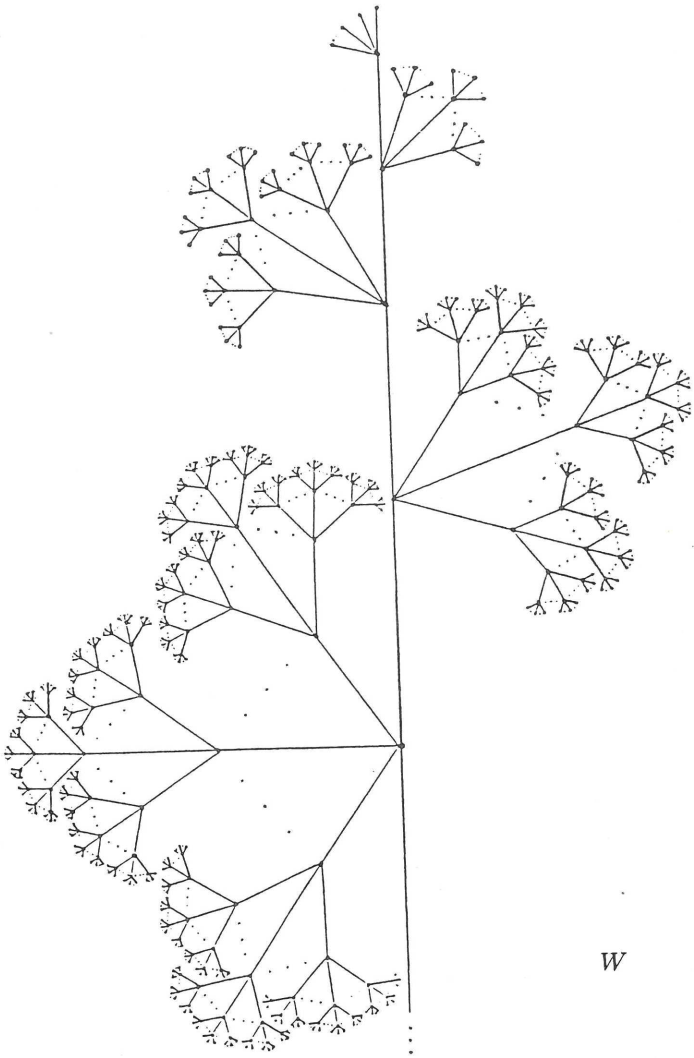
4. Examples.

In all the categories considered in the following examples, the way we define $\mathcal{M}$ and $\mathcal{P}$ allows us to identify the $\mathcal{M}/_X$-morphisms and the $\mathcal{P}_X$-morphisms with the subsets and the elements of the underlying set, respectively. In order to simplify notations, we will sometimes identify them, as well as the $\mathcal{X}$-objects and their underlying sets.

Example 1. Let $\mathcal{X}$ be the category $\mathcal{T}op$ of topological spaces, $\mathcal{M}$ the class of embeddings, C the usual closure operator in $\mathcal{T}op$ and P the terminal object (i.e. the one-element topological space).

It is easily seen that, in this case, the prereflection (S, ζ) defined in the last section is already a reflection, that is, the $\mathcal{T}op_0$-reflection is obtained at the first step of our iteration of ζ.

With the $\mathcal{T}op_1$-reflection, the situation is completely different. Indeed, next we define, for each ordinal α, a topological space for which the $\mathcal{T}op_1$-reflection is reached exactly at the α-step of the iteration of η. Previously we present a sketch of such a space in the case of α being the first limit ordinal, ω.



W

It is possible to define a topology on W such that, in the resulting topological space, η identifies, in each step, the extreme knots with their closest neighbours, until we obtain, at the ω-iteration, the $\mathcal{T}op_1$-reflection of W: the one-element space. This is what we present below for any fixed ordinal α.

In order to define such a space we first introduce some auxiliary definitions and notations.

We consider $\alpha+1$ with the order topology, i.e. for each $\beta\in\alpha+1$, a subset U of $\alpha+1$ is a neighbourhood of β if there exists $\beta_1\leq\beta$ such that $(\beta_1,\beta]\subseteq U$, or, in the case of β being 0, if $\beta\in U$.

For each $\beta\in\alpha+1$ we consider

$$\mathcal{V}^0_\beta:=\{V\subseteq\alpha+1\mid V=U\setminus\{\beta\}, U \text{ being a neighbourhood of } \beta\}.$$

Let E be an infinite countable set and $*$ a fixed element of E. Let F be the set of all maps f with codomain E and domain a final section of α, that is, $\mathrm{dom}(f)=\alpha\backslash\beta$, with $\beta\in\alpha+1$. This ordinal β will be denoted by L_f. Given $f,g\in F$, we shall say that $f\preceq g$ if $L_f\leq L_g$ and the restriction of f to the domain of g coincides with g. Moreover, we shall denote by f' the element of F that 'follows' f, that is, $f'=f_{|\alpha\backslash(L_f+1)}$.

Let $G:=\{f\in F\mid \text{for each } \gamma\in\mathrm{dom}(f) \text{ there exists } V\in\mathcal{V}^0_\gamma \text{ such that } f_{|V}\equiv *\}$ (by $f_{|V}\equiv *$ we mean that, for each $\gamma\in\mathrm{dom}(f)$, $f(\gamma)=*$).

Now, let X be the topological space whose underlying set is $\{f\in G\mid L_f \text{ is not a limit ordinal}\}$, and, for each $M\subseteq X$, the closure of M, $\overline{M}$, is the least subset of X containing M and satisfying the following conditions, for $f\in X$:

(i) if $L_f=0$ and $f\in M$, then $f'\in\overline{M}$;

(ii) if $L_f=\gamma+1$ $(\gamma>0)$ and there exists $h:\alpha\setminus\gamma\to E$ such that $h'=f$ and, for each $\beta<\gamma$, $\{g\in M\mid g\preceq h \text{ and } L_g\geq\beta\}$ is infinite, then $f\in\overline{M}$, and $h\in\overline{M}$ whenever $h\in X$ (i.e. L_h is not a limit ordinal).

Hence, given $f\in X$, $\{f\}$ is closed if $L_f\neq 0$ and $\overline{\{f\}}=\{f,f'\}$ if $L_f=0$. Furthermore, it is easy to see that

$$\begin{aligned}\eta_X:X&\to TX\\ f&\mapsto\eta_X(f)=\begin{cases} f' & \text{if } L_f=0\\ f & \text{otherwise}\end{cases}\end{aligned}$$

where $\{f\in X\mid L_f\neq 0\}$ is the underlying set of TX and, for each subset M of TX, its closure on $TX,\overline{M}$, is the least subset of TX containing M and such that, for $f\in TX$:

(i^1) if $L_f=1$ and $f\in M$, then $f'\in\overline{M}$;

(ii^1) if $L_f=\gamma+1$ $(\gamma>1)$ and there exists $h:\alpha\setminus\gamma\to E$ such that $h'=f$ and, for each $\beta<\gamma$, $\{g\in M\mid g\preceq h \text{ and } L_g\geq\beta\}$ is infinite, then $f\in\overline{M}$, and $h\in\overline{M}$ whenever $h\in TX$.

As it is easily verified from its description, TX does not belong to $\mathcal{T}op_1$ whenever α is greater than 1, and so, the iteration of η goes on. Indeed, the process does not stop until the α-step, and, for each $\delta < \alpha$,

$$\eta_X^\delta : X \to T^\delta X$$
$$f \mapsto \eta_X^\delta(f) = \begin{cases} f_{|\alpha\setminus\delta} & \text{if } L_f < \delta \\ f & \text{otherwise} \end{cases}$$

where $T^\delta X$ has as underlying set $\{f \in X \mid L_f \geq \delta\} \cup \{f \in G \mid L_f = \delta\}$ and, for each $M \subseteq T^\delta X$, its closure, $\overline{M}$, is the least subset of $T^\delta X$ containing M and satisfying the following conditions, for $f \in T^\delta X$:

(i^δ) if $L_f = \delta$ and $f \in M$, then $f' \in \overline{M}$;
(ii^δ) if $L_f = \gamma + 1$ ($\gamma > \delta$) and there exists $h : \alpha \setminus \gamma \to E$ such that $h' = f$ and, for each $\beta < \gamma$, $\{g \in M \mid g \preceq h \text{ and } L_g \geq \beta\}$ is infinite, then $f \in \overline{M}$, and $h \in \overline{M}$ whenever $h \in T^\delta X$.

Finally, the $\mathcal{T}op_1$-reflection of X is given by $\eta_X^\alpha : X \to T^\alpha X \cong P$, that is, the $\mathcal{T}op_1$-reflection of X is the one-element topological space.

Example 2. Let $\mathcal{X}$ be the category of $\mathcal{P}r\mathcal{T}op$ of Čech (= pretopological) spaces, i.e. the category whose objects are sets equipped with an additive closure operation and whose morphisms are maps which are continuous with respect to the closure operations (cf. [4]). Let $\mathcal{M}$ be the class of extremal monomorphisms, C the usual closure operator on $\mathcal{P}r\mathcal{T}op$ and P the terminal object. In this situation, the $\mathcal{T}_0$-reflection behaves like the $\mathcal{T}op_1$-reflection. In fact, for each ordinal α, consider the pretopological space $A = (A, c_A)$ where A is the underlying set of the topological space defined in Example 1, and, for each $M \subseteq A$, $c_A(M)$ is the least subset of A satisfying the following conditions, for $f \in A$:

(i$'$) if $L_f = 0$ and $f \in M$, then $f' \in c_A(M)$;
(ii$'$) if $L_f = \gamma + 1$ ($\gamma > 0$) and there exists $h : \alpha \setminus \gamma \to E$ such that $h' = f$ and, for each $\beta < \gamma$, $\{g \in M \mid g \preceq h \text{ and } L_g \geq \beta\}$ is infinite, then $f \in c_A(M)$, and $h \in c_A(M)$ whenever $h \in A$ (i.e. L_h is not a limit ordinal);
(iii$'$) if there exists $g \in M$ such that $f \preceq g$, then $f \in c_A(M)$.

The $\mathcal{T}_0$-reflection of A is also reached at the α-step and it is given by $\zeta_A^\alpha : A \to S^\alpha A \cong P$ (for each $\delta \leq \alpha$, the construction of ζ_A^α is similar to the construction of η_X^δ in Example 1).

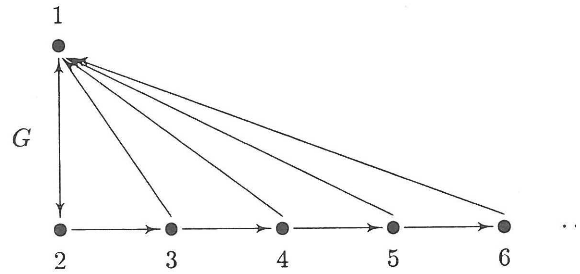
Example 3. The closure operator of the example above is additive but it is not idempotent. Next we will show that, even when C is additive and idempotent, the prereflection (S, ζ) is not always the $\mathcal{T}_0$-reflection.

Let $\mathcal{X}$ be the category $\mathcal{G}rph$ of oriented graphs, that is, its objects are pairs (X, K), where X is a set and K a subset of $X \times X$, and $f \in \mathcal{G}rph((X, K), (X', K'))$ if and only if it is a map from X to X' such that, for each $(x, y) \in K$, $(f(x), f(y)) \in K'$. Let $\mathcal{M}$ be the class of extremal monomorphisms and C the closure operator on $\mathcal{G}rph$ defined as follows: for each $G = (X_G, K_G)$ in $\mathcal{G}rph$ and $M \subseteq X_G$, an element x

of X_G belongs to $c_G(M)$ if and only if $x \in M$ or there exists $x_1, x_2, \dots, x_n$ in X_G such that $x_n \in M$ and, for $x_0 = x$, (x_{i-1}, x_i) and (x_i, x_{i-1}) belong to K_G, for each $i \in \{1, 2, \dots, n\}$. It is easy to verify that C is an idempotent and additive closure operator.

Let $P = (\{a\}, \emptyset)$. We thus have that $\mathcal{P} = \{G \in \mathcal{G}rph \mid G \cong P \text{ or } G \cong T\}$, with T the terminal object of $\mathcal{G}rph$.

Consider $G = (X_G, K_G)$, with $X_G = \mathbb{N}$, and, for each $(n, m) \in \mathbb{N} \times \mathbb{N}$, $(n, m) \in K_G$ if and only if $m = 1$ or $m = n$ or $m = n + 1$.



It is easily seen that, for each $m < \omega$,

$$\zeta_G^m : G \to S^m G = G$$
$$n \mapsto \zeta_G^m(n) = \begin{cases} 1 & \text{if } n \leq m \\ n - m & \text{otherwise} \end{cases}$$

and that

$$\zeta_G^\omega : G \to T = (\{a\}, \{(a, a)\})$$
$$n \mapsto a.$$

In conclusion, the $\mathcal{T}_0$-reflection of G is obtained at the ω-iteration of ζ.

5. Connections with other notions of separation.

Several authors have investigated concepts of T_0-objects, mainly in the context of topological categories. They were studied, for instance, by Brümmer [3], Harvey [9], Hoffmann [10], Marny [15], Weck-Schwarz [22] and [23], and Hušek and Pumplün [12].

This is essentially due to the fact that in $\mathcal{T}op$ the T_0-objects have nice characterizations, namely: a topological space X is T_0 if and only if

(1) X is not a cogenerator,
(2) every initial source with domain X is a monosource,
(3) X does not contain a non-trivial indiscrete subspace,

which can be easily thought in more general settings.

Moreover, condition (3) relates the study of separation with another interesting subject: disconnectednesses. These were introduced by Arhangel'skiĭ and Wiegandt [1], and then studied by various authors (see [16], [17], [21], [2], [19], [14] and [11]).

In general one cannot expect to establish connections between these notions and the notions of T_0- and T_1-object we introduced in Section 2. Indeed, in our approach T_0- and T_1-objects depend on the choice of a closure operator and a class $\mathcal{P}$ which determines 'points', and the others do not.

The following result illustrates this assertion.

Proposition 5.1 ([5]). *If $\mathcal{X}$ is a topological category, then any extremal epireflective subcategory of $\mathcal{X}$ is the subcategory of T_0-objects (resp. T_1-objects) for a suitable choice of closure operator and points.*

However, under additional assumptions, one can establish some connections, namely:

Proposition 5.2 ([5]). *If in $\mathcal{X}$ monosources separate points, then any T_0-object is non-cogenerator if and only if $\mathcal{T}_0$ is a proper subcategory of $\mathcal{X}$.*

Proposition 5.3 ([5]). *If $\mathcal{X}$ is a topological category and $\mathcal{P} = \{X \in \mathcal{X} \mid X$ is a terminal object$\}$, then $\mathcal{T}_0$ (resp. $\mathcal{T}_1$) is a disconnectedness whenever the closure operator C is hereditary (resp. weakly hereditary).*

Additional results on this subject can be found in [5].

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