Quadratic functionals with a variable singular end point

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Abstract. In this paper we introduce the definition of coupled point with respect to a (scalar) quadratic functional on a noncompact interval.

In terms of coupled points we prove necessary (and sufficient) conditions for the non-negativity of these functionals.

Keywords: quadratic functional, singular quadratic functional, Euler-Lagrange equation, conjugate point, coupled point, singularity condition

Classification: 49B10, 34C10, 34A10

1. Introduction.

Quadratic functionals play a crucial role in the theory of the second variation in the calculus of variations. V. Zeidan and P. Zezza [5], [6] introduced the concept of *coupled point* for variational problems with variable end points, which have the following accessory boundary value problem:

Minimize

$$I(\eta)=\frac{1}{2}(\eta^T(a),\eta^T(b))\Gamma\begin{pmatrix}\eta(a)\\ \eta(b)\end{pmatrix}+\frac{1}{2}\int_a^b(\eta^TP\eta+2\eta'^TQ\eta+\eta'^TR\eta')\,ds \tag{1}$$

over all absolutely continuous $\eta(\cdot):[a,b]\to\mathbf{R}^n$, subject to the boundary conditions

$$D\begin{pmatrix}\eta(a)\\ \eta(b)\end{pmatrix}=0 \tag{2}$$

where $P(\cdot),Q(\cdot),R(\cdot)$ are $n\times n$ matrices, Γ and D are $2n\times 2n$ and $2n\times r$ matrices, $P,Q,R\in L^\infty[a,b]$, $R(\cdot)\geq\alpha I$ a.e., $\alpha>0$, P,R,Γ are symmetric and $\operatorname{rank}D=r$, $0\leq r\leq 2n$.

The idea of coupled point is based on the fact that a solution of the corresponding Euler-Lagrange equation is continuously extended as a *constant*. The coupled point is derived by the per partes method and by the classical lemma of the calculus of variations and is described in terms of a solution of the Euler-Lagrange equation satisfying boundary and so called coupling conditions. This definition has the same meaning of the Jacobi condition for the conjugate and focal point case where one or both end points are fixed. It means that the non-existence of a point coupled with

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b or with a in (a, b) is a necessary condition for the functional (1) being positive semidefinite.

The sufficient condition was proved in [7], [8] by Hilbert space (index theory) method, where the crucial role is played by the ellipticity of the investigated functional.

To extend the definition of coupled point to noncompact interval, we have the theory of singular quadratic functionals with zero boundary conditions. The "singularity" of these functionals is caused both by the coefficient functions and by the class of admissible functions which is larger than the one on the compact interval. Here, the non-existence of conjugate (focal) point with singular end point together with the so called *singularity condition* give necessary and sufficient conditions for the singular functional being positive semidefinite. The major contribution to this theory has been made by Morse and Leighton [1] who established an extension of the conjugate point theory in the scalar case. This work is continued in [2], [3]; other references about singular functionals can be found in [4].

Summarizing, on one hand, we have in the compact interval case two methods – differential equations theory for necessary condition and index theory for sufficient condition, on the other hand, the application of index theory in the singular case even with zero boundary conditions is still a question.[1]

Hence we start here with scalar case where the boundary conditions (2) can be classified into three types

$$D = \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix} \qquad \begin{matrix} i = 0 \\ i = 1 \end{matrix} \qquad \begin{matrix} \text{(focal point case)} \\ \text{(conjugate point case)} \end{matrix}$$

$$D = 0 \qquad \text{(unconstrained case)}$$

$$\text{otherwise} \qquad \text{(non-separated case, e.g. periodic case)}$$

The aim of this paper is to extend the definition of coupled point to a noncompact interval. To this goal, we describe coupled points in terms of Riccati equation for the compact interval case and in this way we approach the singular end point case.

2. The variational problem and the Euler-Lagrange equation.

Let $r(\cdot), p(\cdot) \in AC[a, \infty)$, $r(\cdot) > 0$, and α, $\gamma \in \mathbf{R}$. Let us consider the following quadratic functionals[2]

$I(\eta; a, b) = \alpha\eta^2(a) + \gamma\eta^2(b) + \int_a^b [r(s)\eta'^2(s) - p(s)\eta^2(s)]\, ds,$
over all $\eta \in W^{1,2}\,[a, b]$;

$I(\eta; a, \infty)\colon = \liminf_{t\to\infty} I(\eta; a, t)$
over all $\eta \in W^{1,2}_{\text{loc}}\,[a, \infty)$;

$J(\xi; a, b) = \gamma\xi^2(a) + \int_a^b [r(s)\xi'^2(s) - p(s)\xi^2(s)]\, ds,$
over all $\xi \in W^{1,2}\,[a, b]$ such that $\xi(a) = \xi(b)$;

[1] From the point of view of index theory, the fact that singular functionals are only weakly lower semi-continuous in the space $W^{1,2}$ is substantial.

[2] The functional is considered to be an operator defined on the given class of admissible functions.

$J(\xi; a, \infty) = \liminf_{t\to\infty} J(\xi; a, t)$
over all $\xi \in W^{1,2}_{\text{loc}}[a, \infty)$ such that $\xi(a) = \lim_{t\to\infty} \xi(t)$.

Problem. We seek necessary (and) sufficient conditions for the nonnegativity of the objective functionals.

To be more clear, let $(\mathcal{X})$ be one of the following statements

$(\mathcal{P})$	$I(\eta; a, b) \geq 0$	(*compact interval, unconstrained case*)
$(\mathcal{S})$	$I(\eta; a, \infty) \geq 0$	(*noncompact interval, unconstrained case*)
$(\mathcal{P}\,\mathcal{P})$	$J(\xi; a, b) \geq 0$	(*compact interval, periodic case*)
$(\mathcal{P}\,\mathcal{S})$	$J(\xi; a, \infty) \geq 0$	(*noncompact interval, periodic case*)

We study the validity of the statement $(\mathcal{X})$.

Remark 1. i) If a quadratic functional is bounded from below for all admissible functions then its infimum over this class of admissible functions is zero.

ii) The problems on a noncompact interval concern functionals with singular end point and they are formulated on the half-line for stressing the singularity (at $t = \infty$). Under the transformation $x = \frac{1}{s}$ we get the problem with the singularity at $s = 0$ which is treated for conjugate and focal points in [1], [2], [3]. On the other hand, the conjugate point problem of [1], [2], [3] can be rewritten as the problem of finding conditions for

$$(\mathcal{S}_1)\begin{cases} \liminf_{t\to\infty} \int_a^t [r(s)\eta'^2(s) - p(s)\eta^2(s)]\, ds \geq 0 \\ \text{over all } \eta \in W^{1,2}_{\text{loc}}[a, \infty) \text{ such that } \eta(a) = 0 = \lim_{t\to\infty} \eta(t) \end{cases}$$

and the focal point case $(\mathcal{S}_2)$ is the one with the only boundary condition $\eta(a) = 0$.[3]

Agreement. Throughout the paper we use the bold bracket $\boldsymbol{(\ )}$ to indicate the alternative cases.

Since our goal is to study the nonnegativity of quadratic functionals with variable end points, in accordance with [5], [6], we test the validity of $(\mathcal{X})$ for constant functions.

Definition 1. The functional $I(\eta; a, b)\,\boldsymbol{(}I(\eta; a, \infty)\boldsymbol{)}$ and $J(\xi; a, b)\,\boldsymbol{(}J(\xi; a, \infty)\boldsymbol{)}$ is said to be *regular* if

$$\mathbf{k} := \alpha + \gamma - \boldsymbol{(}\limsup_{b\to\infty}\boldsymbol{)} \int_a^b p(s)\, ds \geq 0 \quad \text{and}$$

$$\mathbf{k} := \gamma - \boldsymbol{(}\limsup_{b\to\infty}\boldsymbol{)} \int_a^b p(s)\, ds \geq 0, \qquad \text{respectively.}$$

The Euler-Lagrange equation associated with the objective functionals is

$$(r(t)y')' + p(t)y = 0,\ t \in I = [a, b]\ \boldsymbol{(}I = [a, \infty)\boldsymbol{)}\ \text{(a.e.)}. \tag{3}$$

[3] In the terminology of [1-3], such functions are called "A-admissible "and "F-admissible functions", respectively.

It is well-known that the quadratic functional $I(\eta;a,b)$ over all η such that $\eta(a) = 0 = \eta(b)$ is nonnegative if and only if (3) is disconjugate on (a,b). Let us recall that if t_1 and t_2 are distinct values on I then these values are said to be (mutually) *conjugate* relative to (3) if there exists a nontrivial solution y of (3) such that $y(t_1) = 0 = y(t_2)$. If there exist no pair of conjugate points on I then (3) is said to be *disconjugate* on I. A point $t_2 > t_1$ is said to be a (right) *focal* point with t_1 if there exists a non-trivial solution y of (3) such that $y(t_2) = 0$, $y'(t_1) = 0$.

It is also known that the disconjugacy is related to the existence of solutions of a Riccati equation in the case of focal or conjugate points. If (3) is disconjugate on a subinterval I_0 of I then there exists a solution y of (3) for which $y(t) \neq 0$ on I_0. Then the function $w(t) = r(t)y'(t)y^{-1}(t)$ is a solution of the Riccati differential equation

$$w'(t) + r^{-1}(t)w^2(t) + p(t) = 0, \quad t \in I_0(\text{ a.e.}). \tag{4}$$

Conversely, if w is a solution of (4) on I_0 and if $y(t) = \delta \exp \int_a^t r^{-1} w\, ds$, $\delta \neq 0$, then $y(t) \neq 0$ on I_0, y is a solution of (3) and $w = ry'y^{-1}$ on I_0. Moreover, if $y(t) \neq 0$ on $[a,b]$, the boundary conditions $r(a)y'(a) = \alpha y(a)$, $r(b)y'(b) = \gamma y(b)$ and $y(a) = y(b)$ are equivalent to $w(a) = \alpha$, $w(b) = \gamma$ and $\int_a^b r^{-1} w\, ds = 0$, respectively.

It means that in the conjugate and focal point case the existence of a solution of the Riccati equation can be used to provide necessary and sufficient conditions for a nonnegativity of the quadratic functional.

A similar statement can be proved in the case of free-end-points and periodic conditions.

Lemma 1. (i) *$I(\eta;a,b) \geq 0$ if and only if the solution w of (4) satisfying $w(a) = \alpha$ exists on $[a,b]$ and $w(b) + \gamma \geq 0$.*

(ii) *$J(\xi;a,b) \geq 0$ holds if and only if there exists a solution w of (4) satisfying $\int_a^b r^{-1}(s)w(s)\, ds = 0$ and $w(b) - w(a) + \gamma \geq 0$.*

Proof: (i) Let the solution of (4) satisfying $w(a) = \alpha$ exist on $[a,b]$ and $w(b) \geq -\gamma$. Then the validity of $(\mathcal{P})$ follows from the identity

$$\begin{aligned} I(\eta;a,b) &= \alpha\eta^2(a) + \gamma\eta^2(b) + w\eta^2 \,|_a^b + \int_a^b r^{-1}(r\eta' - w\eta)^2\, ds = \\ &= \eta^2(a)(\alpha - w(a)) + \eta^2(b)(\gamma + w(b)) + \int_a^b r^{-1}(r\eta' - w\eta)^2\, ds, \end{aligned} \tag{5}$$

which holds for every $\eta \in W^{1,2}[a,b]$, see [4, pp. 73-78]).

Conversely, let $(\mathcal{P})$ hold. Suppose that the solution of (4) satisfying $w(a) = \alpha$ does not exist on the whole interval $[a,b]$, i.e., there exists $c \in (a,b]$ such that $\lim_{t\to c-} w(t) = -\infty$. Then there exists $d < c$ such that $\int_a^d r^{-1}w^2\, ds > \alpha + \gamma - \int_a^b p\, ds$. Let y be the corresponding solution of (3), i.e. $w = ry'y^{-1}$. Define

$$\eta(t) = \begin{cases} y(t) & t \in [a,d] \\ y(d) & t \in [d,b]. \end{cases}$$

Then $\eta \in W^{1,2}[a,b]$ and

$$\begin{aligned} I(\eta; a, b) &= y^2(a)\alpha + y^2(d)\gamma + \int_a^d (r\,y'^2 - p\,y^2)\,ds - y^2(d)\int_d^b p\,ds = \\ &= y^2(a)\alpha + y^2(d)\gamma + ry'y\,|_a^d - y^2(d)\int_d^b p\,ds = \\ &= y^2(a)[\alpha - w(a)] + y^2(d)\big[\gamma + w(d) - \int_d^b p\,ds\,\big] = \\ &= y^2(d)\big[\gamma + \alpha - \int_a^b p\,ds - \int_a^d r^{-1}w^2\,ds\,\big] < 0, \end{aligned}$$

which is a contradiction.

Hence w exists on the whole interval $[a,b]$ and the condition $w(b)+\gamma \geq 0$ follows immediately from (5).

(ii) Proof of the necessity. First we prove that $J(\xi; a, b) \geq 0$ implies the disconjugacy of the equation (3) on $[a,b]$. Suppose that b is the first conjugate point with a and y is a solution of (3) such that $y(a) = 0 = y(b)$. Let $t_1 \in (a,b)$ be sufficiently close to a, $t_2 = \sup\{t \in (a,b) | y(t_1) = y(t)\}$. Define

$$\xi(t) = \begin{cases} y(t_1) & t \in [a, t_1] \\ y(t) & t \in [t_1, t_2] \\ y(t_1) & t \in [t_2, b]. \end{cases}$$

Then ξ is an admissible function, for which it holds

$$\begin{aligned} J(\xi; a, b) &= \gamma y^2(t_1) - y^2(t_1)\Big[\int_a^{t_1} p\,dt + \int_{t_2}^b p\,dt\Big] + \int_{t_1}^{t_2} r\xi'^2 - p\xi^2\,dt = \\ &= y^2(t_1)\Big[\gamma - \int_a^{t_1} p\,dt - \int_{t_2}^b p\,dt\Big] + y^2(t_1)[w(t_2) - w(t_1)], \end{aligned}$$

where $w = ry'y^{-1}$ exists with regard to the property of y on the whole interval (a,b). Since $\lim_{t\to a+} w(t) = \infty, \lim_{t\to b-} w(t) = -\infty$, choosing t_1 and t_2 sufficiently close to a and b, respectively, we get

$$w(t_2) - w(t_1) + \gamma - \int_a^{t_1} p\,dt - \int_{t_2}^b p\,dt < 0.$$

Hence $J(\xi; a, b) < 0$, which is a contradiction and a, b are not conjugate points.

Now, let u, v be the solutions of (3) satisfying initial conditions $u(a) = 0$, $u'(a) = 1$, $v(b) = 0$, $v'(b) = -1$. Then u, v are linearly independent and positive on $(a,b]$ and $[a,b)$, respectively, and the solution $y = \frac{u(t)}{u(b)} + \frac{v(t)}{v(a)}$ is positive on $[a,b]$ and satisfies $y(a) = y(b)$. Thus the function $w = ry'y^{-1}$ exists on $[a, b]$ and satisfies $\int_a^b r^{-1}w\,dt = 0$ and $w(b) - w(a) + \gamma \geq 0$ since for all admissible functions ξ

$$J(\xi; a, b) = \xi^2(a)\big[w(b) - w(a) + \gamma\big] + \int_a^b r^{-1}(r\xi' - w\xi)^2\,dt$$

and for $\xi = y(a)\exp\int_a^t r^{-1}w\,ds$ the integral term vanishes.

The proof of the sufficiency is now obvious. □

3. Survey of results.

Here we present a survey of our results which are proved in Sections 4 and 5.

Theorem. *Let $(\mathcal{X})$ be of type $(\mathcal{P}), (\mathcal{P}\,\mathcal{P}), (\mathcal{S})$. $(\mathcal{X})$ holds if and only if*

(1) *the functional is regular*
(2) *there is no coupled point relative to the functional and, moreover, in case $(\mathcal{S})$ the singularity condition is satisfied.*

These results can be summarized in the following table (for the definition of coupled point, see Definitions 2 and 3, for the singularity condition, see Theorem 2).

	compact interval	noncompact interval
unconstrained case	$(\mathcal{P}) \Leftrightarrow (1), (2)$ (Theorem 1)	$(\mathcal{S}) \Leftrightarrow (1), (2)$ (Theorem 2)
periodic case	$(\mathcal{P}\,\mathcal{P}) \Leftrightarrow (1), (2)$ (Theorem 3)	$(\mathcal{P}\,\mathcal{S}) \Rightarrow (1), (2)$ (Theorem 4)

Important Remark. The necessary condition (2) for $(\mathcal{P})$, $(\mathcal{P}\,\mathcal{P})$ has been proved in [5], [6] by the per partes method and by the fundamental lemma of the calculus of variations (the fact that the functional is zero for some admissible function which is not extremal leads to the contradiction with uniqueness theorem for ordinary differential equations). On the other hand, the necessary condition (2) for $(\mathcal{S})$, $(\mathcal{P}\,\mathcal{S})$ does not seem to be suitable to be proved by this lemma. The sufficient condition for $(\mathcal{P})$, $(\mathcal{P}\,\mathcal{P})$ has been proved in [7], [8] by index theory and in Theorems 1 and 3 it is proved by Riccati equation.

The importance of Riccati equation method for functionals on the noncompact interval is in the fact that the fundamental lemma of the calculus of variations as well as index theory are not convenient here.

4. Coupled points in terms of Riccati equation: unconstrained case.

In accordance with [5], a coupled point is defined in case of free-end points on the compact interval (noncompact interval) by the following

Definition 2. Let $y(\cdot)$ be a nonzero solution of (3) satisfying the condition[4]

$$r(a)y'(a) - \alpha y(a) = 0. \tag{6}$$

[4]The boundary problem (3), (6) has always a solution which is determined uniquely up to the multiplicative constant. A similar remark holds for Definition 3, see Section 5.

A point $c \in [a,b)$ $\big(c \in [a,\infty)\big)$ is said to be *coupled point* with a relative to $I(\eta;a,b)$ $\big(I(\eta;a,\infty)\big)$ if

$$(7_1) \qquad r(c)y'(c) + \gamma\, y(c) - \big(\limsup_{b\to\infty}\big) \int_c^b p(s)\,ds\, y(c) = 0$$

$$(8_1) \qquad y(\cdot) \not\equiv y(c) \quad \text{on } [c,b] \quad \big([c,\infty)\big).$$

Remark 2. The point a is coupled with a relative to $I(\eta;a,b)$ $\big(I(\eta;a,\infty)\big)$ if and only if $\mathbf{k} = 0$ (from Definition 1) and $y \equiv$ const is not extremal on $[a,b]$ $\big([a,\infty)\big)$.

The *first coupled point* with a is understood the point a coupled with a or a coupled point $c \in (a,b)$ such that there is no coupled point on $[a,c)$.

Now, we are in a position to formulate the first coupled point in terms of Riccati equation. For simplicity, $I(\eta;a,\cdot)$ denotes $I(\eta;a,b)$ or $I(\eta;a,\infty)$.

Lemma 2. *Let $I(\eta;a,\cdot)$ be regular. A point c is the first coupled point with a relative to $I(\eta;a,\cdot)$ if and only if there exists $d \in (c,b)\big((c,\infty)\big)$ such that the solution w of* (4) *satisfying $w(a) = \alpha$ is defined on $[a,d]$ whereby*

$$(7_2) \qquad \int_a^c r^{-1}(s)\, w^2(s)\,ds = \mathbf{k}$$

$$(8_2) \qquad \int_c^d r^{-1}(s) w^2(s)\,ds > 0$$

where $\mathbf{k}$ is from Definition 1.

Proof: We shall prove the lemma for the functional $I(\eta;a,b)$. The proof of the conclusion for $I(\eta;a,\infty)$ is almost the same and hence will be omitted.

First, let us remember the following argument: If w is a solution of (4) on $[a,d]$ such that $w(a) = \alpha$ and if $y(t) = \delta \exp \int_a^t r^{-1} w\,ds$, $\delta \neq 0$, then $y(t) \neq 0$ on $[a,d]$, y is a solution of (3), (6) on $[a,d]$, $w = ry'y^{-1}$ and

$$(9) \qquad w(t) = \alpha - \int_a^t r^{-1}(s)\, w^2(s)\,ds - \int_a^t p(s)\,ds \quad \text{for } t \in [a,d].$$

I. If a is coupled with a then in view of Remark 2 the condition (7_2) is trivially satisfied and also (8_2) holds.

Let $c \in (a,b)$ be the first coupled point with a and y the corresponding solution of (3) from Definition 2. Obviously $y(c) \neq 0$. Indeed, if $y(c) = 0$ then $y'(c) = 0$ and $y(t) \equiv 0$ which would be a contradiction with the uniqueness theorem. First, suppose that $w(t)$ satisfying $w(a) = \alpha$ exists on all $[a,c]$. Since $y(c) \neq 0$ we can divide (7_1) by $y(c)$ and we get

$$w(c) + \gamma - \int_c^b p(s)\,ds = 0.$$

Substituting for $w(c)$ in (9), we get (7_2).

We shall prove that $w(t)$ really exists on all $[a,c]$. Suppose there exists $t_0 \in (a,c]$ such that $|w(t_0-)| = \infty$. Then $\lim_{t\to t_0-} \int_a^t r^{-1}(s)\, w^2(s)\, ds = \infty$ and there exists $d \in (a,t_0)$ such that

$$\int_a^d r^{-1}(s)\, w^2(s)\, ds = \mathbf{k} = \alpha + \gamma - \int_a^d p(s)\, ds - \int_d^b p(s)\, ds.$$

From (9) and the fact $y(d) \neq 0$ it holds

$$r(d)y'(d) + \gamma\, y(d) - \int_d^b p(s)\, ds\, y(d) = 0,$$

i.e. $d \in (a,c)$ is a coupled point with a, which is a contradiction.

II. Let w exist on $[a,c]$ and (7_2), (8_2) hold. Then a solution y of (3) given by $y(t) = y(a) \exp \int_a^t r^{-1} w\, ds$ satisfies (7_1) and, because $w(t) \not\equiv 0$ on (c,d) for some $d > c$, also (8_1). □

Theorem 1. *Let $I(\eta; a, b)$ be regular. $I(\eta; a, b) \geq 0$ if and only if there exists no coupled point $c \in [a,b)$ with a relative to $I(\eta; a, b)$.*

Proof: I. The necessity. In [5] it has been proved that if $I(\eta; a, b) \geq 0$ then there exists no coupled point $c \in (a,b)$ with a. Hence, to complete the proof of the necessity we need to show that a is not coupled with a. Let a be coupled with a. By Lemma 2, there exists $d > a$ such that the solution w of (4) satisfying $w(a) = \alpha$ is defined on $[a,d]$ and $\int_a^d r^{-1} w^2\, ds > 0$. Now we proceed by the similar way as in the proof of Lemma 1 (i). Let y be the corresponding solution of (3), i.e. $w = ry'y^{-1}$, and define

$$\eta(t) = \begin{cases} y(t) & t \in [a,d] \\ y(d) & t \in [d,b]. \end{cases}$$

Then $I(\eta; a, b) = y^2(d)\big(-\int_a^d r^{-1} w^2\, ds\big) < 0$, a contradiction.

II. The sufficiency. Suppose there exists no coupled point $c \in (a,b)$ with a. Let w be the solution of (4) satisfying $w(a) = \alpha$. Then w exists on all $[a,b]$. Indeed, if there exists $e \in (a,b]$ such that $\int_a^e r^{-1} w^2\, ds = \infty$ then there exists $c \in (a,b)$ such that (7_2), (8_2) hold, i.e. by Lemma 2 c is the first coupled point with a, a contradiction.

By Lemma 2, either $\int_a^b r^{-1} w^2\, ds < \mathbf{k}$ or there exists $c \in (a,b)$ such that (7_2) holds but $\int_c^b r^{-1} w^2\, ds = 0$.

In the first case we have $w(b) = \alpha - \int_a^b r^{-1} w^2\, ds - \int_a^b p\, ds = \alpha + \gamma - \int_a^b p\, ds - \int_a^b r^{-1} w^2\, ds - \gamma > -\gamma$, and $I(\eta; a, b)$ is positive definite by Lemma 1.

In the second case $w(b) = -\gamma$ and $I(\eta; a, b) \geq 0$ with the equality either for $\eta(t) \equiv 0$ or $\eta(t) = \eta(a) \exp \int_a^t r^{-1} w\, ds$. □

Theorem 2. *Let $I(\eta; a, \infty)$ be regular. $I(\eta; a, \infty) \geq 0$ if and only if there exists no coupled point $c \in [a, \infty)$ with a relative to $I(\eta; a, \infty)$ and the singularity condition is satisfied, i.e.*

$$\liminf_{t\to\infty} \eta^2(t)[w(t) + \gamma] \geq 0$$

for each admissible function $\eta(t)$ for which $I(\eta; a, \infty)$ is finite. Here $w(t)$ is the solution of (4) *such that $w(a) = \alpha$, which exists because there is no coupled point with a.*

Proof: Let $w(t)$ be the solution of (4) satisfying $w(a) = \alpha$. If this solution does not exist on all $[a, \infty)$, then by the same argument as in the proof of Theorem 1 there exists $c \in (a, \infty)$ coupled with a relative to $I(\eta; a, \infty)$.

Denote $z[\eta] = r^{-1}(r\eta' - w\eta)^2$.

I. The sufficiency. If there exists no coupled point $c \in (a, \infty)$ with a then, proceeding in the same way as in the proof of Theorem 1, we get $\liminf_{b\to\infty} w(b) = \alpha + \gamma - \limsup_{b\to\infty}\left[\int_a^b p\,ds + \int_a^b r^{-1}w^2\,ds\right] - \gamma \geq \mathbf{k} - \lim_{b\to\infty}\int_a^b r^{-1}w^2\,ds - \gamma \geq -\gamma$. Now, the validity of the singularity condition implies

$$I(\eta; a, \infty) = \liminf_{b\to\infty}\left[\eta^2(b)(w(b) + \gamma) + \int_a^b z[\eta]\,dt\right] \geq$$
$$\geq \liminf_{b\to\infty} \eta^2(b)(w(b) + \gamma) + \lim_{b\to\infty}\int_a^b z[\eta]\,dt \geq 0$$

for every admissible function η.

II. The necessity. Let $I(\eta; a, \infty) \geq 0$ for every admissible function η. At first, suppose there exists the first coupled point $c \in [a, \infty)$ with a. Then, by Lemma 2, there exists $d > c$ such that for the solution w of (4) satisfying $w(a) = \alpha$ we have $\int_a^d r^{-1}w^2\,ds > \mathbf{k}$. Again, proceeding in the similar way as in the proof of Lemma 1 (i), let y be the corresponding solution of (3), and define

$$\eta(t) = \begin{cases} y(t) & t \in [a, d] \\ y(d) & t \in [d, \infty). \end{cases}$$

Then η is admissible and

$$I(\eta; a, \infty) = y^2(a)[\alpha - w(a)] + y^2(d)\left[\gamma + w(d) - \limsup_{b\to\infty}\int_d^b p\,ds\right] =$$
$$= y^2(d)\left[\gamma + \alpha - \limsup_{b\to\infty}\int_a^b p\,ds - \int_a^d r^{-1}w^2\,ds\right] < 0,$$

which is a contradiction.

Now suppose that there exists no coupled point and the singularity condition does not hold, i.e. there exists $\bar\eta(t)$ such that

$$I(\bar\eta; a, \infty) = L < \infty, \quad \liminf_{t\to\infty} \bar\eta^2(t)[w(t) + \gamma] < 0.$$

Recall that the nonexistence of coupled point implies that the solution w of (4) satisfying $w(a) = \alpha$ exists on all $[a, \infty)$.

Let $d \in (a, \infty)$ be such that $\bar{\eta}(d) \neq 0$ and let y be the solution of (3) satisfying (6) and $y(d) = \bar{\eta}(d)$. Observe that such a solution always exists. Indeed, let u be a nontrivial solution of (3) satisfying (6) and $y(t) = u(t)\frac{\bar{\eta}(d)}{u(d)}$. Since (3) is homogeneous, $y(t)$ is the solution of (3) whose existence we needed to prove.

Then the function

$$\tilde{\eta}(t) = \begin{cases} y(t) & t \in [a, d] \\ \bar{\eta}(t) & t \in [d, \infty) \end{cases}$$

is admissible and

$$\begin{aligned} I(\tilde{\eta}; a, \infty) &= \liminf_{b \to \infty} \left[\bar{\eta}^2(b)(w(b) + \gamma) + \int_a^b z[\tilde{\eta}]\, dt \right] = \\ &= \liminf_{b \to \infty} \left[\bar{\eta}^2(b)(w(b) + \gamma) + \int_d^b z[\bar{\eta}]\, dt \right]. \end{aligned}$$

We distinguish two cases:

(i) $\liminf_{b \to \infty} \bar{\eta}^2(b)[w(b) + \gamma] = -l^2 > -\infty, l \neq 0$

(ii) $\liminf_{b \to \infty} \bar{\eta}^2(b)[w(b) + \gamma] = -\infty$.

In the first case, the fact that $I(\bar{\eta}; a, \infty) < \infty$ implies that $\int_a^\infty z[\bar{\eta}]dt < \infty$ and hence there exists d such that $\int_d^\infty z[\bar{\eta}]\, dt < l^2$. Thus $I(\tilde{\eta}; a, \infty) < 0$, which is a contradiction.

In the second case we have $\int_a^\infty z[\bar{\eta}]\, dt = \infty$ and hence, there exists $e \in [a, \infty)$ such that

$$\int_a^e z[\bar{\eta}]\, dt = I(\bar{\eta}; a, \infty).$$

Then $I(\bar{\eta}; a, \infty) = \liminf_{b \to \infty} \left[\bar{\eta}^2(b)(w(b) + \gamma) + \int_e^b z[\bar{\eta}]\, dt \right] + \int_a^e z[\bar{\eta}]\, dt$, hence $\liminf_{b \to \infty} \left[\bar{\eta}^2(b)(w(b) + \gamma) + \int_e^b z[\bar{\eta}]dt \right] = 0$.

Thus

$$\begin{aligned} I(\tilde{\eta}; a, \infty) &= \liminf_{b \to \infty} \left[\bar{\eta}^2(b)(w(b) + \gamma) + \int_d^b z[\bar{\eta}]\, dt \right] = \\ &= \liminf_{b \to \infty} \left[\bar{\eta}^2(b)(w(b) + \gamma) + \int_e^b z[\bar{\eta}]\, dt \right] + \int_d^e z[\bar{\eta}]\, dt = \int_d^e z[\bar{\eta}]\, dt. \end{aligned}$$

Choosing $d > e$ we have $I(\tilde{\eta}; a, \infty) < 0$, a contradiction. □

Remark 3. Obviously, the singularity condition is satisfied on the compact interval $[a, b]$. In case of zero singular end points the singularity condition coincides with the one of Morse and Leighton.

5. Coupled points in terms of Riccati equation: periodic case.

In accordance with [5], a coupled point is defined in the periodic case on the compact interval (noncompact interval) by the following

Definition 3. Let $y(\cdot)$ be a nonzero solution of (3) satisfying the condition $y(a) = y(c) \neq 0$. A point $c \in [a,b)$ $\big(c \in [a,\infty)\big)$ is said to be *coupled point* with a relative to $J(\xi;a,b)$ $\big(J(\xi;a,\infty)\big)$ if

$$r(c)y'(c) - r(a)y'(a) + \gamma\, y(a) - \big(\limsup_{b\to\infty}\big) \int_c^b p(s)\,ds\, y(c) = 0$$
$$y(\cdot) \not\equiv y(c) \quad \text{on } [c,b] \quad \big([c,\infty)\big).$$

The *first coupled point* with a is understood a point a coupled with a or a coupled point $c \in (a,b)$ such that there is no coupled point on $[a,c)$.

Remark 4. Like in the unconstrained case, the point a is coupled with a relative to $J(\eta;a,b)$ $\big(J(\eta;a,\infty)\big)$ if and only if $\mathbf{k} = 0$ (from Definition 1) and $y \equiv \text{const}$ is not extremal on $[a,b]$ $\big([a,\infty)\big)$.

In the following lemmas $J(\eta;a,\cdot)$ denotes $J(\eta;a,b)$ or $J(\eta;a,\infty)$.

Lemma 3. *Let $J(\xi;a,\cdot)$ be regular. If c is the first coupled point with a relative to $J(\xi;a,\cdot)$ then* (3) *is disconjugate on $[a,c]$.*

Proof: The statement may be proved by index theory, see [8, Corollary 3.1], but to keep the method consistent, we prove it by Riccati technique.

Suppose that there exists $t_0 \in (a,c]$ which is the (first) conjugate point with a relative to (3). Then for every $x \in (a,t_0)$ there exists a solution $y_x(t)$ of (3) for which $y_x(a) = y_x(x)$ and $y_x(t) \neq 0$ for $t \in [a,x]$ (see the proof of Lemma 1 (ii)). Let $w_x(t) = r(t)y_x'(t)y_x^{-1}(t)$ and $F(x) = \int_a^x r^{-1}w_x^2\,ds$. According to the continuous dependence on the initial conditions, the function $F(x)$ is continuous and $F(x) \to \infty$ as $x \to t_0-$ (since $w_x(t) \to r(t)y'(t)y^{-1}(t)$ as $x \to t_0$, where $y(t)$ is the solution of (3) for which $y(a) = 0 = y(t_0)$).

Consequently, there exists $e \in (a,t)$ such that $F(e) = \mathbf{k}$, and $\int_e^d r^{-1}w_e^2\,ds > 0$ for some $d > e$, i.e. e is coupled point with a relative to $J(\xi;a,b)$, a contradiction. $\square$

From this proof it follows immediately

Lemma 4. *Let $J(\xi;a,\cdot)$ be regular. A point c is a first coupled point with a relative to $J(\xi;a,\cdot)$ if and only if there exists $d \in (c,b)$ $\big((c,\infty)\big)$ such that the solution w of* (4) *satisfying $\int_a^c r^{-1}w\,ds = 0$ is defined on $[a,d]$ whereby* (7_2) *and* (8_2) *hold.*

Lemma 5. *Let $J(\xi;a,\cdot)$ be regular. Let $c \in [a,b)$ be the first coupled point with a relative to $J(\xi;a,\cdot)$. Then there exists $d \in (c,b)$ $\big((c,\infty)\big)$ such that for the solution w_1 of* (4) *satisfying $\int_a^d r^{-1}w_1\,ds = 0$ we have $\int_a^d r^{-1}w_1^2\,ds > \mathbf{k}$.*

Proof: Let c be the first coupled point with a relative to $J(\xi;a,b)$ and let w be the solution of (4) satisfying $\int_a^c r^{-1}w\,ds = 0$. By Lemma 4 there exists $d \in (c,b)$ such that $\int_a^d r^{-1}w^2\,ds > \mathbf{k}$. Let w_1 be the solution of (4) for which $\int_a^d r^{-1}w_1\,ds = 0$.

Then

$$\int_a^d r^{-1} w_1^2\, ds - \int_a^c r^{-1} w^2\, ds = \int_a^d r^{-1} w_1^2\, ds - \int_a^d r^{-1} w^2\, ds + \int_c^d r^{-1} w^2\, ds >$$
$$> \int_a^d r^{-1} w_1^2\, ds - \int_a^d r^{-1} w^2\, ds = \omega.$$

We shall prove that $\omega > 0$. Let u and v be the corresponding solution of (3) to w and w_1, i.e. $w = ru'u^{-1}$, $w_1 = rv'v^{-1}$ and $u(a) = u(c) \neq 0$, $v(a) = v(d) \neq 0$ (the nonzero boundary conditions are ensured by Lemma 3). Then

$$w_1 - w = r \cdot \frac{v'u - vu'}{uv}. \tag{10}$$

In view of this, it holds

$$\begin{aligned}\omega &= w_1(d) - w_1(a) - w(d) + w(a) = \\ &= \frac{r(a)v(a)(v'(a) - u'(a))}{v(a)u(a)} \Big[\frac{1}{u(d)v(d)} - \frac{1}{u(a)v(a)}\Big] = \\ &= \frac{r(a)(v'(a) - u'(a))(u(a) - u(d))}{u^2(a)u(d)v(d)} = \frac{r(a)(v'(a) - u'(a))(v(d) - u(d))}{u^2(a)u(d)v(d)}.\end{aligned}$$

If $v'(a) > u'(a)$ then $v(t) > u(t)$ on $[a, d]$, since the existence of $\xi \in (a, d)$ such that $v(\xi) = u(\xi)$ implies that the solution $v(t) - u(t)$ has two zeros in $[a, d]$, which contradicts the disconjugacy of (3) on this interval.

Similarly, if $v'(a) < u'(a)$, then $v(t) < u(t)$ on $[a, d]$. Consequently, $\omega > 0$ and $\int_a^d r^{-1} w_1^2\, ds > \int_a^c r^{-1} w^2\, ds = \mathbf{k}$.

The proof for $J(\xi; a, \infty)$ is the same. □

Theorem 3. *Let $J(\xi; a, b)$ be regular. $J(\xi; a, b) \geq 0$ if and only if there exists no coupled point $c \in [a, b)$ with a relative to $J(\xi; a, b)$.*

Proof: I. The necessity. Like the unconstrained case; in [5] it has been proved that if $J(\eta; a, b) \geq 0$ then there exists no coupled point $c \in (a, b)$ with a. Hence we prove that a is not coupled with a. Let a be coupled with a. By Lemma 5 there exists $d > a$ such that the solution w of (4) satisfying $\int_a^d r^{-1} w\, ds = 0$ we have $\int_a^d r^{-1} w^2\, ds > 0$. Let y be the corresponding solution of (3). Then for the function

$$\eta(t) = \begin{cases} y(t) & t \in [a, d] \\ y(d) & t \in [d, b] \end{cases}$$

it holds $J(\eta; a, b) = y^2(a)\big(-\int_a^d r^{-1} w^2\, ds\big) < 0$, a contradiction.

II. Suppose there exists no point $c \in [a, b)$ coupled with a. By Lemma 3, (3) is disconjugate on $[a, b)$ and if b is conjugate point with a, then by the same way as in the proof of Lemma 3 it would exist $\tilde{c} \in [a, b)$ coupled with a relative to $J(\xi; a, b)$.

Therefore (3) is disconjugate on $[a,b]$. By the same argument as in Lemma 1 (ii) there exists for every $c \in (a,b]$ a solution w_c of (4) for which $\int_a^c r^{-1} w_c\, ds = 0$. Since $[a,b)$ does not contain a coupled point with a relative to $J(\xi;a,b)$, by Lemma 4 for every $c \in [a,b)$ it holds $\int_a^c r^{-1} w_c^2\, ds < \mathbf{k}$ or $\int_a^c r^{-1} w_c^2\, ds = \mathbf{k}$ and $\int_c^d r^{-1} w_c^2\, ds = 0$ for every $d > c$. Hence also $\int_a^b r^{-1} w_b^2\, ds \le \mathbf{k}$, i.e., $w_b(b) - w_b(a) + \gamma \ge 0$ and the conclusion holds in view of Lemma 1 (ii). □

Theorem 4 (Necessary condition)**.** *Let $J(\xi;a,\infty)$ be regular. If $J(\xi;a,\infty) \ge 0$ then there is no coupled point $c \in [a,\infty)$ with a relative to $J(\xi;a,\infty)$.*

Proof: Suppose there exists $c \in [a,\infty)$, the first coupled point with a relative to $J(\xi;a,\infty)$. Then, by Lemma 5, there exists $d > c$ such that for the solution w of (4) satisfying $\int_a^d r^{-1} w\, ds = 0$ we have $\int_a^d r^{-1} w^2\, ds > \mathbf{k}$. Let y be the corresponding solution of (3). Proceeding in the same way as in the first part of the proof of Theorem 3, for the admissible function

$$\eta(t) = \begin{cases} y(t) & t \in [a,d] \\ y(d) & t \in [d,\infty), \end{cases}$$

it holds $J(\eta;a,\infty) = y^2(a)\big[-\int_a^d r^{-1} w^2\, ds + \gamma - \limsup_{b\to\infty} \int_a^b p\, ds\big] < 0$, which is a contradiction. □

Example. The following example illustrates the behaviour of the functional $I(\eta;a,\infty)$ and shows the importance of the coupled point in terms of Riccati equation for solving the concrete variational problem.

Consider the functional

$$I(\eta;a,t) = \alpha\eta^2(a) + \gamma\eta^2(t) + \int_a^t [s^2\eta'^2(s) + \frac{1}{s^2}\,\eta^2(s)]\, ds\,, \quad a > 0.$$

We seek conditions on α, γ, a such that $\liminf_{t\to\infty} I(\eta;a,t) \ge 0$ over all $\eta \in W^{1,2}_{\mathrm{loc}}\,[a,\infty)$.

The regularity condition gives $\mathbf{k} = \alpha + \gamma + \frac{1}{a} \ge 0$.

The corresponding Euler-Lagrange equation

$$(t^2 y')' - \frac{1}{t^2}\, y = 0$$

has solutions $e^{-1/t}$, $e^{1/t}$. Hence, the general solutions of the Riccati equation

$$w' + \frac{w^2}{t^2} - \frac{1}{t^2} = 0$$

are

$$w(t) = \frac{ce^{-1/t} - e^{1/t}}{ce^{-1/t} + e^{1/t}} \quad \text{and} \quad w(t) = 1.$$

The condition $w(a) = \alpha$ gives $c = \frac{1+\alpha}{1-\alpha} e^{2/a}$ (for $\alpha \neq 1$) and hence $w(\infty) = \lim_{t\to\infty} w(t) = \frac{(1+\alpha)e^{2/a}-1+\alpha}{(1+\alpha)e^{2/a}+1-\alpha}$. According to Theorem 2, for being the functional nonnegative the singularity condition must hold and it should not exist a coupled point with a.

Let us consider two examples of the parameters α, γ, a.

1. Let $\alpha = 2$. Then $w(\infty) = \frac{3e^{2/a}+1}{3e^{2/a}-1} =: f(a)$.

If $\gamma < -2$ then the functional is negative on $[a,\infty)$ for every $a > 0$.

If $\gamma > -1$ then the functional is positive on $[a,\infty)$ for every $a > 0$.

If $\gamma \in (-2,-1)$ then the equation $f(a)+\gamma = 0$ has a solution $a_0 \in (0,\infty)$ and *the functional is indefinite for $a \in (0,a_0)$ and is nonnegative for $a \geq a_0$.* Let $a \in (0,a_0)$ be fixed. Then there exists a coupled point c with a and by Definition 2, its value is the solution of the equation $w(c) + \gamma + \frac{1}{c} = 0$, that is

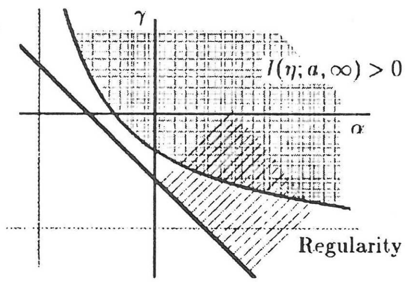
$$\frac{-3e^{2/a} - e^{2/c}}{-3e^{2/a} + e^{2/c}} + \gamma + \frac{1}{c} = 0.$$

2. Let $a = 2$. The functional is nonnegative if and only if the following conditions are satisfied

(i) $\alpha + \gamma \geq -\frac{1}{2}$ (the regularity condition)

(ii) $\gamma > -\frac{e-1+\alpha(1+e)}{e+1+\alpha(e-1)}$ (nonexistence of coupled points).

By other words, the inequalities (i), (ii) with the opposite sign give the domain of parameters α, γ, for which there exists a coupled point with a, that is, the functional is indefinite, see the picture.



Concluding remark. The following problems remain open

(i) The singularity condition in periodic case and hence the sufficient condition for the validity of $(\mathcal{P}\,\mathcal{S})$.

(ii) In case of the compact interval, the nonexistence of coupled point with a and of coupled point with b are equivalent. It arises the question of this symmetry on noncompact interval, i.e. how to define the coupled point with ∞.

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