Existence via partial regularity for degenerate systems of variational inequalities with natural growth

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Abstract. We prove the existence of a partially regular solution for a system of degenerate variational inequalities with natural growth.

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0. Introduction.

In this note we are concerned with degenerate systems of variational inequalities of the form

(0.1)
$$\begin{cases} \text{find } u \in \mathbb{K} \text{ such that } \int_\Omega |Du|^{p-2} Du \cdot D(v-u)\,dx \geq \int_\Omega f(\cdot, u, Du)\cdot(v-u)\,dx \\ \text{holds for all } v \in \mathbb{K}\,. \end{cases}$$

Here the right-hand side f is of natural growth with respect to the third argument, i.e.

$$|f(x,y,Q)| \leq a \cdot |Q|^p\,, \tag{0.2}$$

and the class $\mathbb{K}$ is defined by Dirichlet boundary conditions and a constraint of the form $\operatorname{Im}(u) \subset K$ for a convex set $K \subset \mathbb{R}^N$. We assume that the standard smallness condition (relating the growth constant a and $\operatorname{diam} K$)

$$a < (\operatorname{diam} K)^{-1} \tag{0.3}$$

is satisfied (compare [5]) under which we want to establish the existence of a solution of (0.1). Since we do not impose any variational structure it is not immediately obvious in which way an existence proof should be carried out. In the quadratic case $p = 2$ it is well known (see [5]) that (0.3) implies apriori bounds in Hölder spaces for solutions of (0.1) which in turn can be used to get existence of a function $u \in \mathbb{K}$ solving (0.1). On the other hand the author recently showed (compare [3]) that at least partial regularity is true for arbitrary exponents $p > 2$. In this note we combine the methods and prove existence via partial regularity: in the first step (0.1) is replaced by a sequence of variational inequalities with corresponding nonlinearity f_k defined as a suitable truncation of f. By Schauder's fixed point theorem we find a solution $u_k \in \mathbb{K}$ and we show uniform (partial) regularity on Ω

apart of a closed set Σ of vanishing $\mathcal{H}^{n-p}$-measure which in turn implies that the weak $H^{1,p}$-limit u of the sequence $\{u_k\}$ is a solution of (0.1) on $\Omega - \Sigma$, i.e. (0.1) holds for all $v \in \mathbb{K}$ such that $\operatorname{spt}(u-v) \subset\subset \Omega - \Sigma$. Using a capacity argument one finally sees that u is actually a solution of (0.1) on the whole domain Ω being in addition of class $C^{1,\varepsilon}(\Omega - \Sigma)$ for some $0 < \varepsilon < 1$. We conjecture that the singular set Σ is empty but without any further information we could not establish this more general result, a detailed discussion can be found in [3] where it is shown that for example certain monotonicity properties of u imply $\Sigma = \emptyset$.

1. Notations and statement of the result.

Let $\Omega \subset \mathbb{R}^n$, $n \geq 2$, denote a bounded open set and consider a compact convex region $K \subset \mathbb{R}^N$, $N \geq 1$, which is the closure of a C^2-domain. For exponents $2 \leq p < \infty$ we fix a function u_0 in the Sobolev space $H^{1,p}(\Omega, \mathbb{R}^N)$ with the property $u_0(x) \in K$ a.e. and introduce the convex class

$$\mathbb{K} := \{w \in H^{1,p}(\Omega, \mathbb{R}^N) : w - u_0 \in \overset{\circ}{H}{}^{1,p}(\Omega, \mathbb{R}^N),\ w(x) \in K \text{ a.e. }\}.$$

Moreover, suppose that we are given a (for simplicity) continuous function

$$f : \Omega \times \mathbb{R}^N \times \mathbb{R}^{nN} \ni (x, y, Q) \to f(x, y, Q) \in \mathbb{R}^N$$

for which the growth estimate

$$|f(x, y, Q)| \leq a \cdot |Q|^p \tag{1.1}$$

holds. Here a denotes a positive real number satisfying the smallness condition

$$a < (\operatorname{diam} K)^{-1}.$$

We then look at the variational inequality

(V)

$$\begin{cases} \text{find } u \in \mathbb{K} \text{ such that } \int_\Omega \{|Du|^{p-2} Du \cdot D(v-u) - f(\cdot, u, Du) \cdot (v-u)\}\, dx \geq 0 \\ \text{holds for all } v \in \mathbb{K} \end{cases}$$

for which we prove the following

Theorem. *Suppose that* (1.1), (1.2) *hold. Then* (V) *has at least one solution* $u \in \mathbb{K}$. *For* $p < n$, *there exists a relatively closed subset* Σ *of* Ω *such that* $\mathcal{H}^{n-p}(\Sigma) = 0$ *and with the additional property that* u *is of class* $C^{1,\varepsilon}$ *on* $\Omega - \Sigma$. *In case* $p \geq n$ *the singular set of* u *is empty.*

Comments. 1) Motivated by the quadratic case $p = 2$ treated in [5] we conjecture that the singular set Σ is empty for all exponents p.

2) Clearly it is possible to replace the p-energy $\int_\Omega |Du|^p\, dx$ by a more general splitting functional

$$\int_\Omega \Big(a_{\alpha\beta}(\cdot, u)\, B^{ij}(\cdot, u)\, D_\alpha u^i\, D_\beta u^j\Big)^{p/2}\, dx$$

with smooth elliptic coefficients $a_{\alpha\beta}$, B^{ij} provided we modify (1.2) in an appropriate way.

3) With similar arguments it is possible to include lower order terms in the growth estimate (1.1).

4) It should be noted that for smooth boundary functions u_0 we have partial regularity of u up to the boundary.

2. Approximate problems.

For $k \in \mathbb{N}$ we define

$$f_k(x,y,Q) := \begin{cases} f(x,y,Q) & \text{if } |f(x,y,Q)| \le k \\ k \cdot f(x,y,Q) \cdot |f(x,y,Q)|^{-1} & \text{otherwise} \end{cases}$$

and look at the problem

(2.1)

$$\begin{cases} \text{find } \tilde{u} \in \mathbb{K} \text{ such that } \int_\Omega \Big(|D\tilde{u}|^{p-1} \, D\tilde{u} \cdot D(v-\tilde{u}) - f_k(\cdot, \tilde{u}, D\tilde{u}) \cdot (v-\tilde{u}) \Big) \, dx \ge 0 \\ \text{holds for all } v \in \mathbb{K}\,. \end{cases}$$

In order to solve (2.1) we introduce the operator

$$\begin{aligned} & T : \mathbb{K} \to \mathbb{K}, \\ & \mathbb{K} \ni u \mapsto \text{ the unique solution } w \in \mathbb{K} \text{ of} \\ & \int_\Omega \Big(|Dw|^{p-2} \, Dw \cdot D(v-w) - f_k(\cdot, u, Du) \cdot (v-w) \Big) \, dx \ge 0 \\ & \text{for all } v \in \mathbb{K}\,, \end{aligned}$$

and check that the image $T(\mathbb{K})$ is precompact. For this consider a sequence $\{w_i\} = \{Tu_i\}$ in $T(\mathbb{K})$. Observing $u_0 \in \mathbb{K}$ we get

$$\int_\Omega |Dw_i|^p \, dx \le \int_\Omega |f_k(\cdot, u_i, Du_i)| \cdot |u_0 - w_i| \, dx + \int_\Omega |Dw_i|^{p-1} \, |Du_0| \, dx,$$

hence

$$\sup_{i \in \mathbb{N}} \|w_i\|_{H^{1,p}(\Omega)} < \infty$$

and we may assume (at least for a subsequence)

$$\begin{aligned} w_i &\rightharpoondown: w \in \mathbb{K} && \text{weakly in } H^{1,p}(\Omega), \\ w_i &\to w && \text{strongly in } L^p(\Omega). \end{aligned}$$

On the other hand we have

$$\int_\Omega \Big(|Dw_i|^{p-2} \, Dw_i \cdot (Dw_j - Dw_i) - f_k(\cdot, u_i, Du_i)(w_j - w_i) \Big) \, dx \ge 0,$$

$$\int_\Omega \Big(|Dw_j|^{p-2} \, Dw_j \cdot (Dw_i - Dw_j) - f_k(\cdot, u_j, Du_j)(w_i - w_j) \Big) \, dx \ge 0,$$

so that

$$\int_\Omega |Dw_i - Dw_j|^p \, dx \le c(n,p) \sup |f_k| \cdot \int_\Omega |w_i - w_j| \, dx.$$

This implies $w_i \to w$ strongly in $H^{1,p}(\Omega)$. By Schauder's fixed point theorem [4, Corollary 11.2] there exists at least one solution $\tilde{u} \in \mathbb{K}$ of $\tilde{u} = T\tilde{u}$ which clearly satisfes (2.1).

In the sequel we denote by u_k a solution of (2.1). Since f_k is dominated by f we have the growth condition

$$|f_k(x,y,Q)| \le a \cdot |Q|^p \tag{2.2}$$

which gives (insert u_0 into (2.1)):

$$\Big(1 - a \cdot \operatorname{diam}(K)\Big) \int_\Omega |Du_k|^p \, dx \le \int_\Omega |Du_k|^{p-1} \cdot |Du_0| \, dx$$

and in consequence (after passing to a subsequence)

$$\begin{aligned} u_k &\rightharpoonup: u \in \mathbb{K} \quad && \text{weakly in } H^{1,p}(\Omega), \\ u_k &\to: u && \text{strongly in } L^p(\Omega). \end{aligned}$$

Lemma 2.1. *There exist constants $\varepsilon > 0$, $\alpha \in (0,1)$ and $C > 0$ independent of k with the following properties: If for some ball $B_R(x) \subset \Omega$ we have*

$$\fint_{B_R(x)} |u - (u)_R|^p \, dz < \varepsilon \tag{2.3}$$

then

a) $$u, u_k \in C^{1,\alpha}\Big(B_{R/2}(x)\Big)$$

and

$$|Du_k(x_1) - Du_k(x_2)| + |Du(x_1) - Du(x_2)| \le C \cdot |x_1 - x_2|^\alpha$$

for all $x_1, x_2 \in B_{R/2}(x)$,

b) $$u_k \to u \text{ in } C^{1,\alpha}\Big(B_{R/2}(x)\Big).$$

Proof of Lemma 2.1: We follow [1, Chapter 3] and assume that (2.3) holds for some $\varepsilon > 0$ being determined later. For k sufficiently large we also have

$$\fint_{B_R(x)} |u_k - (u_k)_R|^p \, dz < \varepsilon.$$

Recalling (2.2) and the smallness condition $a < (\operatorname{diam} K)^{-1}$ it is easy to check that a Caccioppoli type inequality

$$R^{p-n} \cdot \int_{B_{3/4R}(x)} |Du_k|^p \, dz \le c \cdot \fint_{B_R(x)} |u_k - (u_k)|^p \, dz$$

holds. Going through the proof of [1, Theorem 3.1] we see

$$u_k \in C^{0,\alpha}\Big(B_{R/2}(x)\Big), \quad [u_k]_{C^{0,\alpha}\left(B_{R/2}(x)\right)} \le C$$

provided ε is small enough (depending on absolute data). From this uniform Hölder bounds for the first derivatives can be deduced along the lines of [1, Theorem 3.2] with obvious simplifications. Part b) of the lemma follows from Arcela's theorem. $\square$

3. Solution of the variational inequality (V).

As in Chapter 2 we let u_k denote a solution of (2.1), and we want to show that the limit function u solves our problem (V).

Let

$$\begin{aligned}\Sigma &:= \{x \in \Omega : \liminf_{\rho \downarrow 0} \fint_{B_\rho(x)} |u - (u)_\rho|^p \, dz > 0\} \\ &\subset \{x \in \Omega : \quad \liminf_{\rho \downarrow 0} \rho^{p-n} \int_{B_\rho(x)} |Du|^p \, dz > 0\}.\end{aligned}$$

By Lemma 2.1 Σ is a relatively closed subset of Ω with $\mathcal{H}^{n-p}(\Sigma) = 0$, especially $\operatorname{cap}_p(\Sigma) = 0$, and we already know $u_k \to u$ in $C^{1,\alpha}$ for compact subsets of $\Omega - \Sigma$. Fix a small ball $B_r(x_0)$ in $\Omega - \Sigma$ and consider a function $w \in \mathbb{K}$ such that $\operatorname{spt}(w - u) \subset B_r(x_0)$. For $\eta \in C_0^1\big(B_r(x_0), [0,1]\big)$, $\eta = 1$ on $B_{r-\delta}(x_0)$, the function $v := (1 - \eta) \cdot u_k + \eta w$ is admissible in (2.1) and by first letting k tend to infinity and then choosing $\delta > 0$ small we arrive at (V) at least for functions w as above. A covering argument then implies

$$\begin{aligned}&\int_\Omega \Big(|Du|^{p-2} Du \cdot D(w - u) - f(\cdot, u, Du) \cdot (w - u)\Big) \, dx \ge 0, \\ &w \in \mathbb{K}, \quad \operatorname{spt}(w - u) \subset \Omega - \Sigma.\end{aligned} \tag{3.1}$$

In order to proceed further we linearize the variational inequality (3.1) where we make use of the smoothness of ∂K: as in [1, Theorem 2.1, 2.2] we get for all $\psi \in C_0^1(\Omega, \mathbb{R}^N)$, $\operatorname{spt}(\psi) \cap \Sigma = \emptyset$,

$$\begin{aligned}&\int_\Omega \Big(|Du|^{p-2} Du \cdot D\psi - f(\cdot, u, Du) \cdot \psi\big) \, dx \\ &\qquad = \int_{\Omega \cap [u \in \partial K]} \psi \cdot N(u) \, b(\cdot, u, Du) \, dx\end{aligned} \tag{3.2}$$

where $N(y)$ is the interior unit normal vectorfield of ∂K and $b(\cdot, u, Du)$ denotes a function with the properties

$$\begin{cases} b(\cdot, u, Du) \ge 0 \quad \text{a.e. on } [u \in \partial K] := \{x \in \Omega : u(x) \in \partial K\}, \\ |b(\cdot, u, Du)| \le \tilde{a} \cdot |Du|^p, \quad \tilde{a} = \tilde{a}(n, p, N, \partial K). \end{cases}$$

Remark. It is easy to check that the functions u_t and v_t defined in the proof of [1, Theorem 2.1] belong to the class $\mathbb{K}$ and are admissible in (3.1) provided the support of the cut-off function η occurring in the definitions of u_t and v_t is disjoint to Σ.

Let $\psi \in C_0^1(\Omega, \mathbb{R}^N)$ denote an arbitrary test vector; since $\operatorname{cap}_p(\Sigma) = 0$ we find a sequence $\eta_\nu \in C^\infty\big(\mathbb{R}^n, [0,1]\big)$ such that $\operatorname{spt} \eta_\nu \cap \Sigma = \emptyset$, $\eta_\nu \to 1$ a.e. and $\int_{\mathbb{R}^n} |D\eta_\nu|^p \, dx \to 0$. Inserting $\eta_\nu \psi$ into (3.2) and passing to the limit $\nu \to \infty$ we get (3.2) for all $\psi \in C_0^1(\Omega, \mathbb{R}^N)$ and, by approximation, for all $\psi \in \mathring{H}^{1,p}(\Omega, \mathbb{R}^N) \cap L^\infty$. We apply this result to $\psi := v - u$ where $v \in \mathbb{K}$ is arbitrary. Observing

$$(v - u) \cdot N(u) \geq 0 \quad \text{a.e. on } [u \in \partial K]$$

(by the convexity of K) we have shown that u is a solution of the variational inequality (V) having the regularity properties stated in our Theorem.

4. Applications to nonlinear elliptic systems: existence of small solutions.

We here consider the problem of finding a solution $u \in H^{1,p}(\Omega, \mathbb{R}^N)$ of the nonlinear Dirichlet problem

$$\begin{cases} -D_\alpha\Big(|Du|^{p-2}\, D_\alpha u\Big) = f(\cdot, u, Du) & \text{on } \Omega, \\ u = u_0 \quad \text{on } \partial\Omega \end{cases}$$

where f satisfies the hypotheses stated in Section 1, especially the estimate (1.1), and u_0 is given in the space $H^{1,p}(\Omega, \mathbb{R}^N)$.

Theorem. *Suppose $u_0 \in L^\infty(\Omega, \mathbb{R}^N)$ and in addition let the smallness condition*

$$a < \Big(2 \cdot \|u_0\|_\infty\Big)^{-1} \tag{4.2}$$

hold. Then problem (4.1) admits at least one solution $u \in H^{1,p}(\Omega, \mathbb{R}^N)$ being of class $C^{1,\varepsilon}$ on $\Omega - \Sigma$ where Σ is a relatively closed subset of Ω such that $\mathcal{H}^{n-p}(\Sigma) = 0$. In case $p \geq n$ we have $\Sigma = \emptyset$.

Remarks. 1) For $p = 2$ it is possible to replace (4.2) by the weaker condition $a < \|u_0\|_\infty^{-1}$. We do not know how to obtain this result for general p.

2) Under certain assumptions on f it can be shown that there are no interior singularities, some ideas will be given after the proof of the Theorem.

3) It turns out that the above solution satisfies the maximum principle $\|u\|_\infty \leq \|u_0\|_\infty$ by the way staying in the convex hull of the boundary values u_0. This corresponds to our results in [2] where we constructed "small" p-harmonic maps of Riemannian manifolds.

Proof of the Theorem: Let $M := \|u_0\|_\infty$ and consider the ball $K := \{z \in \mathbb{R}^N : |z| \leq M + \varepsilon\}$. For ε sufficiently small (4.2) implies

$$a < (\operatorname{diam} K)^{-1}$$

so that there exists a solution $u \in H^{1,p}(\Omega, K)$ of the variational inequality

$$\begin{cases} \int_\Omega |Du|^{p-2}\, Du \cdot D(v-u)dx \geq \int_\Omega f(\cdot, u, Du) \cdot (v-u)\, dx \\ \text{for all } v \in \mathbb{K} \end{cases}$$

with $\mathbb{K}$ being defined in Section 1.

For $\eta \in C_0^1(\Omega, [0,1])$ we let $v := (1-\eta) \cdot u$ which is admissible in (4.3) so that

$$\int_\Omega |Du|^{p-2}\, Du \cdot D(-\eta u)\, dx \geq \int_\Omega (-\eta u) \cdot f(\cdot, u, Du)\, dx,$$

hence

$$\int_\Omega \eta \left[1 - a \cdot \|u\|_\infty\right] |Du|^p\, dx + \int_\Omega |Du|^{p-2}\, \frac{1}{2} \nabla |u|^2 \cdot \nabla \eta\, dx \leq 0$$

and we arrive at

$$\int_\Omega |Du|^{p-2}\, \nabla |u|^2 \cdot \nabla \eta\, dx \leq 0.$$

By approximation this inequality extends to all $\eta \in \mathring{H}^{1,p}(\Omega)$, $\eta \geq 0$, especially we may choose

$$\eta := \max(0, |u|^2 - M^2)$$

since u has boundary values u_0 and $\|u_0\|_\infty = M$. We then get

$$\int_{[|u|>M]} |Du|^{p-2}\, |\nabla \eta|^2\, dx = 0$$

so that $|Du|^{p-2}\, |\nabla \eta|^2 = 0$ a.e. on the set $[|u| > M]$. Since $Du(x) = 0$ implies $\nabla \eta(x) = 0$ we deduce $\nabla \eta = 0$ a.e. on Ω by the way $\eta = 0$ a.e. on Ω, which gives $\|u\|_\infty \leq \|u_0\|_\infty$. For $\psi \in C_0^1(\Omega, \mathbb{R}^N)$ and $0 < t \leq \varepsilon \cdot \|\psi\|_\infty^{-1}$, the function $v := u + t \cdot \psi$ belongs to the class $\mathbb{K}$ so that

$$\int_\Omega |Du|^{p-2}\, Du \cdot D\psi\, dx \geq \int_\Omega f(\cdot, u, Du) \cdot \psi\, dx$$

which proves that u is a solution of (4.1) satisfying the (partial) regularity properties stated in Theorem. $\square$

Let us add a final comment concerning removable singularities. First of all it is easy to check that

$$|Du|^{\frac{p}{2}-1}\, Du \in H^{1,2}_{\mathrm{loc}}(\Omega - \Sigma)$$

and $$|Du|^{p-2}\, Du \in H^{1,\frac{p}{p-1}}_{\mathrm{loc}}(\Omega - \Sigma)$$

so that $$-D_\alpha\Big(|Du|^{p-2}\, D_\alpha u\Big) = f(\cdot, u, Du)$$

holds almost everywhere on $\Omega - \Sigma$. This implies

$$\int_{\Omega-\Sigma}\Big(|Du|^p \cdot \operatorname{div} X - p \cdot |Du|^{p-2}\, D_\alpha u \cdot D_\beta u\, D_\alpha X^\beta\Big)\, dx = \int_{\Omega-\Sigma} p\Big(f(\cdot, u, Du)\cdot D_\beta u\Big) X^\beta\, dx$$

for all $X \in C_0^1(\Omega - \Sigma, \mathbb{R}^n)$. If we impose the structural condition

$$\left\{\begin{array}{l} f(x,y,Q)\cdot Q^\alpha = 0, \quad \alpha = 1,\dots,n, \\ \text{for all } x \in \Omega\,,\ y \in \mathbb{R}^N,\ Q \in \mathbb{R}^{nN} \end{array}\right. \tag{4.4}$$

then

$$\int_{\Omega-\Sigma}\Big(|Du|^p \operatorname{div} X - p \cdot |Du|^{p-2}\, D_\alpha u \cdot D_\beta u\, D_\alpha X^\beta\Big) dx = 0, \quad X \in C_0^1(\Omega - \Sigma, \mathbb{R}^n). \tag{4.5}$$

Suppose now that Σ is discrete. W.l.o.g. we may assume $0 \in \Sigma$ and that 0 is the only singular point in the ball $B_r(0) =: B$. Then it is easy to check that (4.5) implies the standard monotonicity formula

$$s^{p-n}\int_{B_s} |Du|^p dx - t^{p-n}\int_{B_t} |Du|^p\, dx = p\int_{B_s - B_t} |Du|^{p-2}\, |D_r u|^2 \cdot |x|^{p-n}\, dx \tag{4.6}$$

for all balls $B_t \subset B_s \subset B_r$. (In order to justify this, one has to multiply the radial vectorfield X occuring in the proof of (4.6) by a sequence of cut-off functions.) Now, just as in the proof of [3, Theorem 1.2], a blow-up argument relying on (4.6) gives

$$\lim_{\rho\downarrow 0} \rho^{p-n}\int_{B_\rho(0)} |Du|^p\, dx = 0$$

so that 0 is a removable singular point.

Remark. For general singular sets Σ, i.e. $\operatorname{cap}_p(\Sigma) = 0$, it is not obvious if (4.5) is sufficient to prove the monotonicity formula for balls with center $x_0 \in \Sigma$. In the positive case the singular set will be removable.

What is the meaning of the structural condition (4.4)? In the codimension 1 case, i.e. $N = n + 1$, a class of functions f satisfying (4.4) is given by

$$f(x,y,Q) := f_0(x,y,Q)\cdot *Q_1 \wedge \dots \wedge Q_n \,/\, |Q_1 \wedge \dots \wedge Q_n|$$

where $f_0 : \Omega \times \mathbb{R}^N \times \mathbb{R}^{nN} \to \mathbb{R}$ is a scalar function growing of order p in Q and $* : \Lambda_n\mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ denotes the isomorphism between the space of n-vectors in $\mathbb{R}^{n+1}$ and $\mathbb{R}^{n+1}$.

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